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Applications of the Second Kind Chebyshev Polynomials of Bi-Starlike and Bi-Convex λ-Pseudo Functions Associated with Sakaguchi Type Functions

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Abstract

The purpose of this paper is to use the second kind Chebyshev polynomials to introduce a new class of analytic and bi-univalent functions associating bi-starlike and biconvex λ-pseudo functions with Sakaguchi type functions defined in the open unit disk. We determine upper bounds for the initial Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in this class.

1 Introduction

Let $\mathcal{A}$ denote the family of functions f have the type:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$.

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According to the "Koebe One-Quarter Theorem" [6] each function f from S has an inverse f^{-1}, which satisfies

$$f^{-1}(f(z)) = z, \qquad (z \in U)$$

and

$$f\left(f^{-1}(w)\right) = w, \quad \left(|w| < r_0(f), r_0(f) \geq \frac{1}{4}\right)$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots . \quad (1.2)$$

A function $f \in \mathcal{A}$ is named bi-univalent in U if together f and f^{-1} are univalent in U. Let Σ stand for the class of bi-univalent functions in U given by (1.1). Beginning with Srivastava et al. pioneering work [15] on the subject, the large number of works associated with the subject have been presented (see, for example [1, 2, 4, 9, 10, 14–18]. We see that the set Σ is not empty. We see that the functions

$$\frac{z}{1-z}, \quad \frac{1}{2}\log\left(\frac{1+z}{1-z}\right) \quad \text{and} \quad -\log(1-z)$$

are in the set Σ with the corresponding inverse functions

$$\frac{w}{1+w}, \quad \frac{e^{2w}-1}{e^{2w}+1} \quad \text{and} \quad \frac{e^w - 1}{e^w},$$

respectively. But the functions

$$z - \frac{z^2}{2} \quad \text{and} \quad \frac{z}{1-z^2}$$

are not a member of the set Σ.

The problem to find the bound of $|a_n|, (n = 3, 4, \ldots)$ of functions $f \in \Sigma$ is still an open problem.

Sakaguchi [13] introduced the class S_S^* of functions starlike with respect to symmetric points, which consists of functions $f \in S$ satisfying the condition

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z) - f(-z)}\right\} > 0, \quad z \in U.$$

Also, Wang et al. [21] introduced the class $\mathrm{K_s}$ of functions convex with respect to symmetric points, which consists of functions $f \in S$ satisfying the condition

$$\operatorname{Re}\left\{\frac{(zf'(z))'}{(f(z)-f(-z))'}\right\} > 0, \quad z \in U.$$

Frasin [8] introduced and studied the family $S(\mu, m, n)$ consisting of functions $f \in \mathcal{A}$ which satisfy the condition

$$\operatorname{Re}\left\{\frac{(m-n)zf'(z)}{f(mz)-f(nz)}\right\} > \mu,$$

for some $0 \leq \mu < 1; m, n \in \mathbb{C}$ with $m \neq n; |n| \leq 1$ and for all $z \in U$.

Recently, Babalola [3] defined the class $\mathcal{L}_\lambda$ of λ-pseudo-starlike functions which are the functions $f \in \mathcal{A}$ such that

$$\operatorname{Re}\left\{\frac{z\,(f'(z))^\lambda}{f(z)}\right\} > 0, \quad (\lambda \geq 1, z \in U).$$

Let the functions f and g be analytic in U. We say that the function f is said to be subordinate to g, if there exists a Schwarz function w analytic in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$ such that $f(z) = g(w(z))$. This subordination is denoted by $f \prec g$ or $f(z) \prec g(z)$ $(z \in U)$. It is well-known that (see [12]), if the function g is univalent in U, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

The importance of Chebyshev polynomial in numerical analysis is increased in both theoretical and practical points of view. There are four kinds of Chebyshev polynomials. Several researchers dealing with orthogonal polynomials of Chebyshev family, contain mainly results of Chebyshev polynomials of first kind $T_n(t)$, the second kind $U_n(t)$ and their numerous uses in different applications one can refer [5, 7, 11]. The Chebyshev polynomials of the first and second kinds are well-known and they are defined by

$$T_n(t) = \cos n\theta \quad \text{and} \quad U_n(t) = \frac{\sin(n+1)\theta}{\sin\theta} \quad (-1 < t < 1),$$

where n indicates the polynomial degree and $t = \cos n\theta$.

