Earthline Journal of Mathematical Sciences
E-ISSN: 2581-8147
Volume 13, Number 2, 2023, Pages 473-495
https://doi.org/10.34198/ejms.13223.473495

Some New Classes of Harmonic Hemivariational Inequalities

Muhammad Aslam Noor[1,*] and Khalida Inayat Noor[2]

[1] Department of Mathematics, COMSATS University Islamabad, Islamabad, Pakistan
e-mail: noormaslam@gmail.com

[2] Department of Mathematics, COMSATS University Islamabad, Islamabad, Pakistan
e-mail: khalidan@gmail.com

Abstract

Some new classes of harmonic hemivariational inequalities are introduced and investigated in this paper. It has been shown that the optimality conditions of the sum of two harmonic convex functions can be characterized by the harmonic hemivariational inequalities. Several special cases such as harmonic complementarity problems and related harmonic problems are discussed. The auxiliary principle technique is applied to suggest and analyze some iterative schemes for harmonic hemivariational inequalities. We prove the convergence of these iterative methods under some weak conditions. Our method of proof of the convergence criteria is simple compared to other techniques. Results obtained in this paper continue to hold for new and known classes of harmonic variational inequalities and related optimization problems. The ideas and techniques of this paper may inspire further research in various branches of pure and applied sciences.

1 Introduction

Variational inequalities theory, which was introduced by Stampachia [28] in potential theory, provides us with a simple, general and unified framework to

Received: July 16, 2023; Accepted: August 19, 2023; Published: August 26, 2023
2020 Mathematics Subject Classification: Primary: 49J40, 90C33; Secondary: 26D15, 26A51.
Keywords and phrases: harmonic hemivariational inequalities, iterative methods, convergence, harmonic convex functions.

*Corresponding author

study a wide class of problems arising in pure and applied sciences. Variational inequalities have been extended and generalized in several directions using novel and innovative techniques.

It is well known that the minimum of the differentiable convex function F on the convex set $\mathcal{K}$ can be characterized by the variational inequalities. Then the minimum $u \in \mathcal{K}$ of the differentiable convex function F is equivalent to finding $u \in \mathcal{K}$ such that

$$\langle F'(u), v-u\rangle \geq 0, \quad \forall v \in \mathcal{K}, \tag{1.1}$$

which is called the variational inequality (1.1). Here $F'(u)$ is the Frechet derivative of the convex function F at $u \in \mathcal{K}$ in the direction $v-u$.

Noor [14] considered the energy (virtual work) functional $I[v]$ defined as

$$I[v] = F(v) + \phi(v), \quad \forall v \in H, \tag{1.2}$$

where F and ϕ are two different convex functions. The problem (1.2) is called the sum (difference) of two functions, which was considered by Noor [14] in 1975. It has been shown that if the functions F and ϕ are differentiable functions, then the minimum $u \in \mathcal{K}$ of energy functional $I[v]$ defined by (1.2) is equivalent to finding $u \in \mathcal{K}$ such that

$$\langle F'(u), v-u\rangle + \langle \phi'(u), v-u\rangle \geq 0, \quad \forall v \in \mathcal{K}, \tag{1.3}$$

which is called the mildly nonlinear variational inequalities.

If the function ϕ is nonlinear Lipschitz continuous function, then the minimum of the energy functional $I[v]$ defined by (1.2) can be characterized by the inequality

$$\langle F'(u), v-u\rangle + \phi'(u; v-u) \geq 0, \quad \forall v \in \mathcal{K}, \tag{1.4}$$

is known as the Hemivariational inequalities, which was introduced and investigated by Panagiotopoulos [43, 44] with applications in structural analysis.

In passing, it is worth mentioning that problem (1.4) is a special case of problem (1.3). Mildly nonlinear variational inequalities and hemivariational inequalities can be viewed as novel and important generalizations of variational inequalities.

Anderson et al. [4] have investigated several aspects of the harmonic convex functions. Noor et al. [29] have shown that the minimum of the differentiable harmonic convex function F on the harmonic convex set $\mathcal{C}_h$ can be characterized by a class of variational inequalities,

$$\langle F'(u), \frac{uv}{u-v}\rangle \geq 0, \quad \forall v \in \mathcal{C}_h, \tag{1.5}$$

which is called harmonic variational inequality. For the numerical methods, generalizations and other aspects of harmonic variational inequalities, see [11, 29, 30, 33]. Iscan [13] and Noor et al. [9, 32, 34–36, 40] have derived several Hermite-Hadamard type integral inequalities for the harmonic convex functions and their variant forms. It is amazing that the harmonic means have applications in electrical circuits. It is known that the total resistance of a set of parallel resistors is obtained by adding up the reciprocals of the individual resistance values, and then taking the reciprocal of their total. More precisely, if η and ζ are the resistances of two parallel resistors, then the total resistance is computed by the formula: $\frac{1}{\eta} + \frac{1}{\zeta} = \frac{\eta\zeta}{\eta+\zeta}$, which is half the harmonic means. Al-Azemi et al. [1] studied the Asian options with harmonic average, which can be viewed a new direction in the study of the risk analysis and financial mathematics. Noor [20] used the harmonic mean to suggest some iterative methods for solving nonlinear equations. We would like to emphasize that the hemivariational inequalities and harmonic variational inequalities are quite different generalizations of the variational inequalities and related optimizations problems. It is natural to study these different problems in a unified framework. This motivated us to introduce and consider some classes of harmonic hemivariational inequality. Due to structure of these inequalities, it is not possible to extend the usual projection and resolvent techniques for solving harmonic heimivariational inequalities. However, these difficulties can be overcome by using the auxiliary principle, which is mainly due to

