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Common Fixed Point Results for Compatible Mappings of Type (C)

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Abstract

The aim of this paper is to study and generalize common fixed point theorems for four self-mappings compatible of type (C) in the settings of complete metric spaces. The main idea is the extensions and generalizations of some fixed point theorems related to compatible mappings.

1 Introduction and Preliminaries

According to Banach [5] "A self map $T : X \to X$ from a complete metric space X into itself has a fixed, if it satisfy the inequality $d(Tx, Ty) \leq qd(x, y)$ for all $x, y \in X$ and $0 \leq q < 1$". Ever since, its introduction in 1922, sevseral authors have generalized this theorem and results have been extended for fixed points in terms of more than one mappings. In this direction Jungck [12] in 1976, introduced the concept of commuting maps for the purpose of calculating common fixed points on complete metric spaces. Since then many generalizations of fixed point theorems for commuting maps have appeared. Similarly, the idea of compatible mappings was introduced by Jungck [13] in 1986. Jungck et al. [14]

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also introduced compatible mappings of type (A) and generalized several theorems for fixed point resuls. The introduction of the notion of compatibility of mappings by Jungck has proved a turning point and an essential tool in the area of fixed point theory for the improvement of fixed point results. Several types of compatible mappings have now been introduced during the last many years and reults have been generalized and unified in different directions of which we mention names of a few authors viz, Chugh and Kumar [6], Fisher [8], Pant [18], Popa [23], Singh [29].

Our objectives in this paper is the establishement of some common fixed results for four compatible mappings of type (C) in complete metric spaces. The results of this paper are motivated by a paper of Mukherjee [17]. The following fixed point theorem was proved in [17].

Theorem 1.1. *[17] Let f and g be mappings of a complete metric space X into itself with f continuous. Let f and g commute with each other and $g(X) \subseteq f(X)$. Also, let g satisfiy the following conditions:*

$$\begin{aligned}(d(gx, gy)) \leq\ & \alpha_1 d(g(x), f(x)) + \alpha_2 d(g(y), f(y)) \\ & + \alpha_3 d(g(x), f(y)) + \alpha_4 d(g(y), f(x)) \\ & + \alpha_5 d(f(x), f(y))\end{aligned} \tag{1.1}$$

with $\alpha_i \geq 0$ for all i and $\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 < 1$. Then f and g have a unique common fixed point in X.

Definition 1.1. *[1] Let f_1 and f_2 be self mappings of a set X. If $y = f_1\zeta = f_2\zeta = \zeta$ for some $\zeta \in X$, then ζ is called a coincidence point of f_1 and f_2 and y is called a point of coincidence of f_1 and f_2.*

Definition 1.2. *[12] Let (X, d) be a metric space. Two self maps f_1, $f_2 : X \to X$ are said to be commuting mappings if $f_1 f_2 x = f_2 f_1 x$ for all $x \in X$.*

Definition 1.3. *[28] Two self maps f_1 and f_2 of a metric space (X, d) into itself are said to be weakly commuting iff $d(f_1 f_2 x, f_2 f_1 x) \leq d(f_1 x, f_2 x)$ for all $x \in X$.*

Definition 1.4. *[13] Two self maps f_1 and f_2 of a metric space X are said to be compatible if*

$$\lim_{n\to\infty} d(f_1 f_2 x_n, f_2 f_1 x_n) = 0$$

where $x_n \in X$ *is a sequence in* X *such that* $\lim_{n\to\infty} f_1 x_n = \lim_{n\to\infty} f_2 x_n = t \in X$.

Definition 1.5. *[15] Let* (X, d) *be a metric space. Two self maps* f_1, $f_2 : X \to X$ *are said to weakly compatible if they commute at their coincidence points, that is,* $f_1\zeta = f_2\zeta$ *implies* $f_1 f_2 \zeta = f_2 f_1 \zeta$ *for some* $\zeta \in X$.

