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On the Stability of a High Order Stiffly Stable Parameter Dependent Nested Hybrid Linear Multistep Methods for Stiff ODEs

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Abstract

This paper is on the stability of a high order stiffly stable parameter dependent nested hybrid multistep method for the numerical integration of stiff initial value problems (IVPs) in ordinary differential equations (ODEs). The method incorporates one or more off-step points for better stability properties. The stability properties of the methods were investigated and the intervals of absolute stability of the methods with step number $k \leq 6$ are presented using the boundary locus techniques. The method is A-stable and $A(\alpha)$-stable which makes the methods more suitable for stiff initial value problems.

1 Introduction

This paper is on the stability of a high order stiffly stable parameter dependent nested hybrid multistep method. Nested method was introduced in [6] and further works on nested method can be found in [4] and [7-10]. Nested method is an extension of the general linear method (GLM) in [3]. The purpose for the nested method is to by-pass the order barrier in multistage and multivalue algorithm.

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An example of nested multistage method can be found in the work of [2] and [5]. Also, an example of nested multivalue method can be found in the work [11] and [12].

The general form of the k-step higher order is given below:

$$\begin{cases} Y_0 = \sum_{j=0}^{k} \phi_j y_{n+j} + h\lambda_k f_{n+k}; \quad Y_0 = y_{n+c_0} \\ Y_{i+1} = y_{n+k-1} + \sum_{j=0}^{k} \varphi_j f_{n+j} + h\rho_i f(Y_i); \quad i = 0(1)s-1; \quad Y_{i+1} = y_{n+c_{i+1}} \\ y_{n+k} = y_{n+k-1} + \sum_{j=0}^{k-2} \delta_j y_{n+j} + h\theta_{s-1} f(Y_{s-1}) + h\gamma_k (f_{n+k} + a f_{n+k-1}); \ 0 \le c \le k. \end{cases} \tag{1}$$

The stability region of the formula in (1) depends on the choice of the parameter "a". It is obvious that during the application of the methods for the numerical integration of stiff IVPs, errors are bound to occur at some stages of the computation due to inaccuracy inherent in the formula and the arithmetic operations adopted during the computer implementation. The magnitude of the error determines the degree of accuracy and stability of the method [10]. The minimum properties a numerical method could possess are stability properties [1].

Some of the advantages of this work are that it is of high order and stiffly stable and it has a better stability function. The better stability function of the method was made possible because of the present of a parameter "a" which lies between 0 and 1 and it helps to obtain a better stability function thereby leading to a smooth stability curve.

2 Order Conditions

Expanding the local truncation error by Taylor's series technique, the stage method and the output method in (1) at x_n gives the following order conditions:

$$h^{q_0} : \begin{cases} \sum_{j=0}^{k} \phi_j = 1, & q_0 = 0, \\ \sum_{j=1}^{k} j\phi_j + \lambda_k = c_0, & q_0 = 1, \\ \frac{1}{q_0!} \sum_{j=1}^{k} j^{q_0} \phi_j + \frac{1}{(q_0-1)!} k^{q_0-1} \lambda_k = \frac{c_0^{q_0}}{q_0!}; & q_0 = 2, 3, \ldots \end{cases} \tag{2a}$$

The error constant of the method in (1) is

$$C_{q_0+1} = \frac{1}{(q_0+1)!}\sum_{j=0}^{k} j^{q_0+1}\phi_j + \frac{1}{q_0!}k^{q_0}\lambda_k = \frac{c_i^{q_0+1}}{(q_0+1)!} + O\left(h^{q_0+1}\right). \tag{2b}$$

Equations (2a) is the order condition of the first input method Y_0 in (1) while (2b) is the error constants of the first input method Y_0 in (1).

The order conditions and error constants of the second input method Y_{i+1} in (1) are

$$h^{q_i} : \begin{cases} \sum_{j=0}^{k} \varphi_j + \rho_0 = 4 - c_1, \qquad i = 1,2,\ldots,s, \quad i = 0(1)s-1, \\ \sum_{j=1}^{k} j\varphi_j + c_0\rho_0 = 8 - \frac{c_i^2}{2!}, \quad q_i = 1, \\ \frac{1}{q_i!}\sum_{j=1}^{k} j^{q_i}\varphi_j + \rho_0 c_0^{q_i-1} = -\frac{c_0^{q_i}}{q_i!} + \frac{64}{q_i!}; \quad q_i = 2,3,\ldots \end{cases} \tag{3a}$$

The error constant of the method in (1) is

$$C_{q_i+1} = \frac{1}{(q_i+1)!}\sum_{j=0}^{k} j^{q_i+1}\varphi_j + \rho_0 c_0^{q_i} = \frac{c_i^{q_i+1}}{(q_i+1)!} + \frac{64}{(q_i+1)!} + O\left(h^{q_i+1}\right). \tag{3b}$$