Lions et al. [12] and Glowinski et al. [8]. Noor [14,19,20] and Noor et al. [21–31,37] have used this technique to develop some iterative schemes for solving various classes of variational inequalities and equilibrium problems. We point out that this technique does not involve any projection and resolvent of the operator and is flexible. In this paper, we show that the auxiliary principle technique can be applied to suggest and analyze some new classes of inertial iterative methods for solving harmonic hemivariational inequalities. It is worth mentioning that the inertial type methods was suggested by Polyak [46] to speed up the convergence of iterative methods. We also prove that the convergence of these new methods requires pseudomonotonicity, which is weaker conation than monotonicity. As special cases, one obtain several known and new results for hemivariational inequalities, variational inequalities and related optimization problems. Results obtained in this paper, represent an improvement and refinement of the known results for nonconvex variational inequalities.

2 Preliminaries

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|.\|$ respectively. Let K be a nonempty closed convex set in H. Let $j : H \longrightarrow R$ be a locally Lipschitz continuous function.

First of all, we recall the following concepts and results from nonsmooth analysis, see [5].

Definition 2.1. *[1] Let j be locally Lipschitz continuous at a given point $x \in H$ and v be any other vector in H. The Clarke's generalized directional derivative of j at x in the direction v, denoted by $j^0(x; v)$, is defined as*

$$j^0(x; v) = \lim_{t \to 0^+} \sup_{h \to 0} \frac{f(x + h + tv) - f(x + h)}{t}.$$

The generalized gradient of j at x, denoted $\partial j(x)$, is defined to be subdifferential *of the function* $j^0(x; v)$ *at* 0. *That is*

$$\partial j(x) = \{w \in H : \langle w, v\rangle \leq j^0(x; v), \quad \forall v \in H\}.$$

Lemma 2.1. *Let* j *be a locally Lipschitz continuous at a given point* $x \in H$ *with a constant* L. *Then*

(i). $\partial j(x)$ *is a non-empty compact subset of* H *and* $\|\xi\| \leq L$ *for each* $\xi \in \partial j(x)$.

(ii). *For every* $v \in H$, $j^0(x; v) = max\{\langle \xi, c\rangle : \xi \in \partial j(x)\}$.

(iii). *The function* $v \longrightarrow j^0(x; v)$ *is finite, positively homogeneous, subadditive, convex and continuous.*

(iv). $j^0(x; -v) = (-j)^0(x; v)$.

(v). $j^0(x; v)$ *is upper semicontinuous as a function of* $(x; v)$.

(vi). $\forall x \in H$, *there exists a constant* $\alpha > 0$ *such that*

$$|j^0(x; v)| \leq \alpha\|v\|, \quad \forall v \in H.$$

If j is convex on K and locally Lipschitz continuous at $x \in K$, then $\partial j(x)$ coincides with the subdifferential $j'(x)$ of j at x in the sense of convex analysis, and $j^0(x; v)$ coincides with the directional derivative $j'(x; v)$ for each $v \in H$, that is, $j^0(x; v) = \langle j'(x), v\rangle$.

For the sake of completeness and to convey the main ideas, we include the relevant details.

Definition 2.2. *[4] The set* $\mathcal{C}_h$ *is said to be a harmonic convex set, if*

$$\frac{uv}{v + \lambda(u - v)} \in \mathcal{C}_h, \quad \forall u, v \in \mathcal{C}_h, \quad \lambda \in [0, 1].$$

Definition 2.3. *[4] The function* ϕ *on the harmonic convex set* $\mathcal{C}_h$ *is said to* *be exponentially harmonic convex, if*

$$\phi(\frac{uv}{v+\lambda(u-v)}) \leq (1-\lambda)\phi(u) + \lambda\phi(v), \quad \forall u, v \in \mathcal{C}_h \quad \lambda \in [0,1].$$

The function ϕ is said to be harmonic concave function, if and only if, $-\phi$ is harmonic convex function.

We now show that the minimum of a differentiable harmonic convex function on the harmonic convex set $\mathcal{C}_h$ can be characterized by the variational inequality.

Theorem 2.1. *Let ϕ be a differentiable harmonic convex function on the harmonic convex set $\mathcal{C}_h$. Then $u \in \mathcal{C}_h$ is a minimum of ϕ, if and only if, $u \in \mathcal{C}_h$ is the solution of the inequality*

$$\langle \phi'(u), \frac{uv}{u-v} \rangle \geq 0, \quad \forall v \in \mathcal{C}_h. \tag{2.1}$$

The inequality of type (2.1) is called the harmonic variational inequality.