Definition 1.6. *[14] Two self maps* f *and* g *of a metric space* (X, d) *are compatible of type (A), if*

$$\lim_{n\to\infty} d\left(f_1^2 x_n, f_2 f_1 x_n\right) = 0 \text{ and } \lim_{n\to\infty} d\left(f_1 f_2 x_n, f_2^2 x_n\right) = 0$$

where $\{x_n\}$ *is a sequence in* X *such that* $lim_{n\to\infty} f_1 x_n = \lim_{n\to\infty} f_2 x_n = t$ *for some* $t \in X$.

Definition 1.7. *[19] Two self maps* f_1 *and* f_2 *of a metric space* (X, d) *are said to be compatible of type (P), if*

$$\lim_{n\to\infty} d\left(f_1 f_2 x_n, f_2 f_1 x_n\right) = 0$$

whenever $\{x_n\}$ *is a sequence in* X *such that* $\lim_{n\to\infty} f_1 x_n = \lim_{n\to\infty} f_2 x_n = t$ *for some* $t \in X$.

Definition 1.8. *[22] Two self maps* f_1 *and* f_2 *of a metric space* (X, d) *are compatible of type (B), if the given two inequalities are satisfied*

$$\begin{aligned}
\lim_{n\to\infty} d\left(f_1 f_2 x_n, f_2^2 x_n\right) &\leq \frac{1}{2}\left[\lim_{n\to\infty} d\left(f_1 f_2 x_n, f_1 t\right) + \lim_{n\to\infty} d\left(f_1 t, f_1^2 x_n\right)\right] \\
\lim_{n\to\infty} d\left(f_2 f_1 x_n, f_1^2 x_n\right) &\leq \frac{1}{2}\left[\lim_{n\to\infty} d\left(f_2 f_1 x_n, f_2 t\right) + \lim_{n\to\infty} d\left(f_1 t, f_1^2 x_n\right)\right]
\end{aligned}$$

where $\{x_n\}$ *is a sequence in* X *such that* $\lim_{n\to\infty} f_1 x_n = \lim_{n\to\infty} f_2 x_n = t$ *for some* $t \in X$.

Definition 1.9. *[21] Two self maps* f_1 *and* f_2 *of a metric space* (X, d) *are compatible of type (C), if the following inequalities are satisfied*

$$\lim_{n\to\infty} d\left(f_1 f_2 x_n, f_2^2 x_n\right) \leq \tfrac{1}{3} \begin{array}{c}[\lim_{n\to\infty} d\left(f_1 f_2 x_n, f_1 t\right) + \lim_{n\to\infty} d\left(f_1 t, f_1^2 x_n\right) \\ + \lim_{n\to\infty} d\left(f_1 t, f_2^2 x_n\right)]\end{array}$$

and

$$\lim_{n\to\infty} d\left(f_2 f_1 x_n, f_1^2 x_n\right) \leq \frac{1}{3} \begin{array}{c}[\lim_{n\to\infty} d\left(f_2 f_1 x_n, f_2 t\right) + \lim_{n\to\infty} d\left(f_2 t, f_2^2 x_n\right) \\ + \lim_{n\to\infty} d\left(f_2 t, f_1^2 x_n\right)]\end{array}$$

where $\{x_n\}$ *is a sequence in* X *such that* $\lim_{n\to\infty} f_1 x_n = \lim_{n\to\infty} f_2 x_n = t$ *for some* $t \in X$.

Lemma 1.1. *[12] Let* f_1 *and* f_2 *be compatible mappings of a metric space* (X, d) *into itself. If* $\lim_{n\to\infty} f_1 x_n = \lim_{n\to\infty} f_2 x_n = \zeta$, *for some* $\zeta \in X$, *then* $\lim_{n\to\infty} f_2 f_1 x_n = f_1 \zeta$, *if* f_1 *is continuous.*

Proposition 1.1. *Let* (X, d) *be a complete metric space and* $f : X \to X$ *be any mapping. Then* f *is continuous iff* f *is sequentially continuous.*

Proposition 1.2. *Let* f_1 *and* f_2 *be mappings from a complete metric space* (X, d) *into itself. Then the following are equivalent:*

