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Error Bounds and Merit Functions for Exponentially General Variational Inequalities

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Abstract

In this paper, some new classes of classes of exponentially general variational inequalities are introduced. It is shown that the odd-order and nonsymmetric exponentially boundary value problems can be studied in the framework of exponentially general variational inequalities. We consider some classes of merit functions for exponentially general variational inequalities. Using these functions, we derive error bounds for the solution of exponentially general variational inequalities under some mild conditions. Since the exponentially general variational inequalities include general variational inequalities, quasi-variational inequalities and complementarity problems as special cases, results proved in this paper hold for these problems. Results obtained in this paper represent a refinement of previously known results for several classes of variational inequalities and related optimization problems.

1 Introduction

Variational inequalities have been extended and generalized in several directions for studying a wide class of equilibrium problems arising in financial, economics,

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transportation, elasticity, optimization, pure and applied sciences. An important and useful generalization of variational inequalities is called the *general (Noor) variational inequality* introduced by Noor [21] in 1988, which enables us to study the odd-order and nonsymmetric problems in a unified framework, see [21, 24–27, 41–43]. It is well known that the minimum of the differentiable convex functions on the convex set can be characterized by the variational inequalities, the origin of which can be traced back to Stamapcchia [54]. This simple fact has influenced every branch of mathematical and engineering sciences. Convexity theory is a branch of mathematical sciences with a wide range of applications in industry, physical, social, and engineering sciences. Researches in this domain have established important and novel connections with all areas of pure and applied sciences, see [5,9,11,13–15,17,18,20–26,31–43,46,49–54,56,57,59] In recent years, several new generalizations of convex functions have been introduced using novel and innovative ideas to tackle difficult problems, which arise in various fields of pure and applied sciences. Noor and Noor [27–31, 40, 44] studied the exponentially convex functions and their variant forms, the origin of which can be traced back to Bernstein [5]. See also [1–5,27–31,33,40,44,45,48,58] for more details. Noor and Noor [26] proved that the optimality conditions of the differentiable exponentially convex functions can be characterised by the exponentially variational inequality. These facts motivated and inspired Noor and Noor [31] to consider the exponentially general variational inequality. It is established the odd-order exponentially boundary value problems can be studied in the unified frame work of exponentially general variational inequalities. Clearly exponentially general variational inequality include the general variational inequalities and complementarity problems, For the applications, motivations and numerical methods, see [31, 40, 41, 45].

In recent years, much attention has been given to reformulate the variational inequality as an optimization problem. A function which can constitute an equivalent optimization problem is called a merit function. Merit functions are being used in designing new globally convergent algorithms and in analyzing the rate of convergence of some iterative methods. Various merit functions for variational inequalities and complementarity problems have been suggested and proposed by many authors, see [5, 13, 22, 23, 26, 32, 36, 37, 39, 41, 52, 53, 56, 57] and the references therein. Merit functions are used to derive the error bounds, which provide a measure of the distance between a solution set and an arbitrary point. Therefore, error bounds play an important role in the analysis of global or local convergence analysis of algorithms for solving variational inequalities. To the best of our knowledge, no merit functions have been considered for exponentially general variational inequalities.

In this paper, we consider normal residue merit functions, regularized merit functions, difference merit functions and dual merit functions for exponentially general variational inequalities and some other related aspects. We also obtain error bounds for the solution of the exponentially general variational inequalities under some weaker conditions. Since the exponentially general variational inequalities include variational inequalities, quasi-variational inequalities and quasi complementarity problems as special cases, one can deduce the similar results for these problems under weaker conditions. In this respect, our results can be viewed as refinement of the previously known results for exponentially variational inequalities.

2 Preliminaries and Basic Results

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ respectively. Let K be a closed convex set in H and $T, g : H \leftrightarrow H$ be nonlinear operators. We now consider the problem of finding $u \in H : g(u) \in K$ such that

$$\langle e^{Tu}, g(v) - g(u)\rangle \geq 0, \quad \forall v \in H : g(v) \in K. \tag{2.1}$$

Problem (2.1) is called the *exponentially general variational inequality* , which was introduced by Noor and Noor [31].

First of all, we recall the following well known concepts for the sake of completeness.

For the nonsymmetric and odd-order problems, many methods have developed by several authors including Filippov [12] and Tonti [55] to construct the energy functional by introducing the concept of g-symmetry and g-positivity of the operator g. We now introduce some new concepts the for the exponentially operator T.