For the output method y_{n+k} in (1), the order conditions and error constants are

$$h^p : \begin{cases} \sum_{j=0}^{k-2} \delta_j + \gamma_k = 1, \quad p = 0 \\ \sum_{j=1}^{k} j\delta_j + \gamma_k(5+a) = c_4, \quad p = 1 \\ \frac{1}{p!}\sum_{j=1}^{k} j^p\delta_j + \frac{1}{(p-1)!}\theta_{s-1}^{-} c_{s-1}^{p-1} \\ \quad + \frac{1}{(p-1)!}\left(a(-1+k)^{p-1} + k^{p-1}\right)\gamma_k = \frac{k^p}{p!}; \quad p = 2,3,\ldots \end{cases} \tag{4a}$$

and

$$\begin{aligned} C_{p+1} = {} & \frac{1}{(p+1)!}\sum_{j=1}^{k} j^{p+1}\delta_j + \frac{1}{p!}\theta_{s-1}c_{s-1}^{p} \\ & + \frac{1}{p!}\left(a(-1+k)^p + k^p\right)\gamma_k = \frac{k^{p+1}}{(p+1)!} + O\left(h^{p+1}\right) \end{aligned} \tag{4b}$$

respectively.

3 Stability Analysis of the Methods

In this research work, an investigation will be under taken to establish the stability of high order stiffly stable parameter dependent nested hybrid linear multistep methods by applying the method (1) on step numbers k_i's $\forall i = 1(1)6$ to the standard scalar test equation [9]

$$y' = \lambda y, \quad \text{Re}(\lambda) < 0 \tag{5}$$

which gives the general stability polynomial of the algorithms in (1) stated below

$$\pi_1(w,z) = w^{k-i} - \sum_{j}^{k-2} \partial_j w^j - z\theta_{s-1} f\left(R_1\left(Y_{s-1}\right) - z\gamma_1\left(w^k + aw^{k-1}\right) = 0, \quad 0 \leq c \leq k \tag{6a}$$

where

$$Y_0 = \sum_{j=0}^{k} \phi_j w^j + z\beta_k w^k \tag{6b}$$

$$Y_{i+1} = w^{k-1} + \sum_{j=0}^{k} \varphi_j w^j + z\rho_i f\left(Y_i\right). \tag{6c}$$

For better understanding of this work, it is important to give some definitions as it relates to the stability properties of the method (1) and the definitions can be seen in [11].

Definition 1. A numerical method in (1) is said to be zero stable if the roots $|w_i| \leq 1$ of the stability polynomials (6) are inside the unit circle with simple roots on the unit circle, where

$$\rho(w) = \sum_{j=0}^{k} \alpha_j w^j$$

is the first characteristics polynomial for the numerical method in (1).

Definition 2. A numerical method in (1) is said to be A-stable if the region of absolute stability lies in the entire left half plane of the z complex region.

Definition 3. A numerical method is said to be $A(\alpha)$-stable if for some $\alpha \in [0, \pi/2)$, i.e., if α lies between 0 and $\pi/2$ in the region of absolute stability. The largest $\alpha\,(\alpha_{\max})$ is regarded as the angle of absolute stability on the argument of stability.

Definition 4. A numerical method is said to be A-stable if the absolute value of the root(s) of the stability polynomial lies in the open left half of the complex plane of the stability region.

Examples of A-stable methods can be found in the work of [2,4]. Definition of stiff stability show that stiff stability implies $A(\alpha)$-stability.

Derivation of the Method in (1)

For $k = 1, s = 1, p = q + 1 = 3, c_0 = \frac{1}{2}$ and $a = \frac{1}{2}$ gives;

$$\begin{cases} Y_0 = \frac{1}{4}\left(y_n + 3y_{n+1}\right) - \frac{h}{4} f_{n+1}; \quad q_0 = 2, \quad C_3 = \frac{1}{48} \\ y_{n+1} = y_n + \left(\frac{1-a}{2\left(\frac{-1}{2} + \frac{a}{2}\right)}\right) hf\left(Y_0\right); \quad p = 2, \quad C_3 = \frac{1}{24} \end{cases}, \quad a = \frac{1}{2}. \tag{7}$$

Applying the method in (7) for $k = 1$ and $s = 1$ to the test equation in (5) gives the stability polynomial

$$\pi_1(w, z) = w - 1 - z\theta_0\left(R_1(w, z)\right) - z\gamma_1(w + a) = 0, \quad z = \lambda h, \tag{8a}$$

where

$$R_1(w, z) = \sum_{j=0}^{k} \hat{\alpha}_j w^j + z\beta_1 w \tag{8b}$$

setting $z = 0$ in (8a) and (8b) gives the stability polynomial for $k = 1$

$$\pi_1(w) = w - 1 = 0. \tag{9}$$

The boundary locus plot of the stability polynomial for $k = 1$ is given below

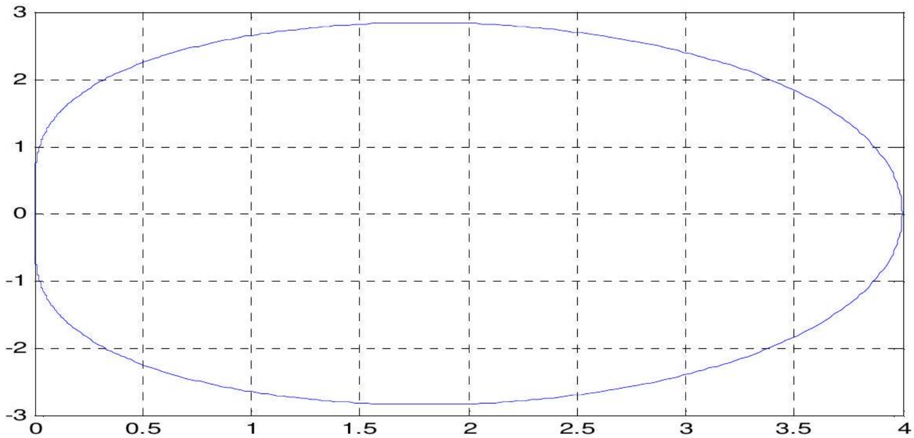

Figure 1: The stability region of the method (1) for $k = 1$.
The stability region is the exterior of the closed curves.