Proof. Let $u \in \mathcal{C}_h$ is a minimum of differentiable harmonic convex function ϕ. Then

$$\phi(u) \leq \phi(v), \quad \forall v \in \mathcal{C}_h. \tag{2.2}$$

Since $\mathcal{C}_h$ is a harmonic convex set, so $\forall u, v \in \mathcal{C}_h, \quad v_\lambda = \frac{uv}{u+\lambda(u-v)} \in \mathcal{C}_h$. Replacing v by v_λ) in (2.3) and diving by λ and taking limit as $\lambda \to 0$, we have

$$0 \leq \frac{\phi(\frac{uv}{u+\lambda(u-v)}) - \phi(u)}{\lambda} = \langle \phi'(u), \frac{uv}{u-v} \rangle$$

the required result (2.1). Conversely, let the function ϕ be exponentially harmonic convex function on the harmonic convex set $\mathcal{C}_h$. Then

$$\frac{uv}{v+\lambda(u-v)} \leq (1-\lambda)\phi(u) + \lambda\phi(v) = \phi(u) + \lambda(\phi(v) - \phi(u)),$$

which implies that

$$\phi(v)-\phi(u) \quad \geq \quad \lim_{\lambda\to 0}\frac{\phi(\frac{uv}{v+\lambda(u-v)})-\phi(u)}{\lambda} = \langle \phi'(u), \frac{uv}{u-v}\rangle \geq 0, \quad \text{using} \quad (2.1).$$

Consequently, it follows that

$$\phi(u) \leq \phi(v), \quad \forall v \in \mathcal{C}_h.$$

This shows that $u \in \mathcal{C}_h$ is the minimum of the differentiability harmonic convex function. □

We would like to mention that Theorem 2.1 implies that harmonic optimization programming problem can be studied via the harmonic variational inequality (2.1).

Using the ideas and techniques of Theorem 2.3, we can derive the following result.

Theorem 2.2. *Let ϕ be a differentiable harmonic convex functions on the harmonic convex set $\mathcal{C}_h$. Then*

$$(i). \qquad \phi(v)-\phi(u) \geq \langle \phi'(u), \frac{uv}{u-v}\rangle, \quad \forall u,v \in \mathcal{C}_h.$$

$$(ii). \qquad \langle \phi'(u)-\phi'(v), \frac{uv}{v-u}\rangle \geq 0, \quad \forall u,v \in \mathcal{C}_h.$$

Motivated by Theorem 2.1 and Theorem 2.2, we introduce some new concepts.

Definition 2.4. *An operator* T *is said to be a harmonic monotone operator, if and only if,*

$$\langle \mathrm{T}u - \mathrm{T}v, \frac{uv}{u-v}\rangle \geq 0, \quad \forall u,v \in H.$$

Definition 2.5. *An operator* T *is said to a harmonic pseudomonotone operator, if*

$$\langle Tu, \frac{uv}{u-v}\rangle \geq 0 \quad \Rightarrow \quad -\langle Tv, \frac{uv}{u-v}\rangle \geq 0, \quad \forall u,v \in H.$$

An harmonic monotone operator is a harmonic pseudomonotone operator, but the converse is not true.

Consider the energy (virtual) functional

$$I[v] = F(v) + \phi(v), \tag{2.3}$$

where $F(v)$ and $\phi(v)$ are two harmonic convex functions.

We now consider the optimality conditions of the energy function $I[v]$ defined by (2.4) under suitable conditions.

Theorem 2.3. *Let F be a differentiable harmonic convex function and $\phi(v)$ be a directionally differentiable harmonic convex functions on the convex set $\mathcal{C}_h$. If $u \in \mathcal{C}_h$ is the minimum of the functional $I[v]$ defined by (2.3), then*

$$\langle F'(u), \frac{uv}{u-v}\rangle + \phi'(u; \frac{uv}{u-v}) \geq 0, \quad \forall v, u \in \mathcal{C}_h. \tag{2.4}$$

Proof. Let $u \in \mathcal{C}_h$ be a minimum of the functional $I[v]$. Then

$$I[u] \leq I[v], \quad \forall v \in K$$

which implies that

$$F(u) + \phi(u) \leq F(v) + \phi(v), \quad \forall v \in \mathcal{C}_h. \tag{2.5}$$

Since $\mathcal{C}_h$ is a convex set, so, $\forall u, v \in \mathcal{C}_h, \quad \lambda \in [0,1]$, $v_t = \frac{uv}{(1-\lambda)v+\lambda u} \in \mathcal{C}_h$.

Taking $v = v_t$ in (2.5), we have

$$F(u) + \phi(u) \leq F(v_t) + \phi(v_t), \quad \forall v \in \mathcal{C}_h. \tag{2.6}$$

This implies that

$$0 \leq F(\frac{uv}{(1-\lambda)v+\lambda u}) - F(u) + \phi(\frac{uv}{(1-\lambda)v+\lambda u}) - \phi(u), \quad \forall v \in \mathcal{C}_h. \tag{2.7}$$

Dividing the above inequality by λ and taking limit as $\lambda \to 0$, we have

$$\begin{aligned} 0 &\leq \frac{F(\frac{uv}{(1-\lambda)v+\lambda u}) - F(u)}{\lambda} + \frac{\phi(\frac{uv}{(1-\lambda)v+\lambda u}) - \phi(u)}{\lambda} \\ &= \langle F'(u), \frac{uv}{u-v}\rangle + \phi'(u; \frac{uv}{u-v}), \end{aligned}$$

which is the required (2.4).