Definition 2.1. *$\forall u, v \in H$, the operator $T : H \longrightarrow H$ with respect to an arbitrary operator g is said to be :*

(a). *exponentially general symmetric, if and only if,*

$$\langle e^{Tu}, g(v)\rangle = \langle g(u), e^{Tv}\rangle.$$

(b). *exponentially general symmetric positive, if and only if,*

$$\langle e^{Tu}, g(u)\rangle \geq 0.$$

(c). *exponentially general symmetric coercive (g-elliptic), if there exists a constant $\alpha > 0$ such that*

$$\langle e^{Tu}, g(u)\rangle \geq \alpha \|g(u)\|^2.$$

Note that exponentially general coercivity implies exponentially general symmetric positivity, but the converse is not true. It is also worth mentioning that there are operators which are not exponentially general symmetric but exponentially general positive. On the other hand, there are exponentially general positive, but not exponentially general symmetric operators. Furthermore, it is well-known [12, 55] that if, for a linear operator T, there exists an inverse operator T^{-1} operator on $R(T)$ with $\overline{R(T)} = H$, then one can find an infinite set of auxiliary operators g such that the operator T is both exponentially general symmetric and exponentially general positive. It is worth mentioning that for $e^{Tu} = \Phi(u)$, where $\Phi : H \to H$ is a nonlinear operator, Definition 2.1 is due to Filippov [12] and Tonti [55].

Applications

To convey an idea of the applications of the exponentially general variational inequalities (2.5), we consider the third-order obstacle boundary value problem of finding u such that

$$\left.\begin{array}{ll} -e^{\frac{d^3v}{dx^3}} \geq f(x) & \text{on } \Omega = [0,1] \\ u \geq \psi(x) & \text{on } \Omega = [0,1] \\ [-e^{\frac{d^3v}{dx^3}} - f(x)][u - \psi(x)] = 0 & \text{on } \Omega = [0,1] \\ u(0) = 0, \quad u'(0) = 0, \quad u'(1) = 0 & \end{array}\right\} \tag{2.2}$$

where $f(x)$ is a continuous function and $\psi(x)$ is the obstacle function. We study the problem (2.2) in the framework of variational inequality approach. To do so, we first define the set K as

$$K = \{v : v \in H_0^2(\Omega) : v \geq \psi \quad \text{on } \Omega\},$$

which is a closed convex set in $H_0^2(\Omega)$, where $H_0^2(\Omega)$ is a Sobolev (Hilbert) space, see [12, 55]. One can easily show that the energy functional associated with the problem (2.2) is

$$\begin{aligned} I[v] &= -\int_0^1 \left(e^{\frac{d^3v}{dx^3}}\right)\left(\frac{dv}{dx}\right)dx - 2\int_0^1 f(x)\left(\frac{dv}{dx}\right)dx, \quad \text{for all } \tfrac{dv}{dx} \in K \\ &= \int_0^1 \left(\frac{d^2v}{dx^2}\right)^2 dx - 2\int_0^1 f(x)\left(\frac{dv}{dx}\right)dx = \langle e^{Tu}, g(v)\rangle - 2\langle f, g(v)\rangle, \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} \langle e^{Tu}, g(v)\rangle &= -\int_0^1 \left(e^{\frac{d^3v}{dx^3}}\right)\left(\frac{dv}{dx}\right)dx = \int_0^1 \left(\frac{d^2u}{dx^2}\right)\left(\frac{d^2v}{dx^2}\right)dx \qquad (2.4) \\ \langle f, g(v)\rangle &= \int_0^1 f(x)\frac{dv}{dx}dx \end{aligned}$$

and $g = \dfrac{d}{dx}$ is the linear operator. It is clear that the operator T defined by (2.4) exponentially general symmetric and exponentially general positive. Using the technique of Noor [24], one can easily show that the minimum $u \in H$ of the functional $I[v]$ defined by (2.2) associated with the problem (2.2) on the closed convex set K can be characterized by the inequality of type

$$\langle e^{Tu}, g(v) - g(u) \rangle \geq \langle f, g(v) - g(u) \rangle, \quad \forall g(v) \in K,$$

which is exactly the exponentially general variational inequality (2.1).

We now discus some important special cases.

(I). If $e^{Tu} = \Phi(u)$, where Φ is a nonlinear operator, the (2.1) reduces to find $u \in K$ such that

$$\langle \Phi(u), g(v) - g(u) \rangle \geq 0, \quad \forall v \in H : g(v) \in K, \tag{2.5}$$

which is called the general variational inequality, introduced and studied by Noor [21]] in 1988.

It is worth mentioning that a wide class of unrelated odd-order and nonsymmetric obstacle, unilateral, contact, free, moving, and equilibrium problems arising in regional, physical, mathematical, engineering and applied sciences can be studied in the unified and general framework of the general variational inequalities (2.5), see [5,21–26,38,39,42,43] and the references therein.

(II). For $g \equiv I$, where I is the identity operator, then the problem (2.5) is equivalent to finding $u \in K$ such that

$$\langle e^{Tu}, v - u \rangle \geq 0 \quad \forall v \in K, \tag{2.6}$$

which is called the exponentially variational inequalities, introduced and studied by Noor et al. [40].

(III). If $e^{Tu} = \Phi(u)$, where Φ is a nonlinear operator, then (2.6) reduces to find $u \in K$ such that

$$\langle \Phi(u), v - u\rangle \geq 0 \quad \forall v \in K, \tag{2.7}$$

which is known as the classical variational inequality introduced and studied by Stampacchia [54] in 1964. For recent state-of-the-art, see [5, 11–15, 17–25, 32–35, 38–40, 42, 43] and the references therein.