Again, for method $k = 2, s = 2, p = q_1 = q_0 + 1 = 4, c_0 = \frac{7}{4}$ and $c_1 = \frac{3}{2}, a = \frac{1}{4}$,

$$
\begin{aligned}
&Y_0 = \frac{1}{256}\left(-3y_n + 28y_{n+1} + 231y_{n+2} - 42hf_{n+2}\right); \quad C_4 = \frac{7}{2048}, \quad q_0 = 3 \\
&Y_1 = y_{n+1} + \frac{1}{96}\left(-hf_n + 30hf_{n+1} - 13hf_{n+2} + 32hf\left(Y_0\right)\right); \quad C_5 = \frac{179}{92160}, \quad q_1 = 4 \\
&y_{n+2} = y_{n+1} + \frac{2}{9}\left(-\frac{1-a}{-2+a}hf_n - \frac{8-5a}{-2+a}hf\left(Y_1\right)\right) - \frac{1}{6(-2+a)}h\left(f_{n+2} + af_{n+1}\right); \\
&\quad C_4 = \frac{1}{84}, \quad p = 3.
\end{aligned} \tag{10}
$$

Applying the method in (10) to (5) when $k = 2$ gives the stability polynomial

$$
\pi_2(w, z) = w^2 - w - z\sum_{j=0}^{0}\delta_j w^j - z\theta_1\left(R_2(w, z)\right) - z\gamma_2\left(w^2 + aw\right) = 0, \quad z = \lambda h \tag{11a}
$$

where

$$
R_1(w, z) = \sum_{j=0}^{2}\phi_j w^j + z\lambda_2 w^2 \tag{11b}
$$

$$R_2(w,z) = \left(w + z\sum_{j=0}^{2}\varphi_j w^j + z\rho_0\left(R_1(w,z)\right)\right) \tag{11c}$$

setting $z = 0$ in (11a), (11b) and (11c) gives the stability polynomial for $k = 2$.

$$\pi_2(w) = w^2 - w = 0. \tag{12}$$

Plotting the stability polynomial $\pi_2(w)$ in boundary locus sense gives the plot in Figure 2.

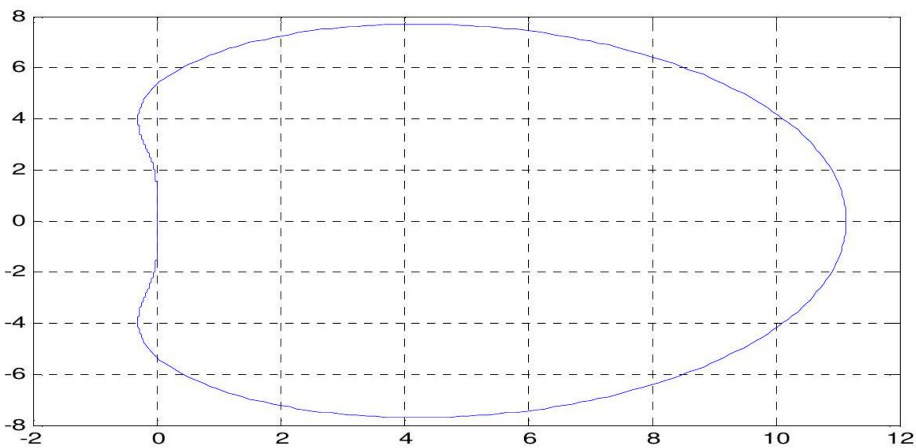

Figure 2: The stability region of the method (1) for $k = 2$.
The stability region is the exterior of the closed curves.

For $k = 3, s = 3$ and $p = q_2 = q_1 = q_0 + 1 = 5, c_0 = \frac{23}{8}, c_1 = \frac{11}{4}$, and $c_2 = \frac{5}{2}, a = \frac{1}{2}$, we have a method of order five,

$$Y_0 = \frac{1}{49152}\left(70y_n - 483y_{n+1} + 2070y_{n+2} + 47495y_{n+3} - 4830hf_{n+3}\right);$$
$$C_5 = \frac{161}{262144}, \quad q_0 = 4,$$

$$
\begin{aligned}
Y_1 &= y_{n+2} + \frac{1143}{235520}hf_n - \frac{1983}{51200}hf_{n+1} + \frac{30549}{71680}hf_{n+2} - \frac{4977}{10240}hf_{n+3} \\
&\quad + \frac{3396}{824025}hf\left(Y_0\right); \quad C_6 = \frac{681}{409600}, \quad q_1 = 5, \\
Y_2 &= y_{n+2} + \frac{179}{63360}hf_n - \frac{319}{13440}hf_{n+1} + \frac{1979}{5760}hf_{n+2} - \frac{601}{5760}hf_{n+3} + \frac{976}{3465}hf\left(Y_1\right); \\
&\quad C_6 = \frac{41}{46080}, \quad q_2 = 5, \\
y_{n+3} &= y_{n+2} - \frac{(-1+a)}{30(-3+a)}hf_n - \frac{(1-3a)}{6(-3+a)}hf_{n+1} - \frac{(19-9a)}{\frac{15}{2}(-3+a)}hf\left(Y_2\right) \\
&\quad + \frac{1}{3(-3+a)}h\left(f_{n+3} + af_{n+2}\right); \quad C_5 = \frac{11}{2880}, \quad p = 4.
\end{aligned} \tag{13}
$$