(i) f_1 *and* f_2 *are compatible,*

(ii) f_1 *and* f_2 *are compatible of type (A),*

(iii) f_1 *and* f_2 *are compatible of type (B),*

(iv) f_1 *and* f_2 *are compatible of type (C).*

Proposition 1.3. *[21] Let* f_1 *and* f_2 *be compatible mappings of type (C) from a metric space* (X, d) *into itself. If* $\lim_{n\to\infty} f_1 x_n = \lim_{n\to\infty} f_2 x_n = t$ *for some* $t \in X$, *then we have the following:*

(i) $\lim_{n\to\infty} f_2 f_2 x_n = f_1 t$ *if* f_1 *is continuous at* t,

(ii) $\lim_{n\to\infty} f_1 f_2 x_n = f_2 t$ *if* f_2 *is continuous at* t,

(iii) $f_1 f_2 t = f_2 f_1 t$ *and* $f_1 t = f_2 t$ *if* f_1 *and* f_2 *are continuous at* t.

2 Common Fixed Point Results

In this section we obtain coincidence and common fixed point theorems for four compatible mappings of type (C) in complete metric spaces. We shall extend the results of Theorem 1.1 to four mappings with the condition of continuity being imposed on two of the mappings h and k.

Theorem 2.1. *Let (X,d) be a complete metric space and let f, g, h and $k : X \to X$ be four self mappings of X into itself satisfying the following conditions:*

(i) $f(X) \subseteq k(X)$, $g(X) \subseteq h(X)$,

(ii) *either h or k is continuous,*

(iii) $\{f,\ k\}$ *and* $\{g,\ h\}$ *are two commuting pairs,*

(iv) *if* $\{f,\ k\}$ *and* $\{g,\ h\}$ *are compatible of type (C) and*

(v) *for all $x, y \in X$ the following inequality is satisfied*

$$\begin{aligned} d(fx, gy) \leq{} & k_1 d(hx, ky) + k_2 d(fx, ky) \\ & + k_3 d(fx, hx) + k_4 d(hx, gy) + k_5 d(gy, ky) \end{aligned} \tag{2.1}$$

where $k_1,\ k_2,\ k_3,\ k_4,\ k_5 \geq 0$, such that $0 \leq k_1 + k_2 + k_3 + 2k_4 + k_5 < 1$. Then f, g, h and k have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be an arbitrary point. We need to prove that the sequence $\{y_n\}$ with initial point x_0 is a Cauchy sequence in X.

$$\begin{cases} y_n = fx_n = kx_{n+1} \\ y_{n+1} = gx_{n+1} = hx_{n+2} \end{cases} \text{for } n = 0, 1,\ 2\ ,\ \ldots\ . \tag{2.2}$$

Now by using condition (2.2) in (2.1) we have

$$\begin{aligned} d(fx_n, gx_{n+1}) \leq{} & k_1 d(hx_n, kx_{n+1}) + k_2 d(fx_n, kx_{n+1}) \\ & + k_3 d(fx_n, hx_n) + k_4 d(hx_n, gx_{n+1}) \\ & + k_5 d(gx_{n+1}, kx_{n+1}) \end{aligned}$$

From the last inequality, we get

$$d(y_{n+1}, y_n) \leq k_1 d(y_n, y_{n-1}) + k_2 d(y_n, y_n) + k_3 d(y_n, y_{n-1}) + k_4 d(y_{n+1}, y_{n-1}) + k_5 d(y_{n+1}, y_n)$$

which yields

$$d(y_{n+1}, y_n) \leq \alpha d(y_n, y_{n-1})$$
$$\text{where } q = \frac{(k_1+k_3+k_4)}{[1-(k_4+k_5)]} < 1$$

$$d(y_{n+1}, y_n) \leq q d(y_n, y_{n-1}). \tag{2.3}$$

Therefore, by induction for all $n \in \mathbb{N}$, we get

$$d(y_{n+1}, y_n) \leq q d(y_n, y_{n-1}) \leq q^2 d(y_{n-1}, y_{n-2}) \leq \ ... \ \leq q^{n+1} d(x_0, x_1).$$