Since F is differentiable harmonic convex function, so

$$F(\frac{uv}{v+\lambda(u-v))}) \leq F(u) + \lambda(F(v) - F(u)), \quad \forall u, v \in \mathcal{C}_h$$

from which, we have

$$F(v) - F(u) \geq \lim_{\lambda \to 0}\{F(\frac{uv}{v+\lambda(u-v)} - F(u)\lambda\} = \langle F'(u), \frac{uv}{u-v}\rangle. \quad (2.8)$$

In a similar way,

$$\phi(v) - \phi(u) \geq \lim_{\lambda \to 0}\{\phi(\frac{uv}{v+\lambda(u-v)} - \phi(u)\lambda\} = \langle \phi'(u), \frac{uv}{u-v}\rangle. \quad (2.9)$$

From (2.9) and (2.8), we have

$$F(v) + \phi(v) - (F(u) + \phi(u)) \geq= \langle F'(u), \frac{uv}{u-v}\rangle + \phi'(u; \frac{uv}{u-v}) \geq 0.$$

Consequently, it follows that $u \in \mathcal{C}_h$ such that

$$F(u) + \phi(u) \leq (F(v) + \phi(v)), \quad \forall v \in \mathcal{C}_h,$$

which shows that $u \in \mathcal{C}_h$ is the minimum of the function $I[v]$ defined by (2.3).

□

Remark 2.1. *The inequality of the type (2.4) is called the harmonic hemivariational inequality. In many applications, the inequality of the type (2.4) may not arise as the minimum of the sum of the two differentiable harmonic convex functions. These facts motivated us to consider more general harmonic hemivariaonal inequality, which contains the inequality (2.4) as a special case.*

For given nonlinear continuous operators $T, A : H \longrightarrow H$, we consider the problem of finding $u \in K$ such that

$$\langle Tu, \frac{uv}{u-v} \rangle + A(u; \frac{uv}{u-v}) \geq 0, \quad p \geq 1, \quad \forall v \in \mathcal{C}_h, \tag{2.10}$$

which is called the *harmonic hemivariational inequality.*

We now discuss some new and known classes of variational inequalities and related optimization problems.

(i). If $A(u; \frac{uv}{u-v}) = \phi'(u; \frac{uv}{u-v})$ denotes directional derivative of the harmonic convex function $\phi(u)$ in the direction $\frac{uv}{u-v}$, then problem (2.10) reduces to finding $u \in \mathcal{C}_h$, such that

$$\langle Tu, \frac{uv}{u-v} \rangle + \phi'(u; \frac{uv}{u-v}) \geq 0, \quad \forall v \in \mathcal{C}_h, \tag{2.11}$$

which is also called the harmonic directional variational inequality.

(ii). For $A(u; v-u) = J^0(u; \frac{uv}{u-v})$, the problem (2.10) reduces to finding $u \in \mathcal{C}$ such that

$$\langle Tu, \frac{uv}{u-v} \rangle + J^0(u; \frac{uv}{u-v}) \geq 0, \quad \forall v \in \mathcal{C}_h, \tag{2.12}$$

which is known as harmonic hemivariational inequality. Hemivariational inequalities have important applications in superpotential analysis of elasticity and structural analysis.

(iii). If $\phi(.)$ is a smooth and convex function, then $\phi'(u; \frac{uv}{u-v}) = \langle \phi'(u), \frac{uv}{u-v} \rangle$, and consequently problem (2.11) is equivalent to finding $u \in \mathcal{C}_h$ such that

$$\langle Tu, \frac{uv}{u-v} \rangle + \langle \phi'(u), \frac{uv}{u-v} \rangle \geq 0, \quad \forall v \in \mathcal{C}_h, \tag{2.13}$$

which is called the nonlinear harmonic variational inequality.

(iv). If $A(u;\frac{uv}{u-v}) = -\langle Au, \frac{uv}{u-v}\rangle$ then the problem (2.10) reduces to finding $u \in \mathcal{C}_h$ such that

$$\langle Tu, \frac{uv}{u-v}\rangle - \langle A(u), \frac{uv}{u-v}\rangle \geq 0, \quad \forall v \in \mathcal{C}_h, \tag{2.14}$$

which is called the mildly nonlinear harmonic variational inequality. It is worth mentioning the problem (2.14) can be viewed as difference of two harmonic monotone operators, see [20].

(v). If $(\mathcal{C}_h)^\star = \{u \in H : \langle u, \frac{uv}{u-v}\rangle \geq 0, \quad \forall v \in \mathcal{C}_h\}$ is a polar harmonic convex cone of the harmonic convex $\mathcal{C}_h$, then problem (2.10) is equivalent to fining $u \in H$, such that

$$\frac{uv}{u-v} \in \mathcal{C}_h, \quad Tu + A(u) \in (\mathcal{C}_h)^\star, \quad \langle Tu + A(u), \frac{uv}{u-v}\rangle = 0, \tag{2.15}$$

is called the general harmonic complementarity problem. For the applications, numerical methods and other aspects of complementarity problems, see [6, 10, 15, 16, 42, 45] the references therein.

(vi). If $\mathcal{C}_h = H$, then problem (2.10) is equivalent to fining $u \in H$, such that

$$\langle Tu + A(u), \frac{uv}{u-v}\rangle = 0, \quad \forall v \in H \tag{2.16}$$

which is called the weak formulation of the mildly nonlinear harmonic boundary value problem. One can easily show that the system of absolute value equations [19, 20] is a special case of the problem (2.16) and complementarity problems, see [2, 9].

(viii). If $A(.;.) = 0$, then problem (2.10) reduces to finding $u \in K$ such that

$$\langle Tu, \frac{uv}{v-u}\rangle \geq 0, \quad \forall v \in \mathcal{C}_h, \tag{2.17}$$

is called the harmonic variational inequality introduced and studied in [29].