Applying the method in (13) to (5) when $k = 2$ gives the stability polynomial

$$
\pi_3(w,z) = w^3 - w^2 - z\sum_{j=0}^{1}\delta_j w^j - z\theta_2\left(R_3(w,z)\right) - z\gamma_3\left(w^3 + aw^2\right) = 0, \quad z = \lambda h \tag{14a}
$$

where

$$
R_1(w,z) = \sum_{j=0}^{3}\phi_j w^j + z\lambda_3 w^3 \tag{14b}
$$

$$
R_2(w,z) = w^2 + z\sum_{j=0}^{3}\varphi_j w^j + z\rho_0\left(\sum_{j=0}^{3}\phi_j w^j + z\lambda_3 w^3\right) \tag{14c}
$$

$$
R_3(w,z) = \left(w^2 + z\sum_{j=0}^{3}\varphi_j w^j + z\rho_1\left(R_2(w,z)\left(R_1(w,z)\right)\right)\right) \tag{14d}
$$

setting $z = 0$ in (14a), (14b), (14c) and (14d) gives the stability polynomial for $k = 3$.

$$
\pi_3(w) = w^3 - w^2 = 0. \tag{15}
$$

Plotting the stability polynomial $\pi_3(w)$ in boundary locus sense gives the plot in Figure 3.

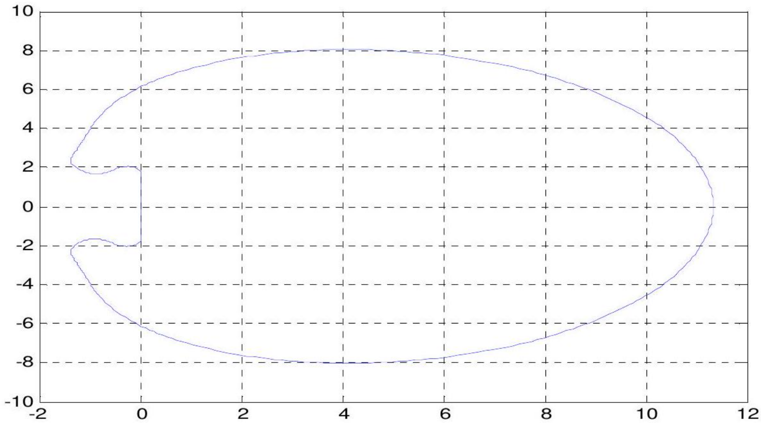

Figure 3: The stability region of the method for $k = 3$.
The stability region is the exterior of the closed curves.

For $k = 4$, $s = 4, p = q_3 = q_2 = q_1 = q_0 + 1 = 6, c_0 = \frac{63}{16}, c_1 = \frac{23}{8}, c_2 = \frac{15}{4}$, and $c_3 = \frac{7}{2}$, gives the method of order six,

$$\begin{aligned}
Y_0 = & -\frac{7285}{33554432} y_n + \frac{3255}{2097152} y_{n+1} - \frac{44415}{8388608} y_{n+2} + \frac{30597}{2097152} y_{n+3} \\
& + \frac{33197745}{33554432} y_{n+4} - \frac{458955}{8388608} h_{n+4}; \quad C_6 = \frac{30597}{268435456}, \quad q_0 = 5, \\
Y_1 = & \, y_{n+3} - \frac{136493}{53084160} hf_n + \frac{675857}{34652160} hf_{n+1} - \frac{2320297}{30474240} hf_{n+2} + \frac{27461}{55296} hf_{n+3} \\
& - \frac{6820457}{5898240} hf_{n+4} + \frac{940562}{590085} hf\left(Y_0\right); \quad C_7 = \frac{20953037}{18119393280}, \quad q_1 = 6, \\
Y_2 = & \, y_{n+3} - \frac{681}{317440} hf_n + \frac{3867}{235520} hf_{n+1} - \frac{669}{10240} hf_{n+2} + \frac{33273}{71680} hf_{n+3} \\
& - \frac{537}{1280} hf_{n+4} + \frac{18876}{24955} hf\left(Y_1\right); \quad C_7 = \frac{34537}{36700160}, \quad q_2 = 6, \\
Y_3 = & \, y_{n+3} - \frac{41}{34560} hf_n + \frac{589}{63360} hf_{n+1} - \frac{131}{3360} hf_{n+2} + \frac{6347}{17280} hf_{n+3} \\
& - \frac{997}{11520} hf_{n+4} + \frac{520}{2079} hf\left(Y_2\right); \quad C_7 = \frac{757}{1548288}, \quad q_3 = 6,
\end{aligned}$$

$$
\begin{aligned}
y_{n+4} = y_{n+3} &- \frac{\frac{1}{4}(57-59a)}{630(-4+a)} hf_n - \frac{4(-232+237a)}{450(-4+a)} hf_{n+1} \\
&- \frac{4(118-119a)}{90(-4+a)} hf_{n+2} - \frac{4(646-251a)}{\frac{1575}{2}(-4+a)} hf\left(Y_3\right) \\
&- \frac{179}{360(-4+a)} h\left(f_{n+4} + af_{n+3}\right); \quad C_6 = \frac{13}{4140}, \quad p = 5.
\end{aligned} \tag{16}
$$