We consider the function

$$H(z,t) = \frac{1}{1-2tz+z^2}, \quad t \in \left(\frac{1}{2}, 1\right], \quad z \in U.$$

We note that if $t = \cos\beta$, where $\beta \in \left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$, then

$$H(z,t) = \frac{1}{1-2\cos\beta z+z^2} = 1 + \sum_{n=1}^{\infty} \frac{\sin(n+1)\beta}{\sin\beta} z^n, \quad z \in U.$$

Therefore

$$H(z,t) = 1 + 2\cos\beta z + \left(3\cos^2\beta - \sin^2\beta\right) z^2 + \cdots, \quad z \in U.$$

In view of [22], we can write

$$H(z,t) = 1 + U_1(t)z + U_2(t)z^2 + \cdots \quad (z \in U, t \in (-1,1)),$$

where

$$U_{n-1} = \frac{\sin(n \arccos t)}{\sqrt{1-t^2}} \quad (n \in \mathbb{N} = \{1, 2, \ldots\})$$

are the Chebyshev polynomials of the second kind. Also, it is known that

$$U_n(t) = 2tU_{n-1}(t) - U_{n-2}(t)$$

and

$$U_1(t) = 2t, \quad U_2(t) = 4t^2 - 1, \quad U_3(t) = 8t^3 - 4t, \ldots. \tag{1.3}$$

The generating function of the first kind of Chebyshev polynomial $T_n(t), t \in [-1,1]$ is given by

$$\sum_{n=0}^{\infty} T_n(t) z^n = \frac{1-tz}{1-2tz+z^2}, \quad z \in U.$$

The Chebyshev polynomials of first kind $T_n(t)$ and of the second kind $U_n(t)$ are connected by

$$\frac{dT_n(t)}{dt} = nU_{n-1}(t), \quad T_n(t) = U_n(t) - tU_{n-1}(t), \quad 2T_n(t) = U_n(t) - U_{n-2}(t).$$

2 Main Results

Definition 2.1. *For $0 \leq \delta \leq 1, \lambda \geq 1, m, n \in \mathbb{C}$ with $m \neq n; |n| \leq 1$ and $t \in \left(\frac{1}{2}, 1\right]$, a function $f \in \Sigma$ is said to be in the class $\mathcal{R}_{\Sigma}(\delta, \lambda, m, n; t)$ if it satisfies the subordinations:*

$$(1-\delta)\frac{(m-n)z\left(f'(z)\right)^{\lambda}}{f(mz)-f(nz)} + \delta\frac{(m-n)\left((zf'(z))'\right)^{\lambda}}{(f(mz)-f(nz))'} \prec H(z,t) = \frac{1}{1-2tz+z^2}$$

and

$$(1-\delta)\frac{(m-n)z\left(g'(w)\right)^{\lambda}}{g(mw)-g(nw)} + \delta\frac{(m-n)\left((wg'(w))'\right)^{\lambda}}{(g(mw)-g(nw))'} \prec H(w,t) = \frac{1}{1-2tw+w^2},$$

where the function $g = f^{-1}$ is given by (1.2).

In particular, if we choose $\delta = 0, m = 1$ and $n = -1$ in Definition 2.1, the family $\mathcal{R}_{\Sigma}(\delta, \lambda, m, n; t)$ reduces to the family $\mathcal{R}_{\Sigma}^{S}(\lambda; t)$ of λ-pseudo bi-starlike functions with respect to symmetrical points which satisfying the following subordinations:

$$\frac{2z\left(f'(z)\right)^{\lambda}}{f(z)-f(-z)} \prec H(z,t) = \frac{1}{1-2tz+z^2}$$

and

$$\frac{2z\left(g'(w)\right)^{\lambda}}{g(w)-g(-w)} \prec H(w,t) = \frac{1}{1-2tw+w^2}.$$

Theorem 2.1. *For $0 \leq \delta \leq 1, \lambda \geq 1, m, n \in \mathbb{C}$ with $m \neq n$; $|n| \leq 1$ and $t \in \left(\frac{1}{2}, 1\right]$, let f be in the class $\mathcal{R}_{\Sigma}(\delta, \lambda, m, n; t)$. Then*

$$|a_2| \leq \frac{2t\sqrt{2t}}{\sqrt{\left|\begin{array}{c} 4\left((2\delta+1)\left(3\lambda-m^2-n^2-mn\right)+(3\delta+1)\left[(m+n)^2-2\lambda(m+n-\lambda+1)\right]\right. \\ \left.-(\delta+1)^2(2\lambda-m-n)^2\right)t^2+(\delta+1)^2(2\lambda-m-n)^2 \end{array}\right|}}$$

and

$$|a_3| \leq \frac{4t^2}{(\delta+1)^2(2\lambda-m-n)^2} + \frac{2t}{(2\delta+1)\left(3\lambda-m^2-n^2-mn\right)}.$$