Now, for any $m, n \in \mathbb{N}$ with $m > n$ and repeated use of (2.3) we have

$$\begin{aligned} d(y_n, y_m) &\leq [d(y_{n+1}, y_n) + d(y_n, y_m)] \\ &\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \ ... \ + \\ &\quad d(y_{m-2}, y_{m-1}) + d(y_{m-1}, y_m) \\ &\leq \left[q^n + q^{n+1} + \ ... \ + q^{m-2} + q^{m-1}\right] d(y_0, y_1) \\ &= q^n \left[1 + q + q^2 + \ ... \ + q^{m-n-2} + q^{m-n-1}\right] d(y_0, y_1) \\ &\leq \frac{q^n}{(1-q)} d(y_0, y_1) \\ &\Rightarrow d(y_n, y_m) \leq \frac{q^n}{(1-q)} d(y_0, y_1). \end{aligned}$$

Thus, $d(y_n, y_m) \to 0$ as $n, m \to \infty$.(since $0 \leq q < 1$), hence $\{y_n\}$ is a Cauchy sequence and by the completeness of X, $\{y_n\}$ converges to some $\zeta \in X$ and hence $\lim_{n\to\infty} y_n = \zeta$.

Next, we shall prove that the point $\zeta \in X$ is the unique common fixed point of f, g, h and k. For this let $\zeta \in X$ be such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} kx_{n+1} = \lim_{n\to\infty} gx_{n+1} = \lim_{n\to\infty} hx_{n+2} = \zeta.$$

Suppose h is continuous then we have

$$h\zeta = h(\lim_{n\to\infty} hx_{n+2}) = \lim_{n\to\infty} h^2 x_{n+2}.$$

Also,

$$h\zeta = h\left(\lim_{n\to\infty} gx_n\right) = \lim_{n\to\infty} hgx_n = \lim_{n\to\infty} ghx_n.$$

Since $\{g, h\}$ is commuting and is compatible of type (C). Hence, $\lim_{n\to\infty} d\left(ghx_n, hgx_n\right) = 0$ and therefore, $\lim_{n\to\infty} ghx_n = h\zeta$ for some $\zeta \in X$. Next we use (2.1) we have

$$\begin{aligned} d\left(fhx_n, gx_{n+1}\right) \leq\, & k_1 d\left(hhx_n, kx_{n+1}\right) + k_2 d\left(fhx_n, kx_{n+1}\right) \\ & + k_3 d\left(fhx_n, hhx_n\right) + k_4 d\left(hhx_n, gx_{n+1}\right) \\ & + k_5 d\left(gx_{n+1}, kx_{n+1}\right). \end{aligned}$$

On taking limit as $n \to \infty$ we get

$$\begin{aligned} d\left(h\zeta, \zeta\right) & \leq k_1 d\left(h\zeta, \zeta\right) + k_2 d\left(h\zeta, \zeta\right) \\ & \quad + k_3 d\left(h\zeta, h\zeta\right) + k_4 d\left(h\zeta, \zeta\right) + k_5 d\left(\zeta, \zeta\right) \\ & = \left(k_1 + k_2 + k_4\right) d\left(h\zeta, \zeta\right) \\ & \leq \left(k_1 + k_2 + k_3 + 2k_4 + k_5\right) d\left(h\zeta, \zeta\right). \end{aligned}$$

Thus,

$$d\left(h\zeta, \zeta\right) \leq \left(k_1 + k_2 + k_3 + 2k_4 + k_5\right) d\left(h\zeta, \zeta\right).$$

Since, $(k_1 + k_2 + k_3 + 2k_4 + k_5) < 1$ so we get

$$h\zeta = \zeta.$$

Again using continuity of k, we have

$$k\zeta = k(\lim_{n\to\infty} kx_{n+1}) = \lim_{n\to\infty} k^2 x_{n+1}$$

and

$$k\zeta = k\left(\lim_{n\to\infty} fx_n\right) = \lim_{n\to\infty} kfx_n = \lim_{n\to\infty} fkx_n.$$