Also, applying the method in (16) to (5) for $k = 4$ gives the stability polynomial

$$\pi_4(w,z) = w^4 - w^3 - z\sum_{j=0}^{2}\delta_j w^j - z\theta_3\left(R_4(w,z)\right) + z\gamma_4\left(w^4 + aw^3\right) = 0, \quad z = \lambda h \tag{17a}$$

where

$$R_1(w,z) = \sum_{j=0}^{4}\phi_j w^j + z\lambda_4 w^4 \tag{17b}$$

$$R_2(w,z) = w^3 + z\sum_{j=0}^{4}\varphi_j w^j + z\rho_0\left(\sum_{j=0}^{4}\phi_j w^j + z\lambda_4 w^4\right) \tag{17c}$$

$$R_3(w,z) = w^3 + z\sum_{j=0}^{4}\varphi_j w^j + z\rho_1\left(w^3 + z\sum_{j=0}^{4}\varphi_j w^j + z\rho_0\left(\sum_{j=0}^{4}\phi_j w^j + z\lambda_4 w^4\right)\right) \tag{17d}$$

$$R_4(w,z) = \left(w^3 + z\sum_{j=0}^{4}\varphi_j w^j + z\rho_2\left(R_3(w,z)\left(R_2(w,z)\left(R_1(w,z)\right)\right)\right)\right) \tag{17e}$$

setting $z = 0$ in (17a), (17b), (17c), (17d) and (17e) gives the stability polynomial for $k = 4$.

$$\pi_4(w) = w^4 - w^3 = 0. \tag{18}$$

The boundary locus plot of the stability polynomial for $k = 4$ is shown in Figure 4.

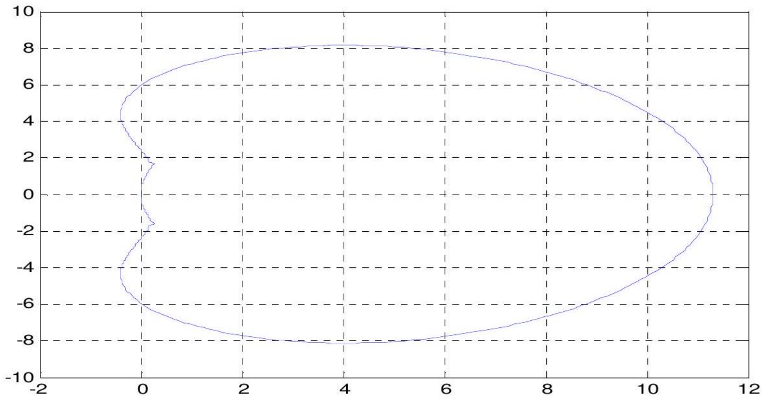

Figure 4: The stability region of the method for $k = 4$.
The stability region is the exterior of the closed curves.

For $k = 5, s = 5, p = q_4 = q_3 = q_2 = q_1 = q_0 + 1 = 7, c_2 = \frac{39}{8}, c_3 = \frac{19}{4}$, and $c_4 = \frac{9}{2}$ gives the method of order seven,

$$\begin{aligned} Y_0 =& \frac{1570863}{42949672960} y_n - \frac{9833355}{34359738368} y_{n+1} + \frac{4381881}{4294967296} y_{n+2} - \frac{19822795}{8589934592} y_{n+3} \\ &+ \frac{40285035}{8589934592} y_{n+4} + \frac{171257055123}{171798691840} y_{n+5} - \frac{249767217}{8589934592} h f_{n+5}; \\ & C_7 = \frac{11893677}{549755813888}, q_0 = 6, \end{aligned}$$

$$\begin{aligned} Y_1 =& y_{n+4} + \frac{151234285}{99589554176} h f_n - \frac{2928519075}{238639120384} h f_{n+1} + \frac{826756935}{17850957824} h f_{n+2} \\ &- \frac{2352809825}{19730006016} h f_{n+3} + \frac{31979005155}{58250493952} h f_{n+4} - \frac{4530823545}{1879048192} h f_{n+5} \\ &+ \frac{1680435380}{582790173} h f\left(Y_0\right); \quad C_8 = \frac{21529183055}{26938034880512}, \quad q_1 = 7, \end{aligned}$$

$$\begin{aligned} Y_2 =& y_{n+4} + \frac{20953037}{14910750720} h f_n - \frac{19334231}{1698693120} h f_{n+1} + \frac{191274881}{4435476480} h f_{n+2} \\ &- \frac{327513697}{2925527040} h f_{n+3} + \frac{302153677}{566231040} h f_{n+4} - \frac{197301587}{188743680} h f_{n+5} \\ &+ \frac{68317816}{46616715} h f\left(Y_1\right); \quad C_8 = \frac{212521897}{289910292480}, \quad q_2 = 7, \end{aligned}$$