For the recently applications, numerical methods, sensitivity analysis and local uniqueness of solutions of harmonic variational inequalities and related optimization problems, see [3, 4, 7–28, 30, 37, 40–46] and the references therein.

This show that the problem (2.10) is quite and unified one. Due to the structure and nonlinearity involved, one has to consider its own. It is an open problem to develop unified numerical implementation numerical methods for solving the harmonic variational inequalities.

3 Main Results

In this section, we use the auxiliary principle technique, which is mainly due to Glowinski et al. [8] as developed in [14, 19–25, 29–31, 40, 41], to suggest and analyze some inertial iterative methods for solving harmonic hemivariational inequalities (2.10).

For a given $u \in \mathcal{C}_h$ satisfying (2.10), consider the problem of finding $w \in \mathcal{C}_h$ such that

$$\langle \rho T(w+\eta(u-w)), \frac{uw}{u-w}\rangle + \langle w-u, v-w\rangle + A(w; \frac{uw}{u-w}) \geq 0, \quad \forall v \in \mathcal{C}_h, \tag{3.1}$$

where $\rho > 0, \eta \in [0,1]$ are constants.

Inequality of type (3.1) is called the auxiliary harmonic hemivariational inequality.

If $w = u$, then w is a solution of (2.10). This simple observation enables us to suggest the following iterative method for solving (2.10).

Algorithm 3.1. *For a given $u_0 \in K$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned}&\langle \rho T(u_{n+1} + \eta(u_n - u_{n+1})), \frac{u_n u_{n+1}}{u_n - u_{n+1}}\rangle + \langle u_{n+1} - u_n, v - u_{n+1}\rangle \\ &\geq -A(u_{n+1}; \frac{u_n u_{n+1}}{u_n - u_{n+1}}), \quad \forall v \in \mathcal{C}_h.\end{aligned}$$

Algorithm 3.1 is called the hybrid proximal point algorithm for solving harmonic hemivariational inequalities (2.10).

Special Cases

We now consider some cases of Algorithm 3.1.

(I). For $\eta = 0$, Algorithm 3.1 reduces to:

Algorithm 3.2. *For a given $u_0 \in K$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\langle \rho T(u_{n+1}, \frac{u_n u_{n+1}}{u_n - u_{n+1}}\rangle + \langle u_{n+1} - u_n, v - u_{n+1}\rangle + A(u_{n+1}; \frac{u_n u_{n+1}}{u_n - u_{n+1}}), \forall v \in \mathcal{C}_h. \quad (3.2)$$

(II). If $\eta = 1$, then Algorithm 3.1 reduces to:

Algorithm 3.3. *For a given $u_0 \in K$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\langle \rho T(u_n, \frac{u_n u_{n+1}}{u_n - u_{n+1}}\rangle + \langle u_{n+1} - u_n, v - u_{n+1}\rangle + A(u_{n+1}; \frac{u_n u_{n+1}}{u_n - u_{n+1}}), \forall v \in \mathcal{C}_h.$$

(III). If $\eta = \frac{1}{2}$, then Algorithm 3.1 collapses to:

Algorithm 3.4. *For a given $u_0 \in K$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\langle \rho T(\frac{u_{n+1} + u_n}{2}), \frac{u_n u_{n+1}}{u_n - u_{n+1}}\rangle + \langle u_{n+1} - u_n, v - u_{n+1} + A(u_{n+1}; \frac{u_n u_{n+1}}{u_n - u_{n+1}}), \forall v \in \mathcal{C}_h.$$

which is called the mid-point proximal method for solving the problem (2.10).

If $A(.;.) = 0$, then Algorithm 3.1 reduces to:

Algorithm 3.5. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\langle \rho T(u_{n+1} + \eta(u_n - u_{n+1})), \frac{u_n u_{n+1}}{u_n - u_{n+1}} \rangle + \langle u_{n+1} - u_n, v - u_{n+1} \rangle \quad \forall v \in \mathcal{C}_h.$$

for solving harmonic variational inequality.

For the convergence analysis of Algorithm 3.2, we recall the following concepts and results.

Definition 3.1. *$\forall u, v, z \in H$, an operator $T : H \to H$ is said to be:*
(i). harmonic monotone, iff,

$$\langle Tu - Tv, \frac{uv}{u - v} \rangle \geq 0.$$

(ii) harmonic pseudomonotone with respect to $A(.;.) + \eta\|v - u\|^p, \}$, if and only if,

$$\langle Tu, \frac{uv}{u-v} \rangle + A(u; \frac{uv}{u-v}) \geq 0$$
$$\Longrightarrow$$
$$\langle Tv, \frac{uv}{u-v} \rangle - A(v; \frac{uv}{u-v}) \geq 0.$$

(iii). partially relaxed strongly harmonic monotone, if there exists a constant $\alpha > 0$ such that

$$\langle Tu - Tv, \frac{zv}{z - v} \rangle \geq -\alpha\|z - u\|^2.$$

Note that for $z = u$, partially relaxed strongly harmonic monotonicity reduces to monotonicity. It is known that partially relaxed strongly harmonic monotonicity, but the converse is not true. It is known that harmonic monotonicity implies harmonic pseudomonotonicity; but the converse is not true. Consequently, the class of harmonic pseudomonotone operators is bigger than the one of harmonic monotone operators.