Since $\{f, k\}$ is compatible and compatibility implies compatibility of type (C) so, $\lim_{n\to\infty} d\left(kfx_n, fkx_n\right) = 0$ and this yields $\lim_{n\to\infty} fkx_n = k\zeta$. Now, from (2.1) we obtain by putting $x = x_n$, $y = kx_{n+1}$ and taking limit as $n \to \infty$ to obtain

$$\begin{aligned} d\left(fx_n, gkx_{n+1}\right) \leq\ & k_1 d\left(hx_n, kkx_{n+1}\right) + k_2 d\left(fx_n, kkx_{n+1}\right) \\ & + k_3 d\left(fx_n, hx_n\right) + k_4 d\left(hx_n, gkx_{n+1}\right) \\ & + k_5 d\left(gkx_{n+1}, kkx_{n+1}\right) \end{aligned}$$

$$\begin{aligned} d\left(\zeta, k\zeta\right) &\leq k_1 d\left(\zeta, k\zeta\right) + k_2 d\left(\zeta, k\zeta\right) + k_3 d\left(\zeta, \zeta\right) \\ &\quad + k_4 d\left(\zeta, k\zeta\right) + k_5 d\left(k\zeta, k\zeta\right) \\ &= \left(k_1 + k_2 + k_4\right) d\left(\zeta, k\zeta\right) \\ &\leq \left(k_1 + k_2 + k_3 + 2k_4 + k_5\right) d\left(\zeta, k\zeta\right). \end{aligned}$$

Hence,

$$d\left(\zeta, k\zeta\right) \leq \left(k_1 + k_2 + k_3 + 2k_4 + k_5\right) d\left(\zeta, k\zeta\right).$$

But $\left(k_1 + k_2 + k_3 + 2k_4 + k_5\right) < 1$ which implies that

$$k\zeta = \zeta.$$

And using (2.1) again replacing x by ζ and y by x_{n+1} and take limit as $n \to \infty$

$$\begin{aligned} d\left(f\zeta, gx_{n+1}\right) \leq\ & k_1 d\left(h\zeta, kx_{n+1}\right) + k_2 d\left(f\zeta, kx_{n+1}\right) \\ & + k_3 d\left(f\zeta, h\zeta\right) + k_4 d\left(h\zeta, gx_{n+1}\right) + k_5 d\left(gx_{n+1}, kx_{n+1}\right) \end{aligned}$$

$$\begin{aligned} d\left(f\zeta, \zeta\right) \leq\ & k_1 d\left(h\zeta, \zeta\right) + k_2 d\left(f\zeta, \zeta\right) + k_3 d\left(f\zeta, h\zeta\right) \\ & + k_4 d\left(h\zeta, \zeta\right) + k_5 d\left(\zeta, \zeta\right). \end{aligned}$$

Now, replace $h\zeta$ by ζ we get

$$\begin{aligned} d(f\zeta,\zeta) &\leq k_1 d(\zeta,\zeta) + k_2 d(f\zeta,\zeta) + k_3 d(f\zeta,\zeta) \\ &\quad + k_4 d(\zeta,\zeta) + k_5 d(\zeta,\zeta) \\ &\leq (k_2+k_3)\, d(f\zeta,\zeta). \end{aligned}$$

And so,

$$d(f\zeta,\zeta) \leq (k_2+k_3)\, d(f\zeta,\zeta).$$

Since, $0 \leq (k_2+k_3) < 1$ and therefore,

$$f\zeta = \zeta.$$

Lastly, from (2.1) we have

$$\begin{aligned} d(f\zeta,g\zeta) &\leq k_1 d(h\zeta,k\zeta) + k_2 d(f\zeta,k\zeta) \\ &\quad + k_3 d(f\zeta,h\zeta) + k_4 d(h\zeta,g\zeta) + k_5 d(g\zeta,k\zeta). \end{aligned}$$

Now, we are using $f\zeta = h\zeta = k\zeta = \zeta$ to get

$$\begin{aligned} d(\zeta,g\zeta) &\leq k_1 d(\zeta,\zeta) + k_2 d(\zeta,\zeta) \\ &\quad + k_3 d(\zeta,\zeta) + k_4 d(\zeta,g\zeta) + k_5 d(g\zeta,\zeta) \\ &= k_4 d(\zeta,g\zeta) + k_5 d(g\zeta,\zeta) \\ &= (k_4+k_5)\, d(g\zeta,\zeta) \\ &\Rightarrow d(\zeta,g\zeta) \leq (k_4+k_5)\, d(g\zeta,\zeta) \\ &\Rightarrow g\zeta = \zeta. \end{aligned}$$