$$\begin{aligned}
Y_3 =& y_{n+4} + \frac{34537}{29818880}hf_n - \frac{670599}{71106560}hf_{n+1} + \frac{951159}{26378240}hf_{n+2} - \frac{21893}{229376}hf_{n+3} \\
&+ \frac{7971207}{16056320}hf_{n+4} - \frac{858693}{2293760}hf_{n+5} + \frac{1579568}{2270905}hf\left(Y_2\right); \\
&C_8 = \frac{1217123}{2055208960}, \quad q_3 = 7,
\end{aligned}$$

$$\begin{aligned}
Y_4 =& y_{n+4} + \frac{757}{1225728}hf_n - \frac{4933}{967680}hf_{n+1} + \frac{35417}{1774080}hf_{n+2} - \frac{62941}{1128960}hf_{n+3} \\
&+ \frac{374357}{967680}hf_{n+4} - \frac{24131}{322560}hf_{n+5} + \frac{63104}{276507}hf\left(Y_3\right); \\
&C_8 = \frac{13025}{43352064}, \quad q_4 = 7,
\end{aligned}$$

$$\begin{aligned}
y_{n+5} =& y_{n+4} - \frac{(53-58a)}{3240(-5+a)}hf_n + \frac{(-90+97a)}{840(-5+a)}hf_{n+1} + \frac{(37-39a)}{120(-5+a)}hf_{n+2} \\
&+ \frac{589}{1080} + \frac{79}{36(-5+a)}hf_{n+3} + \frac{8\left(475+\frac{948}{-5+a}\right)}{2835}hf\left(Y_4\right) \\
&- \frac{79}{120(-5+a)}h\left(f_{n+5}+af_{n+4}\right); \quad C_7 = \frac{373}{362880}, \quad p = 6.
\end{aligned} \tag{19}$$

Applying the method in (19) to (5) for $k = 5$ gives the stability polynomial

$$\pi_5(w,z) = w^5 - w^4 - z\sum_{j=0}^{3}\delta_j w^j - z\theta_4\left(R_5(w,z)\right) + z\gamma_5\left(w^5 + aw^4\right) = 0, \quad z = \lambda h \tag{20a}$$

where

$$R_1(w,z) = \sum_{j=0}^{5}\phi_j w^j + z\lambda_5 w^5 \tag{20b}$$

$$R_2(w,z) = w^4 + z\sum_{j=0}^{5}\varphi_j w^j + z\rho_0\left(\sum_{j=0}^{5}\phi_j w^j + z\lambda_5 w^5\right) \tag{20c}$$

$$R_3(w,z) = w^4 + z\sum_{j=0}^{5}\varphi_j w^j + z\rho_1\left(w^4 + z\sum_{j=0}^{5}\varphi_j w^j + z\rho_0\left(\sum_{j=0}^{5}\phi_j w^j + z\lambda_5 w^5\right)\right) \tag{20d}$$

$$R_4(w,z) = w^4 + z\sum_{j=0}^{5}\varphi_j w^j + z\rho_2\left(w^4 + z\sum_{j=0}^{5}\varphi_j w^j + z\rho_1\left(w^4 + z\sum_{j=0}^{5}\varphi_j w^j + z\rho_0\left(\sum_{j=0}^{5}\phi_j w^j + z\lambda_5 w^5\right)\right)\right) \tag{20e}$$

$$R_5(w,z) = \left(w^4 + z\sum_{j=0}^{5}\varphi_j w^j + z\rho_3\left(R_4(w,z)\left(R_3(w,z)\left(R_2(w,z)\left(R_1(w,z)\right)\right)\right)\right)\right) \tag{20f}$$

setting $z = 0$ in (20a), (20b), (20c), (20d), (20e) and (20f) gives the stability polynomial for $k = 5$.

$$\pi_5(w) = w^5 - w^4 = 0. \tag{21}$$

The boundary locus plot of the stability polynomial for $k = 5$ is given in Figure 5.

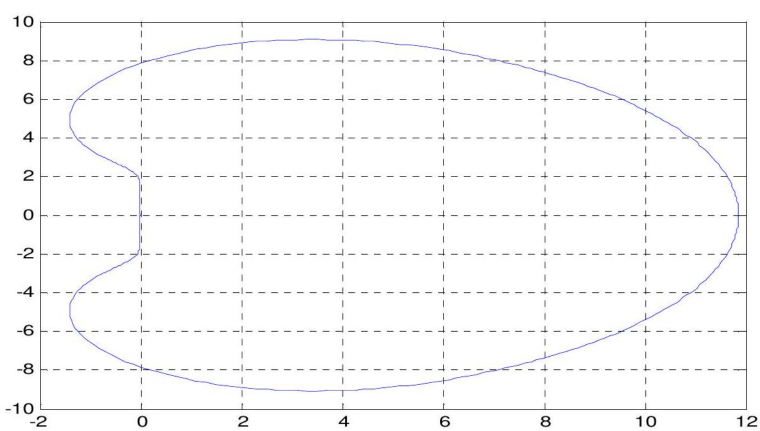

Figure 5: The stability region of the method for $k = 5$.
The stability region is the exterior of the closed curves.