(IV). If $K^* = \{u \in H : \langle u, v\rangle \geq 0, \quad \forall v \in K\}$ is a polar (dual) cone of a convex cone K in H, then problem (2.1) is equivalent to finding $u \in H$ such that

$$g(u) \in K, \quad e^{Tu} \in K^* \quad \text{and} \quad \langle e^{Tu}, g(u)\rangle = 0, \tag{2.8}$$

which is known as the exponentially general complementarity problem.

For $g(u) = m(u) + K$, where m is a point-to-point mapping, problem (2.8) is called the implicit exponentially (quasi) complementarity problem. If $g \equiv I$, then problem (2.8) is known as the exponentially generalized complementarity problem. Such problems have been studied extensively in the literature, see [9, 21, 22, 26] references. For suitable and appropriate choice of the operators and spaces, one can obtain several classes of variational inequalities and related optimization problems.

We now recall the following well known result and concepts.

Lemma 2.1. *[15] For a given $z \in H$, $u \in K$ satisfies the inequality*

$$\langle u - z, v - u\rangle \geq 0, \quad \forall v \in K, \tag{2.9}$$

if and only if,

$$u = P_K[z],$$

where P_K is the projection of H onto K. Also, the projection operator P_K is nonexpansive.

Definition 2.2. *[31] $\forall u, v \in H$, the operator $T : H \longrightarrow H$ with respect to an arbitrary operator g is said to be:*

(a) *strongly exponentially monotone, if and only if, there exists a constant $\alpha > 0$ such that*

$$\langle e^{Tu} - e^{Tv}, g(u) - g(v) \rangle \geq \alpha \|g(u) - g(v)\|^2.$$

(b) *exponentially general monotone, if and only if f,*

$$\langle e^{Tu} - e^{Tv}, g(u) - g(v) \rangle \geq 0.$$

(c) *exponentially general pseudomonotone, if and only if,*

$$\langle e^{Tu}, g(v) - g(u) \rangle \geq 0 \Longrightarrow \quad \langle e^{Tv}, g(v) - g(u) \rangle \geq 0.$$

(d) *exponentially general Lipschitz continuous, if there exists a constant $\beta > 0$ such that*

$$\|e^{Tu} - e^{Tv}\| \leq \beta \|g(u) - g(v)\|.$$

(e) *exponentially general hemicontinuous, if $\forall u, v \in H$, the mapping $t \longrightarrow \langle e^{T(g(u)+t(g(v)-g(u)),g(v)} \rangle$ is continuous for $t \in [0, 1]$.*

From (a) and (d), we have $\alpha \leq \beta$. For $g = I$, the identity operator, Definition 2.2 reduces to the well-known definition of strongly exponentially monotonicity, exponentially monotonicity, exponentially pseudomonotonicity and exponentially Lipschitz continuity of T. It is well-known [18] that exponentially monotonicity implies exponentially pseudomonotonicity, but the converse is not true.

Remark 2.1. *We would like to point out that, if the operator T is strongly exponentially monotone with a constant $\alpha > 0$, then*

$$\alpha \|u - v\|^2 \leq \langle e^{Tu} - e^{Tv}, u - v \rangle \leq \|e^{Tu} - e^{Tv}\| \|u - v\|,$$

implies that

$$\|e^{Tu} - e^{Tv}\| \geq \alpha \|u - v\|, \quad \forall u, v \in H.$$

In this case, we say that the operator T is strongly exponentially nonexpanding with a constant $\alpha > 0$. Note that the strongly exponentially general monotonicity implies nonexpandicity, but not conversely. It is clear that if the operator T is strongly exponentially general monotone and g is strongly exponentially nonexpanding, then

$$\langle e^{Tu} - e^{Tv}, g(u) - g(v)\rangle \geq \alpha\|g(u) - g(v)\|^2 \geq \alpha\|u - v\|^2, \quad \forall u, v, \in H.$$

Lemma 2.2. *Let the operator T be exponentially general pseudomonotone and exponentially general hemicontinuous and let the operator g be convex. Then problem (2.1) is equivalent to finding $u \in H : g(u) \in K$ such that*

$$\langle e^{Tv}, g(v) - g(u)\rangle \quad \geq 0, \quad \forall v \in H : \quad g(v) \in K. \tag{2.10}$$

Proof. Let $u \in H : g(u) \in K$ be a solution of (2.1). Then

$$\langle e^{Tu}, g(v) - g(u)\rangle \quad \geq 0, \quad \forall v \in H : \quad g(v) \in K,$$

which implies, using the exponentially general pseudomonotonicity of T,

$$\langle e^{Tv}, g(v) - g(u)\rangle \quad \geq 0, \quad \forall v \in H : \quad g(v) \in K.$$