Therefore, ζ is a common fixed point of four self-mappings f, g, h and k compatible of type (C). Hence,

$$f\zeta = g\zeta = h\zeta = k\zeta = \zeta.$$

Uniqueness: For the unicity of common fixed point let us assume that ξ is another common fixed point of f, g, h and k different from ζ. Then by inequality (2.1) we have

$$\begin{aligned} d(\zeta,\xi) = d(f\zeta,g\xi) &\leq k_1 d(h\zeta,k\xi) + k_2 d(f\zeta,k\xi) \\ &\quad + k_3 d(f\zeta,h\xi) + k_4 d(h\zeta,g\xi) + k_5 d(g\xi,k\xi). \end{aligned}$$

From which it follows that

$$d(\zeta,\xi) = 0 \Leftrightarrow \zeta = \xi.$$

Thus, ζ is the unique common fixed point of f, g, h and k. □

The following example support our results.

Example 2.1. Let $X = [0,1]$ with the usual metric $d(x,y) = |x-y|$ for all $x, y \in X$. We can define self maps f, g, h and $k : X \to X$ by

$$f(x) = 1,\ g(x) = \tfrac{x+1}{2},\ h(x) = \tfrac{4x-2}{2} \text{ and } k(x) = 2x-1.$$

Then f, g, h and k are all continuous. Hence, we can see that $\{f,k\}$ and $\{g,h\}$ are compatible of type (C), since they commute at coincident point $x = 1$ with $f(X) \subseteq k(X) = \{1\} \subseteq [-1,1]$ and $g(X) \subseteq h(X) = \left[\frac{1}{2},1\right] \subseteq [-1,1]$.

Now, if $\{x_n\} = 1 + \frac{1}{n} \subseteq [0,1]$ is a sequence in X, then

$$\lim_{n\to\infty} f(x_n) = 1 \text{ and } \lim_{n\to\infty} k(x_n) = 1.$$

As a result, we have fx_n, $kx_n \to \zeta = 1$

$$\begin{aligned} 0 &= \lim_{n\to\infty} |fkx_n - kkx_n| \\ &\le \frac{1}{3}\left[\lim_{n\to\infty} |fkx_n - f(1)| + \lim_{n\to\infty} |f(1) - ffx_n| + \lim_{n\to\infty} |f(1) - kkx_n|\right] = 0 \end{aligned}$$

also

$$\begin{aligned} 0 &= \lim_{n\to\infty} |kfx_n - ffx_n| \\ &\le \frac{1}{3}\left[\lim_{n\to\infty} |kfx_n - k(1)| + \lim_{n\to\infty} |k(1) - kkx_n| + \lim_{n\to\infty} |k(1) - ffx_n|\right] = 0. \end{aligned}$$

Therefore, the pair $\{f,k\}$ is compatible of type (C). Again, using the same sequence $x_n = 1 + \frac{1}{n}$ for $n \ge 1$, we have

$$\lim_{n\to\infty} gx_n = \lim_{n\to\infty} hx_n = \zeta \to 1.$$

Hence,

$$
\begin{aligned}
0 &= \lim_{n\to\infty} |ghx_n - hhx_n| \\
&\le \frac{1}{3}\left[\lim_{n\to\infty} |ghx_n - g(1)| + \lim_{n\to\infty} |g(1) - ggx_n| + \lim_{n\to\infty} |g(1) - hhx_n|\right] = 0.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
0 &= \lim_{n\to\infty} |hgx_n - ggx_n| \\
&\le \frac{1}{3}\left[\lim_{n\to\infty} |hgx_n - h(1)| + \lim_{n\to\infty} |h(1) - hhx_n| + \lim_{n\to\infty} |h(1) - ggx_n|\right] = 0.
\end{aligned}
$$

Thus, g and h are also compatible of type (C). Hence, all the hypothesis of Theorem 2.1 are satisfied with 1 as the unique common fixed point of the mappings f, g, h and k.