Proof. Let $f \in \mathcal{R}_\Sigma(\delta, \lambda, m, n; t)$. Then there exists two analytic functions $u, v : U \to U$ given by

$$u(z) = u_1 z + u_2 z^2 + u_3 z^3 + \cdots \quad (z \in U) \tag{2.1}$$

and

$$v(w) = v_1 w + v_2 w^2 + v_3 w^3 + \cdots \quad (w \in U), \tag{2.2}$$

with $u(0) = v(0) = 0, |u(z)| < 1, |v(w)| < 1, z, w \in U$ such that

$$(1-\delta)\frac{(m-n)z\left(f'(z)\right)^\lambda}{f(mz)-f(nz)}+\delta\frac{(m-n)\left((zf'(z))'\right)^\lambda}{(f(mz)-f(nz))'} = 1+U_1(t)u(z)+U_2(t)u^2(z)+\cdots \tag{2.3}$$

and

$$(1-\delta)\frac{(m-n)z\left(g'(w)\right)^\lambda}{g(mw)-g(nw)}+\delta\frac{(m-n)\left((wg'(w))'\right)^\lambda}{(g(mw)-g(nw))'} = 1+U_1(t)v(w)+U_2(t)v^2(w)+\cdots \tag{2.4}$$

Combining (2.1), (2.2), (2.3) and (2.4), we obtain

$$\begin{aligned}(1-\delta)&\frac{(m-n)z\left(f'(z)\right)^\lambda}{f(mz)-f(nz)}+\delta\frac{(m-n)\left((zf'(z))'\right)^\lambda}{(f(mz)-f(nz))'} \\ &= 1+U_1(t)u_1 z+\left[U_1(t)u_2+U_2(t)u_1^2\right]z^2+\cdots\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}(1-\delta)&\frac{(m-n)z\left(g'(w)\right)^\lambda}{g(mw)-g(nw)}+\delta\frac{(m-n)\left((wg'(w))'\right)^\lambda}{(g(mw)-g(nw))'} \\ &= 1+U_1(t)v_1 w+\left[U_1(t)v_2+U_2(t)v_1^2\right]w^2+\cdots.\end{aligned} \tag{2.6}$$

It is well-known that if $|u(z)| < 1$ and $|v(w)| < 1, z, w \in U$, then

$$|u_i| \le 1 \quad \text{and} \quad |v_i| \le 1 \quad \text{for all } i \in \mathbb{N}. \tag{2.7}$$

Comparing the corresponding coefficients in (2.5) and (2.6), after simplifying, we have

$$(\delta+1)(2\lambda-m-n)a_2 = U_1(t)u_1, \tag{2.8}$$

$$\begin{aligned}&(2\delta+1)\left(3\lambda-m^2-n^2-mn\right)a_3\\&\quad+(3\delta+1)\left[(m+n)^2-2\lambda(m+n-\lambda+1)\right]a_2^2\\&=U_1(t)u_2+U_2(t)u_1^2,\end{aligned}\tag{2.9}$$

$$-(\delta+1)(2\lambda-m-n)a_2=U_1(t)v_1\tag{2.10}$$

and

$$\begin{aligned}&(2\delta+1)(3\lambda-m^2-n^2-mn)\left(2a_2^2-a_3\right)\\&\quad+(3\delta+1)\left[(m+n)^2-2\lambda(m+n-\lambda+1)\right]a_2^2\\&=U_1(t)v_2+U_2(t)v_1^2.\end{aligned}\tag{2.11}$$

It follows from (2.8) and (2.10) that

$$u_1=-v_1\tag{2.12}$$

and

$$2(\delta+1)^2(2\lambda-m-n)^2a_2^2=U_1^2(t)\left(u_1^2+v_1^2\right).\tag{2.13}$$

If we add (2.9) to (2.11), we find that

$$\begin{aligned}&(2\delta+1)\left(3\lambda-m^2-n^2-mn\right)+(3\delta+1)\left[(m+n)^2-2\lambda(m+n-\lambda+1)\right]\\&2\left((2\delta+1)\left(3\lambda-m^2-n^2-mn\right)+(3\delta+1)\left[(m+n)^2-2\lambda(m+n-\lambda+1)\right]\right)a_2^2\\&\qquad=U_1(t)\left(u_2+v_2\right)+U_2(t)\left(u_1^2+v_1^2\right).\end{aligned}\tag{2.14}$$