Conversely let $u \in H$ be such that (2.10) hold. For $t \in [0, 1], u, v \in H, \quad v_t = u + t(v - u) \in H$. Taking $v = v_t$ in (2.10) and using the convexity of the operator g, we have

$$0 \leq t\langle e^{Tv_t}, g(v) - g(u)\rangle.$$

Dividing the above inequality by t and letting $t \longrightarrow 0$, we have

$$\langle Tu, g(v) - g(u)\rangle \quad \geq 0, \quad \forall v \in H : \quad g(v) \in K,$$

the required (2.1). □

Remark 2.2. *Inequality of type (2.10) is called the dual exponentially general variational inequality. From Lemma 2.2, it is clear that the solution sets of both problems (2.1) and (2.10) are equivalent. Lemma 2.2 plays an important part in the approximation of the exponentially general variational inequalities. Lemma 2.2 can be viewed as a natural generalization of a Minty's Lemma [15].*

We now studies those conditions under which the exponentially general variational inequality (2.1)) has a unique solution, which is the main motivation our next result.

Theorem 2.1. *Let T be a strongly exponentially general monotone with constant $\alpha > 0$ and exponentially general Lipschitz continuous operator with constant $\beta > 0$. Let g be a strongly general nonexpanding and general Lipschitz continuous operator with constants $\sigma > 0$ and $\delta > 0$, respectively. If there exists a constant $\rho > 0$ such that*

$$|\rho - \frac{\alpha}{\beta^2}| \quad < \quad \frac{\sqrt{\alpha^2\delta^2 - \delta(\delta-\sigma)}}{\beta^2\delta}, \quad \delta > \sqrt{\delta(\delta-\sigma)}, \tag{2.11}$$

then the exponentially general variational inequality (2.1) has a unique solution.

Proof. **(a). Uniqueness.** Let $u_1 \neq u_2 \in H$ be two solutions of (2.1). Then, we have

$$\langle e^{Tu_1}, g(v) - g(u_1)\rangle \quad \geq \quad 0, \quad \forall g(v) \in K, \tag{2.12}$$

$$\langle e^{Tu_2}, g(v) - g(u_2)\rangle \quad \geq \quad 0, \quad \forall g(v) \in K. \tag{2.13}$$

Taking $v = u_2$ in (2.12) and $v = u_1$ in (2.13), adding the resultants, we have

$$\langle e^{Tu_1} - e^{Tu_2}, g(u_1) - g(u_2)\rangle \leq 0.$$

Since T is strongly exponentially general monotone and g is strongly general nonexpanding, there exists a constant $\alpha > 0$, such that

$$\alpha\|u_1 - u_2\|^2 \leq \quad \alpha\|g(u_1) - g(u_2)\|^2 \leq \langle^{Tu_1} - e^{Tu_2}, g(u_1) - g(u_2)\rangle \leq 0,$$

which implies that $u_1 = u_2$, the uniqueness of the solution of (2.1).

(b). Existence. We now use the auxiliary principle technique to prove the existence of a solution of (2.1). For a given $u \in H : g(u) \in K$, we consider the problem of finding a unique $w \in H : g(w) \in K$ such that

$$\langle \rho e^{Tu} + g(w) - g(u), g(v) - g(w)) \rangle \geq 0, \quad \forall v \in H : g(v) \in K, \tag{2.14}$$

where $\rho > 0$ is a constant. The inequality of type (2.14) is called the auxiliary exponentially variational inequality associated with the problem (2.1). It is clear that the relation (2.14) defines a mapping $u \longrightarrow w$. It is enough to show that the mapping $u \longrightarrow w$ defined by the relation (2.14) has a fixed point belonging to H satisfying the exponentially general variational inequality (2.1). Let $w_1 \neq w_2$ be two solutions of (2.14) related to $u_1, u_2 \in H$ respectively. It is sufficient to show that for a well chosen $\rho > 0$,

$$\|w_1 - w_2\| \leq \theta \|u_1 - u_2\|,$$

with $0 < \theta < 1$, where θ is independent of u_1 and u_2. Taking $v = w_2$ (respectively w_1) in (2.14) related to u_1 (respectively u_2), adding the resultant, we have

$$\langle g(w_1) - g(w_2), g(w_1) - g(w_2) \rangle \leq \langle g(u_1) - g(u_2) - \rho(e^{Tu_1} - e^{Tu_2}), g(w_1) - g(w_2) \rangle,$$

from which we have

$$\begin{aligned} & \|g(w_1) - g(w_2)\|^2 \\ \leq \; & \|g(u_1) - g(u_2) - \rho(e^{Tu_1} - e^{Tu_2})\|^2 \\ \leq \; & \|g(u_1) - g(u_2)\|^2 - 2\rho\langle g(u_1) - g(u_2), Tu_1 - Tu_2 \rangle + \rho^2 \|e^{Tu_1} - e^{Tu_2}\|^2 \\ \leq \; & \|g(u_1) - g(u_2)\|^2 - 2\rho\alpha \|g(u_1) - g(u_2)\|^2 + \rho^2\beta^2 \|g(u_1) - g(u_2)\|^2 \\ \leq \; & (1 - 2\rho\alpha + \rho^2\beta^2)\|g(u_1) - g(u_2)\|^2, \end{aligned}$$

since T is both strongly exponentially general monotone and exponentially general Lipschitz continuous operator with constants $\alpha > 0$ and $\beta > 0$ respectively. Now using the general strongly nonexpandicity with constant $\sigma > 0$ and general Lipschitz continuity with constant $\delta > 0$ of g, we have

$$\|w_1 - w_2\| \leq \theta \|u_1 - u_2\|,$$

where

$$\theta = \sqrt{\frac{\delta}{\sigma}(1 - 2\rho\alpha + \rho^2\beta^2)}.$$

From (2.11), it follows that $\theta < 1$ showing that the mapping defined by (2.14) has a fixed point belonging to H, which is the solution of (2.1), the required result. $\square$