For $k=6, s=6, p=q_5=q_4=q_3=q_2=q_1=q_0+1=8, c_0=\frac{3839}{64}, c_1=\frac{191}{32}, c_2=\frac{95}{16}, c_3=\frac{47}{8}$, $c_4=\frac{23}{4}, c_5=\frac{11}{2}$ and $a=\frac{1}{4}$ gives the method of order eight,

$$\begin{aligned}
Y_0 = &-\frac{920819977}{14073748855328}y_n+\frac{9950051601}{175921860444160}y_{n+1}-\frac{62236597269}{281474976710656}y_{n+2}\\
&+\frac{9232305005}{17592186044416}y_{n+3}-\frac{124963246485}{140737488355328}y_{n+4}+\frac{50382007313}{35184372088832}y_{n+5}\\
&-\frac{1406111442098517}{1407374883553280}y_{n+6}-\frac{1058022153573}{70368744177664}hf_{n+6};\\
&C_8=\frac{151146021939}{36028797018963968},\quad q_0=7,
\end{aligned}$$

$$\begin{aligned}
Y_1 = y_{n+5}&-\frac{154807620931445}{1591806963996426 24}hf_n+\frac{63190231095911}{7365628394471424}hf_{n+1}\\
&-\frac{135853459989697}{3925256511160320}hf_{n+2}+\frac{8617444950372269}{99228175627714560}hf_{n+3}\\
&-\frac{4888541658041807}{29323975112785920}hf_{n+4}+\frac{22765089824959}{38482906972160}hf_{n+5}\\
&-\frac{9927780212406839}{20780769764966 40}hf_{n+6}-\frac{11133442789211}{2116044307146}hf\left(Y_0\right);\\
&C_9=\frac{579847204017222941}{102141639548762849 2800},\quad q_1=8,
\end{aligned}$$

$$\begin{aligned}
Y_2 = y_{n+5}&-\frac{21529183055}{22969485099008}hf_n+\frac{26368680175}{3186865733632}hf_{n+1}-\frac{510362966625}{15272903704576}hf_{n+2}\\
&+\frac{47985404975}{571230650368}hf_{n+3}-\frac{8342970875}{51539607552}hf_{n+4}+\frac{1087915714125}{1864015806464}hf_{n+5}\\
&-\frac{268443523825}{120259084288}hf_{n+6}+\frac{42776282075}{15901846149}hf\left(Y_1\right);\\
&C_9=\frac{537129214325}{985162418487296},\quad q_2=8,
\end{aligned}$$

$$\begin{aligned}
Y_3 = y_{n+5}&-\frac{78297541}{90596966400}hf_n+\frac{607633957}{79524003840}hf_{n+1}-\frac{62270551}{2013265920}hf_{n+2}\\
&+\frac{166619460683}{212902871040}hf_{n+3}-\frac{9466381001}{62411243520}hf_{n+4}+\frac{2851105271}{5033164800}hf_{n+5}\\
&-\frac{17453299073}{18119393280}hf_{n+6}+\frac{11822902}{8632725}hf\left(Y_2\right);\\
&C_9=\frac{277769237291}{556627761561600},\quad q_3=8,
\end{aligned}$$

$$\begin{aligned}
Y_4 = y_{n+5} &- \frac{1217123}{1724907520}hf_n + \frac{1493419}{238551040}hf_{n+1} - \frac{28986429}{1137704960}hf_{n+2} \\
&+ \frac{13694887}{211025920}hf_{n+3} - \frac{4720003}{36700160}hf_{n+4} + \frac{1926315}{36700160}hf_{n+5} \\
&- \frac{2504393}{7340032}hf_{n+6} + \frac{9910172}{15247505}hf\left(Y_3\right); \\
C_9 &= \frac{37641013}{93952409600}, \quad q_4 = 8,
\end{aligned}$$

$$\begin{aligned}
Y_5 = y_{n+5} &- \frac{13025}{35610624}hf_n + \frac{5351}{1634304}hf_{n+1} - \frac{11621}{860160}hf_{n+2} + \frac{68239}{1935360}hf_{n+3} \\
&- \frac{63461}{860160}hf_{n+4} + \frac{173617}{430080}hf_{n+5} - \frac{514019}{7741440}hf_{n+6} + \frac{63104}{276507}hf\left(Y_4\right); \\
C_9 &= \frac{2961281}{14863564800}, \quad q_4 = 8,
\end{aligned}$$

$$\begin{aligned}
y_{n+6} = y_{n+5} &+ \frac{-453+533a}{36960(-6+a)}hf_n + \frac{8304-9619a}{90720(-6+a)}hf_{n+1} + \frac{9(-263+298a)}{7840(-6+a)}hf_{n+2} \\
&+ \frac{984-1079a}{1680(-6+a)}hf_{n+3} + \frac{24487}{30240} + \frac{915}{224(-6+a)}hf_{n+4} \\
&+ \frac{16\left(19087 + \frac{49410}{-6+a}\right)}{218295}hf\left(Y_5\right) - \frac{183}{224(-6+a)}h\left(f_{n+6} + af_{n+5}\right); \\
C_8 &= \frac{-1177}{947520}, \quad p = 7.
\end{aligned} \tag{22}$$