Definition 3.2. *The operator $A(.;.)$ is called harmonic monotone, if and only if,*

$$A(u;\frac{uv}{u-v})+A(v;\frac{uv}{u-v})\leq 0,\quad \forall u,v\in H.$$

Lemma 3.1. $\forall u,v\in H$,

$$2\langle u,v\rangle=\|u+v\|^2-\|u\|^2-\|v\|^2. \tag{3.3}$$

We now consider the convergence criteria of Algorithm 3.2. The analysis is in the spirit of Noor [10]. We include the proof for the sake of completeness and to convey an idea of the technique involved.

Theorem 3.1. *Let $u\in\mathcal{C}_h$ be a solution of (2.10) and let u_{n+1} be the approximate solution obtained from Algorithm 3.2. If the operator T is harmonic pseudomonotone with respect to $A(.;.)$, then*

$$\|u_{n+1}-u\|^2\leq\|u_n-u\|^2-\|u_{n+1}-u_n\|^2. \tag{3.4}$$

Proof. Let $u\in\mathcal{C}_h$ be a solution of (2.10). Then

$$\langle Tv,\frac{uv}{v-u}\rangle-A(v;\frac{uv}{v-u})\geq 0,\quad \forall v\in\mathcal{C}_h. \tag{3.5}$$

since T is a harmonic pseudomonotone operator with respect to $A(.;.)$.

Now taking $v=u_{n+1}$ in (3.5), we have

$$\langle Tu,\frac{uu_{n+1}}{u_{n+1}-u}\rangle-A(u_{n+1};\frac{uu_{n+1}}{u_{n+1}-u}\geq 0. \tag{3.6}$$

Taking $v=u$ in (3.2), we get

$$\langle\rho T(u_{n+1},\frac{u_nu_{n+1}}{u_n-u_{n+1}}\rangle+\langle u_{n+1}-u_n,u-u_{n+1}\rangle+A(u_{n+1};\frac{u_nu_{n+1}}{u_n-u_{n+1}}),$$

which can be written as

$$\langle u_{n+1}-u_n,u-u_{n+1}\rangle\geq\langle\rho Tu_{n+1},\frac{uu_{n+1}}{u_{n+1}-u}\rangle+\rho A(u_{n+1};\frac{uu_{n+1}}{u-u_{n+1}})\geq 0, \tag{3.7}$$

where we have used (3.6).

Setting $u = u - u_{n+1}$ and $v = u_{n+1} - u_n$ in (3.3), we obtain

$$2\langle u_{n+1} - u_n, u - u_{n+1}\rangle = \|u - u_n\|^2 - \|u - u_{n+1}\|^2 - \|u_{n+1} - u_n\|^2. \tag{3.8}$$

Combining (3.7) and (3.8), we have

$$\|u_{n+1} - u\|^2 \leq \|u_n - u\|^2 - \|u_{n+1} - u_n\|^2,$$

the required result (3.4). □

Theorem 3.2. *Let H be a finite dimensional space and all the assumptions of Theorem 3.1 hold. Then the sequence $\{u_n\}_1^\infty$ given by Algorithm 3.2 converges to a solution u of (2.10).*

Proof. Let $u \in K$ be a solution of (2.10). From (3.4), it follows that the sequence $\{||u - u_n||\}$ is nonincreasing and consequently $\{u_n\}$ is bounded. Furthermore, we have

$$\sum_{n=0}^{\infty} \|u_{n+1} - u_n\|^2 \leq \|u_0 - u\|^2,$$

which implies that

$$\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0. \tag{3.9}$$

Let $\hat{u}$ be the limit point of $\{u_n\}_1^\infty$; a subsequence $\{u_{n_j}\}_1^\infty$ of $\{u_n\}_1^\infty$ converges to $\hat{u} \in H$. Replacing w_n by u_{n_j} in (3.2), taking the limit $n_j \longrightarrow \infty$ and using (3.9), we have

$$\langle T\hat{u}, \frac{\hat{u}v}{v - \hat{u}}\rangle + A(\hat{u}; \frac{\hat{u}v}{v - \hat{u}}) \geq 0, \quad \forall v \in \mathcal{C}_h,$$

which implies that $\hat{u}$ solves the harmonic hemivariational inequality (2.10) and

$$\|u_{n+1} - u\|^2 \leq \|u_n - u\|^2.$$

Thus, it follows from the above inequality that $\{u_n\}_1^\infty$ has exactly one limit point $\hat{u}$ and

$$\lim_{n\to\infty}(u_n) = \hat{u}.$$

the required result. □

We again consider the auxiliary principle technique to suggest some hybrid inertial proximal point methods for solving the problem (2.10).

For a given $u \in \mathcal{C}_h$ satisfying (2.10), consider the problem of finding $w \in \mathcal{C}_h$ such that

$$\begin{aligned}&\langle \rho T(w+\eta(u-w)), \frac{uw}{u-w}\rangle + \langle w-u+\alpha(u-u), v-w\rangle \\ &+A((w+\xi(w-u)); \frac{uw}{u-w}) \geq 0, \quad \forall v \in \mathcal{C}_h, \end{aligned} \tag{3.10}$$

where $\rho > 0, \alpha, \xi, \eta \in [0,1]$ are constants.

Clearly, for $w = u, \quad w$ is a solution of (2.10). This fact motivated us to to suggest the following inertial iterative method for solving (2.10).