We note that, if the operator T is exponentially general symmetric and exponentially general positive, then the solution of the auxiliary general variational inequality (2.14) is equivalent to finding the minimum of the functional $I[w]$ on the closed convex set K in H, where

$$I[w] \;=\; \frac{1}{2}\langle g(w) - g(u), g(w) - g(u)\rangle + \rho\langle e^{Tu}, g(w) - g(u)\rangle, \quad \forall u \in H$$

which is a differentiable functional associated with the inequality (2.14). This auxiliary functional can be used to construct a gap (merit) function, whose stationary points solve the exponentially general variational inequality (2.1). In fact, one can easily show that the exponentially general variational inequality (2.1) is equivalent to the optimization problem. This approach is used to suggest and analyze some descent iterative methods for solving variational inequalities.

Definition 2.3. *A function $M : H \longrightarrow R \cup \{+\infty\}$ is called a merit function for the exponentially general variational inequalities (2.1), if and only if,*
(i). $M(u) \geq 0, \quad \forall u \in H.$
(ii). $M(\overline{u}) = 0$, *if and only if,* $\overline{u} \in H$ *solves (2.1).*

3 Main Results

In this section, we consider some merit functions associated with the exponentially general variational inequalities (2.1). Using these merit functions, we obtain some error bounds for the problem (2.1). For this purpose, we need the following result, which can be proved by invoking Lemma 2.1.

Lemma 3.1. *The function $u \in H : g(u) \in K$ is a solution of (2.1), if and only if, $u \in H$ satisfies the relation*

$$g(u) = P_K[g(u) - \rho e^{Tu}], \tag{3.1}$$

where $\rho > 0$ is a constant and g is onto K.

Lemma 3.1 implies that problems (2.1) and (3.1) are equivalent. This alternative formulation is very important from the numerical analysis point of view. This fixed-point formulation has been used to suggest and analyze several iterative schemes for solving exponentially general variational inequalities and related optimization problems. Here we use this equivalence to consider some merit functions.

We now consider the residue vector

$$R_\rho(u) \equiv R(u) := g(u) - P_K[g(u) - \rho e^{Tu}]. \tag{3.2}$$

It is clear from Lemma 3.1 that problem (2.1) has a solution $u \in H$, if and only if, $u \in H$ is a root of the equation

$$R(u) = 0. \tag{3.3}$$

It is known that the normal residue vector $R(u)$ defined by the relation (3.2) is merit function for the exponential general variational inequality (2.1). We use this merit function to derive the global error bounds for the solution of (2.1).

Theorem 3.1. *Let $\overline{u} \in H$ be a solution of (2.1). Let g be both strongly exponentially nonexpanding and exponentially Lipschitz continuous with constants $\sigma > 0$ and $\delta > 0$ respectively. If the operator T is both strongly exponentially general monotone and exponentially general Lipschitz continuous with constants $\alpha > 0$ and $\beta > 0$ respectively, then*

$$(1/k_1)\|R(u)\| \leq \|u - \overline{u}\| \leq k_2\|R(u)\|, \quad \forall u \in H, \tag{3.4}$$

where k_1, k_2 are generic constants.

Proof. Let $\overline{u} \in H : g(\overline{u}) \in K$ be solution of (2.1). Then

$$\langle e^{T\overline{u}}, g(v) - g(\overline{u})\rangle \quad \geq 0, \quad \forall v \in H : g(v) \in K. \tag{3.5}$$

Taking $v = P_K[g(u) - \rho e^{Tu}]$ in (3.5), we have

$$\langle e^{T\overline{u}}, P_K[g(u) - \rho e^{Tu}] - g(\overline{u})\rangle \geq 0. \tag{3.6}$$

Letting $u = P_K[g(u) - \rho e^{Tu}]$, $\quad z = g(u) - \rho e^{Tu}$ and $v = g(\overline{u})$ in (2.9), we have

$$\langle \rho e^{Tu} + P_K[g(u) - \rho e^{Tu}] - g(u), g(\overline{u}) - P_K[g(u) - \rho e^{Tu}]\rangle \geq 0. \tag{3.7}$$

Adding (3.6) and (3.7), we have

$$\langle e^{T\overline{u}} - e^{Tu} + (1/\rho)(g(u) - P_K[g(u) - \rho e^{Tu}]), P_K[g(u) - \rho e^{Tu}] - g(\overline{u})\rangle \geq 0. \tag{3.8}$$