Applying the method in (22) to (5) for $k = 6$ gives the stability polynomial

$$\pi_6(w,z) = w^6 - w^5 - z\sum_{j=0}^{4}\delta_j w^j - z\theta_5\left(R_6(w,z)\right) + z\gamma_6\left(w^6 + aw^4\right) = 0, \quad z = \lambda h \tag{23a}$$

where

$$R_1(w,z) = \sum_{j=0}^{6}\phi_j w^j + z\lambda_6 w^6 \tag{23b}$$

$$R_2(w,z) = w^5 + z\sum_{j=0}^{6}\varphi_j w^j + z\rho_0\left(\sum_{j=0}^{6}\phi_j w^j + z\lambda_6 w^6\right) \tag{23c}$$

$$R_3(w,z) = w^5 + z\sum_{j=0}^{6}\varphi_j w^j + z\rho_1\left(w^5 + z\sum_{j=0}^{6}\varphi_j w^j + z\rho_0\left(\sum_{j=0}^{6}\phi_j w^j + z\lambda_6 w^6\right)\right) \tag{23d}$$

$$R_4(w,z) = w^5 + z\sum_{j=0}^{6}\varphi_j w^j + z\rho_2\left(w^5 + z\sum_{j=0}^{6}\varphi_j w^j + z\rho_1\left(w^5 + z\sum_{j=0}^{6}\varphi_j w^j + z\rho_0\left(\sum_{j=0}^{6}\phi_j w^j + z\lambda_6 w^6\right)\right)\right) \tag{23e}$$

$$R_5(w,z) = w^5 + z\sum_{j=0}^{6}\varphi_j w^j + z\rho_3\left(w^5 + z\sum_{j=0}^{6}\varphi_j w^j + z\rho_2\left(w^5 + z\sum_{j=0}^{6}\varphi_j w^j + z\rho_1\left(w^5 + z\sum_{j=0}^{6}\varphi_j w^j + z\rho_0\left(\sum_{j=0}^{6}\phi_j w^j + z\lambda_6 w^6\right)\right)\right)\right) \tag{23f}$$

$$R_6(w,z) = \left(w^5 + z\sum_{j=0}^{6}\varphi_j w^j + z\widehat{\beta}_4\left(R_5(w,z)\left(R_4(w,z)\left(R_3(w,z)\left(R_2(w,z)\left(R_1(w,z)\right)\right)\right)\right)\right)\right) \tag{23g}$$

setting $z = 0$ in (23a), (23b), (23c), (23d), (23e), (23f) and (23g) gives the stability polynomial for $k = 6$.

$$\pi_6(w) = w^6 - w^5 = 0. \tag{24}$$

Plotting the stability polynomial $\pi_6(w)$ in boundary locus sense gives the plot in Figure 6.

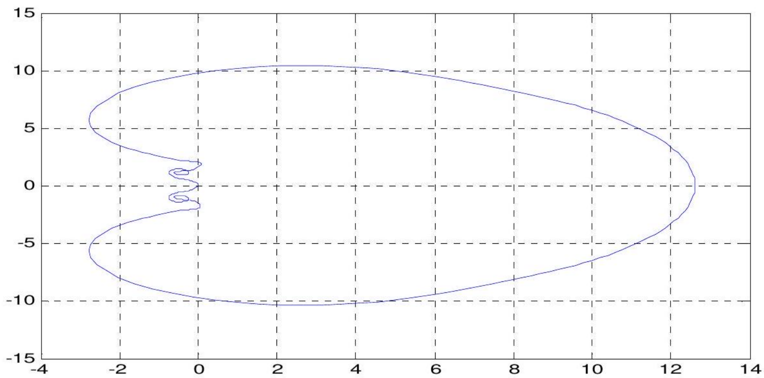

Figure 6: The stability region of the method for $k = 6$.
The stability region is the exterior of the closed curves.

Table 1: The value of "a" for the $k_i, i = 1(1)6$ and the stability properties of the method (1).

k	a	Stability properties
1	1/2	A-Stable $(-\infty, 0]$
2	1/4	$A(\alpha)$-Stable $\left(\alpha = 89.9^0\right)$
3	1/2	$A(\alpha)$-Stable $\left(\alpha = 47.1^0\right)$
4	1/2	$A(\alpha)$-Stable $\left(\alpha = 82.5^0\right)$
5	1/2	$A(\alpha)$-Stable $\left(\alpha = 61.9^0\right)$
6	1/4	$A(\alpha)$-Stable $\left(\alpha = 58.2^0\right)$

4 Discussion of Results

The various boundary locus plots of the stability polynomial of our method on the stability of a high-order stiffly stable parameter dependent nested hybrid linear multistep method for step number $1 \leq k \leq 6$ as shown in Figures 1 to 6 clearly shows that our method is A-stable and $A(\alpha)$-stable.

5 Conclusion

Finally, it has been established from Figures 1 to 6 above that the stability properties of method (1) is found to be of high order and stiffly stable as it is found to be A-stable and $A(\alpha)$-stable in the region of absolute stability. Hence the high order stiffly stable parameter dependent nested hybrid linear multistep method is recommended for all ODE solvers as an instrument for determining the stability of real life problems. The method is therefore suitable for solving stiff initial value problems.

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