Now, we give the special cases of Theorem 2.1 in the form of some corollaries.

3 Corollaries

Corollary 3.1. *Let (X, d) be a complete metric space and f, g, h, $k : X \to X$ be self mappings of X into itself satisfying the following conditions:*

(i) $f(X) \subseteq k(X)$ and $g(X) \subseteq h(X)$,

(ii) either g or h is continuous,

(iii) $\{f,\ k\}$ and $\{g,\ h\}$ are two commuting pairs,

(iv) $\{f,\ g\}$ and $\{f,\ h\}$ are compatible of type (C) and

(v) for all $x, y \in X$ the following inequality is satisfied

$$
\begin{aligned}
d(fx, gy) \le\ & a d(hx, ky) + b[d(fx, hx) + d(gy, ky)] \\
& + c[d(fx, ky) + d(hx, gy)]
\end{aligned}
$$

where $a, b, c \geq 0$ *and* $0 \leq a+2b+2c < 1$, *then* f, g, h *and* k *have a unique common fixed point in* X.

Proof. If $k_1 = a$, $k_2 = k_4 = b$ and $k_3 = k_5 = c$ in Theorem 2.1, then we have a result what is known as generalized contraction. □

Corollary 3.2. *Let* (X, d) *be a complete metric space and let* $f,\ g, h : X \to X$ *be self mappings of* X *into itself satisfying the following conditions:*

(i) $f(X) \subseteq g(X)$ *or* $f(X) \subseteq h(X)$,

(ii) *either* g *or* h *is continuous,*

(iii) $\{f,\ g\}$ *and* $\{f,\ h\}$ *are two commuting pairs,*

(iv) $\{f,\ g\}$ *and* $\{f,\ h\}$ *are compatible of type (C) and*

(v) *for all* $x, y \in X$ *the following inequality is satisfied*

$$d(fx, gy) \leq ad(hx, hy) + b[d(hx, fx) + d(hy, gy)$$

where $a, b \geq 0$ *and* $0 \leq a + 2b < 1$, *then* $f,\ g$ *and* h *have a unique common fixed point in* X.

Proof. If we put $k_1 = a$, $k_3 = k_5 = b$, $k_1 = k_4 = 0$ and $k = h$ in Theorem 2.1, then we have Theorem 2.1 in [4]. □

Corollary 3.3. *Let* (X, d) *be a complete metric space and let* $f,\ g, h : X \to X$ *be commuting self maps of* X *into itself such that*

(i) $f(X) \subseteq g(X)$ *or* $f(X) \subseteq h(X)$,

(ii) *either* g *or* h *is continuous,*

(iii) *the pair*$\{f, g\}$ *and* $\{f, h\}$ *are commuting and compatible of type (C),*

(iv) and satisfy the inequality

$$d(fx, gy) \le ad(hx, hy) + b[d(fx, hy) + d(hx, gy)]$$

for all $x, y \in X$ and $a, b \ge 0$ are nonnegative real numbers such that $a + 2b < 1$, then f, g, h have a unique common fixed point in X.

Proof. If $k_1 = a$, $k_2 = k_4 = b$, $k_3 = k_5 = 0$ and $k = h$ in Theorem 2.1, then we have Theorem 2.3 in [4]. □

Corollary 3.4. *Let (X, d) be a complete metric space and let f and $g : X \to X$ be four self mapings of X into itself satisfying the following conditions:*

(i) $f(X) \subseteq g(X)$

(ii) g is continuous,

(iii) $\{f, g\}$ is a commuting pair,

(iv) $\{f,\ g\}$ are compatible of type (C) and

(v) for all $x, y \in X$ the inequality is satisfied

$$d(fx, fy) \le b[d(fx, gx) + d(fy, gy)]$$

where $0 \le b < \frac{1}{2}$, then f, g have a unique common fixed point in X.

Proof. If $h = g$, $k = f$, $k_3 = k_5 = b$ and $k_1 = k_2 = k_4 = 0$ in Theorem 2.1, then we have the Kannan result [16]. □

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