Since T is a strongly exponentially general monotone and g is nonexpanding, there exists a constant $\alpha > 0$, such that

$$\begin{aligned}
& \alpha\|\overline{u} - u\|^2 \\
\leq \;& \|g(\overline{u}) - g(u)\|^2 \leq \langle e^{T\overline{u}} - e^{Tu}, g(\overline{u}) - g(u)\rangle \\
= \;& \langle e^{T\overline{u}} - e^{Tu}, g(\overline{u}) - P_K[u - \rho e^{Tu}]\rangle + \langle e^{T\overline{u}} - e^{Tu}, P_K[g(u) - \rho e^{Tu}] - g(u)\rangle \\
\leq \;& (1/\rho)\langle g(u) - P_K[g(u) - \rho e^{Tu}], P_K[g(u) - \rho e^{Tu}] - g(u) \\
& + g(u) - g(\overline{u})\rangle + \langle e^{T\overline{u}} - e^{Tu}, P_K[g(u) - \rho e^{Tu}] - g(u)\rangle \\
\leq \;& -(1/\rho)\|R(u)\|^2 + (1/\rho)\|R(u)\|\|g(u) - g(\overline{u})\| + e^{\|T\overline{u}} - e^{Tu}\|\|R(u)\| \\
\leq \;& (1/\rho)(1 + \beta)\|R(u)\|\|g(\overline{u}) - g(u)\| \leq (\delta/\rho)(1 + \beta)\|R(u)\|\|\overline{u} - u\|,
\end{aligned}$$

which implies that

$$\|\overline{u} - u\| \leq k_2\|R(u)\|, \tag{3.9}$$

the right-hand inequality in (3.4) with $k_2 = (\delta/\alpha\delta)(1 + \beta)$ and $\delta > 0$ is the Lipschitz constant of g.

Now from (3.2) and exponentially general Lipschitz continuity of T, we have

$$\begin{aligned}\|R(u)\| &= \|g(u) - P_K[g(u) - \rho e^{Tu}]\| \\ &= \|g(u) - g(\overline{u}) + P_K[g(\overline{u}) - \rho e^{T\overline{u}}] - P_K[g(u) - \rho e^{Tu}]\| \\ &\leq \|g(u) - g(\overline{u})\| + \|g(u) - g(\overline{u}) + \rho(e^{Tu} - e^{T\overline{u}})\| \\ &\leq \{2 + \rho\beta\}\|g(u) - g(\overline{u})\| = k_1\|u - \overline{u}\|,\end{aligned}$$

from which we have

$$(1/(k_1)\|R(u)\| \leq \|u - \overline{u}\|, \tag{3.10}$$

the left-most inequality in (3.4) with $k_1 = (2 + \rho\beta)\delta$, where $\delta > 0$ is the Lipschitz constant of g. Combining (3.9) and (3.10), we obtain the required (3.4). □

Letting $u = 0$ in (3.4), we have

$$(1/k_1)\|R(0)\| \leq \|\overline{u}\| \leq k_2\|R(0)\|. \tag{3.11}$$

Combining (3.4) and (3.11), we obtain a relative error bounds for any point $u \in H$.

Theorem 3.2. *Assume that all the assumptions of Theorem 3.1 hold. If* $0 \neq \overline{u} \in H$ *is the unique solution of (2.1), then*

$$c_1\|R(u)\|/\|R(0)\| \leq \|u - \overline{u}\|/\|\overline{u}\| \leq_2 \|R(u)\|/\|R(0)\|.$$

Note that the normal residue vector (merit function) $R(u)$ is nondifferentiable. To overcome the nondifferentiability, which is a serious drawback of the normal residue merit function, we consider another merit function associated with problem (2.1), which can be viewed as a regularized merit function. We now consider

$$M_\rho(u) = \max_{v\in H: g(v)\in K}\{\langle e^{Tu}, g(u) - g(v)\rangle - (1/2\rho)\|g(u) - g(v)\|^2\}, u \in H : g(u) \in K. \tag{3.12}$$

The function $M_\infty(u)$ is commonly called the gap (merit) function associated with the exponentially general variational inequality (2.1). The function $M_\infty(u)$ has the serious drawback that it is in general nondifferentiable even, if T and g

are differentiable and may not be finite-valued. On the other hand, the function $M_\rho(u)$ which is called the regularized merit function, is finite-valued everywhere and is differentiable whenever T and g are differentiable.

We note that if $g = I$, an identity operator, then the merit function (3.12) reduces to the well known regularized merit function for the classical exponentially variational inequalities (2.6), that is,

$$M_\rho(u) := \max_{v\in K}\{\langle e^{Tu}, u - v\rangle - (1/2\rho)\|u - v\|^2\}, \quad u \in K. \tag{3.13}$$

Thus it is clear that the function $M_\rho(u)$ defined by (3.12) can be viewed as a natural generalization of the regularized merit function associated with the exponentially variational inequalities.