Algorithm 3.6. *For given $u_0, u_1 \in K$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned}&\langle \rho T(u_{n+1}+\eta(u_n-u_{n+1})), \frac{u_n u_{n+1}}{u_n-u_{n+1}}\rangle + \langle u_{n+1}-u_n+\alpha(u_n-u_{n-1}, v-u_{n+1}\rangle \\ &\geq -A((u_{n+1+\xi(u_n-u_{n+1})}; \frac{u_n u_{n+1}}{u_n-u_{n+1}}), \quad \forall v \in \mathcal{C}_h.\end{aligned}$$

which is known as the inertial iterative method.

Note that for $\alpha = 0, \quad \xi = 0$, Algorithm 3.6 is exactly the Algorithm 3.1. Using essentially the technique of Theorem 3.1 and Noor [10], one can study the convergence analysis of Algorithm 3.6.

For different and appropriate values of the parameters, ξ, η, ζ, α, the operators T, A and spaces, one can obtain a wide class of inertial type iterative methods for solving the harmonic hemivaritaional inequalities and related optimization problems.

Conclusion

Some new classes of harmonic hemivariational inequalities are introduced in this paper. It is shown that several important problems such as harmonic complementarity problems, system of harmonic absolute value problems and related problems can be obtained as special cases. The auxiliary principle technique is applied to suggest several inertial type methods for solving harmonic hemivariational inequalities with suitable modifications. We note that this technique is independent of the projection and the resolvent of the operator. Moreover, we have studied the convergence analysis of these new methods under weaker conditions. Using the technique of Noor [19], one can introduce the concept of well-posedness for harmonic hemivariational inequalities and obtain some results. We have only considered the theoretical aspects of the hybrid inertial iterative methods. It is an interesting problem to implement these methods numerically and compare with other iterative schemes.

Author contributions

All authors contributed equally such as formulation, writing, discussion and editing.

Conflict interest

The authors declare that they have no competing interests.

Acknowledgments

The authors wish to express their deepest gratitude to their respected teachers, students, colleagues, collaborators, referees and friends, who have directly or indirectly contributed in the preparation of this paper.

References

[1] F. Al-Azemi and O. Calin, Asian options with harmonic average, *Appl. Math. Inform. Sci.* 9 (2015), 1-9.

[2] G. Alirezaei and R. Mazhar, On exponentially concave functions and their impact in information theory, *J. Inform. Theory Appl.* 9(5) (2018), 265-274. https://doi.org/10.1109/ita.2018.8503202

[3] F. Alvarez, Weak convergence of a relaxed and inertial hybrid projection-proximal point algorithm for maximal monotone operators in Hilbert space, *SIAM J. Optim.* 14 (2003), 773-782. https://doi.org/10.1137/s1052623403427859

[4] G. D. Anderson, M. K. Vamanamurthy and M. Vuorinen, Generalized convexity and inequalities, *J. Math. Anal. Appl.* 335 (2007), 1294-1308. https://doi.org/10.1016/j.jmaa.2007.02.016

[5] F. H. Clarke, *Optimization and Nonsmooth Analysis*, John Wiley and Sons, New York, 1983.

[6] R. W. Cottle, Jong-S. Pang and R. E. Stone, *The Linear Complementarity Problem*, SIAM Publication, 1992.

[7] F. Giannessi, A. Maugeri and M. S. Pardalos, *Equilibrium Problems: Nonsmooth Optimization and Variational Inequality Models*, Kluwer Academic, Dordrecht, Holland, 2001. https://doi.org/10.1007/b101835

[8] R. Glowinski, J-L. Lions and R. Tremolieres, *Numerical Analysis of Variational Inequalities*, North-Holland, Amsterdam, Holland, 1981.

[9] I. Iscan, Hermite-Hadamard type inequalities for harmonically convex functions, *Hacettepe, J. Math. Stats.* 43(6) (2014), 935-942. https://doi.org/10.15672/hjms.20164516901

[10] S. Karamardian, Generalized complementarity problems, *J. Opt. Theory Appl.* 8 (1971), 161-168.

[11] A. G. Khan, M. A. Noor, M. Pervez and K. I. Noor, Relative reciprocal variational inequalities, *Honam J. Math.* 40(3) (2018), 509-519.

[12] J. L. Lions and G. Stampacchia, Variational inequalities, *Commun. Pure Appl. Math.* 20 (1967), 493-519. https://doi.org/10.1002/cpa.3160200302

[13] B. B. Mohsen, M. A. Noor, K. I. Noor and M. Postolcahe, Strongly convex functions of higher order involving bifunction, *Mathematics* 7 (2019), 1028. https://doi.org/10.3390/math7111028

[14] M. Aslam Noor, On Variational Inequalities, PhD Thesis, Brunel University, London, UK, 1975.

[15] M. A. Noor, Fixed point approach for complementarity problems, *J. Math. Anal. Appl.* 133 (1988), 437-448.

[16] M. A. Noor, General variational inequalities, *Appl. Math. Letters* 1 (1988), 119-121.

[17] M. A. Noor, Quasi variational inequalities, *Appl. Math. Letters* 1(4) (1988), 367-370.

[18] M. A. Noor, New approximation schemes for general variational inequalities, *J. Math. Anal. Appl.* 251 (2000), 217-229. https://doi.org/10.1006/jmaa.2000.7042

[19] M. A. Noor, Some developments in general variational inequalities, *Appl. Math. Comput.* 152 (2004), 199-277.