Substituting the value of $u_1^2+v_1^2$ from (2.13) in the right hand side of (2.14), we get

$$\begin{aligned}&2\Big|\left((2\delta+1)\left(3\lambda-m^2-n^2-mn\right)+(3\delta+1)\left[(m+n)^2-2\lambda(m+n-\lambda+1)\right]\right)\\&\qquad-(\delta+1)^2(2\lambda-m-n)^2\frac{U_2(t)}{U_1^2(t)}\Big|a_2^2=U_1(t)\left(u_2+v_2\right).\end{aligned}\tag{2.15}$$

Further computations using (1.3), (2.7) and (2.15), we obtain

$$|a_2|\le\frac{2t\sqrt{2t}}{\sqrt{\left|\begin{array}{c}4\left((2\delta+1)\left(3\lambda-m^2-n^2-mn\right)+(3\delta+1)\left[(m+n)^2-2\lambda(m+n-\lambda+1)\right]\right.\\\left.-(\delta+1)^2(2\lambda-m-n)^2\right)t^2+(\delta+1)^2(2\lambda-m-n)^2\end{array}\right|}}.$$

Next, if we subtract (2.11) from (2.9), we deduce that

$$2(2\delta+1)\left(3\lambda-m^2-n^2-mn\right)\left(a_3-a_2^2\right)=U_1(t)\left(u_2-v_2\right)+U_2(t)\left(u_1^2-v_1^2\right). \tag{2.16}$$

In view of (2.12) and (2.13), we get from (2.16)

$$a_3=\frac{U_1^2(t)\left(u_1^2+v_1^2\right)}{2(\delta+1)^2(2\lambda-m-n)^2}+\frac{U_1(t)\left(u_2-v_2\right)}{2(2\delta+1)\left(3\lambda-m^2-n^2-mn\right)}.$$

Thus applying (1.3), we obtain

$$|a_3|\leq\frac{4t^2}{(\delta+1)^2(2\lambda-m-n)^2}+\frac{2t}{(2\delta+1)\left(3\lambda-m^2-n^2-mn\right)}.$$

□

If we put $\delta=0, m=1$ and $n=-1$ in Theorem 2.1, we conclude the following result.

Corollary 2.1. *For $\lambda\geq 1$ and $t\in\left(\frac{1}{2},1\right]$, let f be in the class $\mathcal{R}_{\Sigma}^{S}(\lambda;t)$. Then*

$$|a_2|\leq\frac{t\sqrt{2t}}{\sqrt{\left|\left(\lambda-2\lambda^2-1\right)t^2+\lambda^2\right|}}$$

and

$$|a_3|\leq\frac{t^2}{\lambda^2}+\frac{2t}{3\lambda-1}.$$

If we put $\delta=\lambda=1, m=1$ and $n=-1$ in Theorem 2.1, we obtain the result for well-known class $\mathcal{F}_{\Sigma}^{sc}(t)$ which was studied recently by Wanas and Majeed [20].

Corollary 2.2. *[20] For $t\in\left(\frac{1}{2},1\right]$, let f be in the class $\mathcal{F}_{\Sigma}^{sc}(t)$. Then*

$$|a_2|\leq\frac{t\sqrt{t}}{\sqrt{|2-5t^2|}}$$

and

$$|a_3|\leq\frac{t(3t+4)}{12}.$$

If we put $\lambda = 1$ in Corollary 2.1, we obtain the result for well-known class $\mathfrak{D}_{\Sigma}^{S}(1,t)$ which was considered recently by Wanas [19].

Corollary 2.3. *[19] For $t \in \left(\frac{1}{2}, 1\right]$, let f be in the class $\mathfrak{D}_{\Sigma}^{S}(1,t)$. Then*

$$|a_2| \leq \frac{t\sqrt{2t}}{\sqrt{|2t^2 - 1|}}$$

and

$$|a_3| \leq t(t+1).$$

3 Conclusion

The primary objective was to define the family $\mathcal{R}_{\Sigma}(\delta, \lambda, m, n; t)$ of analytic and bi-univalent functions associating bi-starlike and bi-convex λ-pseudo functions with Sakaguchi type functions which governed by the second kind Chebyshev polynomials. We generated initial Taylor-Maclaurin coefficients inequalities for functions in this family.

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