We note that the function $M_\rho(u)$ can be written as

$$M_\rho(u) = \langle e^{Tu}, g(u) - P_K[g(u) - \rho e^{Tu}]\rangle - \frac{1}{\rho}\|g(u) - P_K[g(u) - \rho e^{Tu}]\|^2, \quad \forall u \in H : g(u) \in K. \tag{3.14}$$

We now show that the function $M_\rho(u)$ defined by (3.13) is a merit function and this is the main motivation of our next result.

Theorem 3.3. *$\forall u \in H$ and $\rho < 1$, we have*

$$M_\rho(u) \geq \frac{1}{2\rho}\|R(u)\|^2. \tag{3.15}$$

Clearly $M_\rho(u) \geq 0, \forall u \in H$. In particular, we have $M_\rho(u) = 0$, iff, $u \in H$ is a solution of (2.1).

Proof. Setting $v = g(u)$, $u = P_K[g(u) - \rho e^{Tu}]$ and $z = g(u) - \rho e^{Tu}$ in (2.9), we have

$$\langle \rho e^{Tu} - (g(u) - P_K[g(u) - \rho e^{Tu}]), g(u) - P_K[g(u) - \rho e^{Tu}]\rangle \geq 0,$$

which implies that

$$\langle e^{Tu}, R(u)\rangle \geq \frac{1}{\rho}\|R(u)\|^2. \tag{3.16}$$

Combining (3.14) and (3.16), we have

$$M_\rho(u) = \langle e^{Tu}, R(u)\rangle - \frac{1}{2\rho}\|R(u)\|^2 \geq \frac{1}{\rho}\|R(u)\|^2 - \frac{1}{2\rho}\|R(u)\|^2 = \frac{1}{2\rho}\|R(u)\|^2,$$

the required result (3.15). Clearly we have $M_\rho(u) \geq 0, \quad \forall u \in H : g(u) \in K$.

Now if $M_\rho(u) = 0$, then clearly $R(u) = 0$. Hence by Lemma 3.1, we see that $u \in H$ is a solution of (2.1). Conversely, if $u \in H$ is a solution of (2.1), then $g(u) = P_K[g(u) - \rho e^{Tu}]$ by Lemma 3.1. Consequently, from (3.12), we see that $M_\rho(u) = 0$, the required result. □

From Theorem 3.3, we see that the function $M_\rho(u)$ defined by (3.12) is a merit function for the exponentially general variational inequalities (2.1). It is clear that the regularized merit function is differentiable whenever T and g are differentiable. We now derive the error bounds without using the exponentially Lipschitz continuity of the T.

Theorem 3.4. *Let $\overline{u} \in H$ be a solution of (2.1). Let T be a strongly exponentially monotone with a constant $\alpha > 0$. If g is strongly nonexpanding with a constant $\sigma > 0$, then*

$$M_\rho(u) \geq \frac{2\rho}{2\rho\alpha - 1}\sigma\|u - \overline{u}\|^2, \quad \forall u \in H. \tag{3.17}$$

Proof. From (3.12) and the strongly exponentially monotonicity of T, we have

$$\begin{aligned} M_\rho(u) &\geq \langle e^{Tu}, g(u) - g(\overline{u})\rangle - \frac{1}{2\rho}\|g(u) - g(\overline{u})\|^2 \\ &\geq \langle e^{T\overline{u}}, g(u) - g(\overline{u})\rangle + \alpha\|g(u) - g(\overline{u})\|^2 - \frac{1}{2\rho}\|g(u) - g(\overline{u})\|^2. \end{aligned} \tag{3.18}$$

Taking $v = u$ in (3.5), we have

$$\langle e^{T\overline{u}}, g(u) - g(\overline{u})\rangle \geq 0. \tag{3.19}$$

From (3.18),(3.19) and using the strongly nonexpandicity of g with constant $\sigma > 0$, we have

$$M_\rho(u) \geq \alpha\|g(u) - g(\overline{u})\|^2 - \frac{1}{2\rho}\|g(u) - g(\overline{u})\|^2 = (\alpha - \frac{1}{2\rho}\sigma\|u - \overline{u}\|^2,$$

from which the result (3.17) follows. □

We consider another merit function associated with exponentially general variational inequalities (2.1), which can be viewed as a difference of two regularized merit functions. Here we define the D-merit function by a formal difference of the regularized merit function defined by (3.12). To this end, we consider the following function

$$\begin{aligned} D_{\rho,\mu}(u) &= \max_{v\in H}\{\langle e^{Tu}, g(u) - g(v)\rangle + (1/2\mu)\|g(u) - g(v)\|^2 \\ &\quad -(1/2\rho)\|g(u) - g(v)\|^2\}, \quad \forall v \in H, \end{aligned} \tag{3.20}$$

which is called the D-merit function associated with the exponentially general variational inequalities (2.1). The differentiability of $D_{\rho,\mu}(u)$ immediately follows from that of T and g.