[20] M. A. Noor, Hemivariational Inequalities, *J. Appl. Math. Computing* 17 (2005), 59-72

[21] M. A. Noor and K. I. Noor, Higher order strongly general convex functions and variational inequalities, *AIMS Math.* 5(4) (2020), 3646-3663. https://doi.org/10.3934/math.2020236

[22] M. A. Noor and K. I. Noor, Some classes of higher order general convex functions and variational inequalities, *Montes Taurus J. Pure Appl. Math.* 5(3) (2023), 1-15.

[23] M. A. Noor and K. I. Noor, Higher order generalized variational inequalities and nonconvex optimization, *U.P.B. Sci. Bull. Series A* 85(2) (2023), 77-88.

[24] M. A. Noor and K. I. Noor, Some new trends in mixed variational inequalities, *J. Advanc. Math. Studies* 15(2) (2022), 105-140.

[25] M. A. Noor and K. I. Noor, Higher order strongly biconvex functions and biequilibrium problems, *Advanc. Lin. Algebr. Matrix Theory* 11(2) (2021), 31-53. https://doi.org/10.4236/alamt.2021.112004

[26] M. A. Noor, K. I. Noor and H. M. Y. Al-Bayatti, Higher order variational inequalities, *Inform. Sci. Letters* 11(1) (2022), 1-5.

[27] M. A. Noor and K. I. Noor, General biconvex functions and bivariational inequalities, *Numerical Algeb. Control Optim.* 13(1) (2023), 11-27. https://doi.org/10.3934/naco.2021041

[28] M. A. Noor and K. I. Noor, An introduction to general variational inequalities, Preprint, 2023.

[29] M. A. Noor and K. I. Noor, Harmonic variational inequalities, *Appl. Math. Inform. Sci.* 10(5) (2016), 1811-1814. https://doi.org/10.18576/amis/100522

[30] M. A. Noor and K. I. Noor, Some implicit methods for solving harmonic variational inequalities, *Int. J. Anal. Appl.* 12(1) (2016), 10-14.

[31] M. A. Noor and K. I. Noor, Iterative schemes for solving higher order hemivariational inequalities, *Appl. Math. E-Notes* 24 (2024).

[32] M. A. Noor, K. I. Noor and S. Iftikhar, Integral inequalities for differentiable relative harmonic preinvex functions (survey), *TWMS J. Pure Appl. Math.* 7(1) (2016), 3-19. https://doi.org/10.2298/fil1720575n

[33] M. A. Noor, A. G. Khan, A. Pervaiz and K. I. Noor, Solution of harmonic variational inequalities by two-step iterative scheme, *Turk. J. Inequal.* 1(1) (2017), 46-52.

[34] M. A. Noor, K. I. Noor and S. Iftikhar, Strongly generalized harmonic convex functions and integral inequalities, *J. Math. Anal.* 7(3) (2016), 66-77. https://doi.org/10.15330/cmp.11.1.119-135

[35] M. A. Noor, K. I. Noor and S. Iftikhar, Some characterizations of harmonic convex functions, *Inter. J. Anal. Appl.* 15(2) (2017), 179-187.

[36] M. A. Noor, K. I. Noor, M. U. Awan and S. Costache, Some integral inequalities for harmonically h-convex functions, *U.P.B. Sci. Bull. Series A* 77(1) (2015), 5-16. https://doi.org/10.18576/sjm/050203

[37] M. A. Noor and K. I. Noor, Higher order generalized variational inequalities and nonconvex optimization, *U.P.B. Sci. Bull. Series A* 85(2) (2023), 77-88.

[38] M. A. Noor and K. I. Noor, Some new systems of exponentially general equations, *Advanc. Linear Algeb. Matrix Theory* 12(3) (2022), 67-86. https://doi.org/10.4236/alamt.2022.123004

[39] M. A. Noor, K. I. Noor, A. Hamdi and E. H. El-Shemas, On difference of two monotone operators, *Optim. Letters* 3 (2009), 329-335. https://doi.org/10.1007/s11590-008-0112-7

[40] M. A. Noor, K. I. Noor, and M. Th. Rassias, New trends in general variational inequalities, *Acta Appl. Math.* 170(1) (2020), 981-1046. https://doi.org/10.1007/s10440-020-00366-2

[41] M. A. Noor, K. I. Noor and Th. M. Rassias, Some aspects of variational inequalities, *J. Comput. Appl. Math.* 47 (1993), 285-312. https://doi.org/10.1016/0377-0427(93)90058-j

[42] M. A. Noor and W. Oettli, On general nonlinear complementarity problems and quasi equilibria, *Le Math. (Catania)* 49 (1994), 313-331.

[43] P. D. Panagiotopoulos, Nonconvex energy functions. Hemivariational inequalities and substationarity principles, *Acta Mechanica* 48 (1983), 111-130. https://doi.org/10.1007/bf01170410

[44] P. D. Panagiotopolous, *Hemivariational Inequalities: Applications to Mechanics and Engineering*, Springer Verlag, Berlin, Germany, 1995

[45] M. Patriksson, *Nonlinear Programming and Variational Inequalities: A Unified Approach*, Kluwer Academic Publishers, Dordrecht, Holland, 1999

[46] B. T. Polyak, Some methods of speeding up the convergence of iterative methods, *USSR Comput. Math. Math. Phys.* 4 (1964), 1-17. https://doi.org/10.1016/0041-5553(64)90137-5

[47] G. Stampacchia, Formes bilineaires coercitives sur les ensembles convexes, *Comptes Rendus de l'Academie des Sciences, Paris* 258 (1964), 4413-4416.