The D-merit function defined by (3.20) can be written as

$$\begin{aligned} & D_{\rho,\mu}(u) \\ = \;& \langle e^{Tu}, P_K[g(u) - \mu e^{Tu}] - P_K[g(u) - \rho e^{Tu}]\rangle + (1/2\mu)\|g(u) - P_K[g(u) - \mu e^{Tu}]\|^2 \\ & -(1/2\rho)\|g(u) - P_K[g(u) - \rho e^{Tu}]\|^2 \\ = \;& \langle e^{Tu}, R_\rho(u) - R_\mu(u)\rangle + (1/2\mu)\|R_\mu(u)\|^2 - (1/2\rho)\|R_\rho(u)\|^2, \quad u \in H. \end{aligned} \tag{3.21}$$

It is clear that the $D_{\rho,\mu}(u)$ is everywhere finite. We now show that the function $D_{\rho,\mu}(u)$ defined by (3.20) is indeed a merit function for the exponentially general variational inequalities (2.1) and this is the main motivation of our next result.

Theorem 3.5. $\forall u \in H, \rho > \mu > 0,$ *we have*

$$(\rho - \mu)\|R_\rho(u)\|^2 \geq 2\rho\mu D_{\rho,\mu}(u) \geq (\rho - \mu)\|R_\mu(u)\|^2. \tag{3.22}$$

In particular, $D_{\rho,\mu}(u) = 0$, *if and only if,* $u \in H$ *solves problem (2.1).*

Proof. Taking $v = P_K[g(u) - \mu e^{Tu}], u = P_K[g(u) - \rho e^{Tu}]$ and $z = g(u) - \rho e^{Tu}$ in (2.9), we have

$$\langle P_K[g(u) - \rho e^{Tu}] - g(u) + \rho e^{Tu}, P_K[g(u) - \mu e^{Tu}] - P_K[g(u) - \rho e^{Tu}]\rangle \geq 0,$$

which implies that

$$\langle e^{Tu}, R_\rho(u) - R_\mu(u)\rangle \geq (1/\rho)\langle R_\rho(u), R_\rho(u) - R_\mu(u)\rangle. \tag{3.23}$$

From (3.21) and (3.23), we have

$$\begin{aligned} D_{\rho,\mu}(u) &\geq (1/\rho)\langle R_\rho(u), R_\rho(u) - R_\mu(u)\rangle + (1/2\mu)\|R_\mu(u)\|^2 - (1/2\rho)\|R_\rho(u)\|^2 \\ &= (1/2)(1/\mu - 1/\rho)\|R_\mu(u)\|^2 + (1/\rho)\langle R_\rho(u), R_\rho(u) - R_\mu(u)\rangle \\ &\quad +(1/2\rho)\|R_\rho(u) - R_\mu(u)\|^2 - (1/\rho)\langle R_\mu(u), R_\rho(u) - R_\mu(u)\rangle \\ &= (1/2)(1/\mu - 1/\rho)\|R_\mu(u)\|^2 + (1/2\rho)\|R_\rho(u) - R_\mu(u)\|^2 \\ &\geq (1/2)(1/\mu - 1/\rho)\|R_\mu(u)\|^2, \end{aligned} \tag{3.24}$$

which implies the right-most inequality in (3.22).

In a similar way, by taking $u = P_K[g(u) - \mu e^{Tu}], z = g(u) - \mu e^{Tu}$ and $v = P_K[g(u) - \mu e^{Tu}]$ in (2.9), we have

$$\langle P_K[g(u) - \mu e^{Tu}] - g(u) + \mu e^{Tu}, P_K[g(u) - \rho e^{Tu}] - P_K[g(u) - \mu e^{Tu}]\rangle \geq 0,$$

which implies that

$$\langle e^{Tu}, R_\rho(u) - R_\mu(u)\rangle \geq (1/\mu)\langle R_\mu(u), R_\mu(u) - R_\rho(u)\rangle. \tag{3.25}$$

Consequently, from (3.21) and (3.25), we obtain

$$\begin{aligned} D_{\rho,\mu}(u) &\leq (1/\mu)\langle R_\mu(u), R_\rho(u) - R_\mu(u)\rangle + (1/2\mu)\|R_\mu(u)\|^2 - (1/2\rho)\|R_\rho(u)\|^2 \\ &= (1/2)(1/\mu - 1/\rho)\|R_\rho(u)\|^2 - (1/2\mu)\|R_\rho(u) - R_\mu(u)\|^2 \\ &\leq (1/2)(1/\mu - 1/\rho)\|R_\rho(u)\|^2, \end{aligned} \tag{3.26}$$

which implies the left-most inequality in (3.22). Combining (3.24) and (3.26), we obtain (3.22), the required result. □

Using essentially the technique of Theorem 3.4, we can obtain the following result.

Theorem 3.6. *Let $\overline{u} \in H : g(u) \in K$ be a solution of (2.1). If the operator T is strongly exponentially general monotone with constant $\alpha > 0$ and g is strongly nonexpanding with constant $\sigma > 0$, then*

$$\|u - \overline{u}\|^2 \leq (2\mu\rho)((2\alpha\mu + 1)\rho - \mu)\sigma D_{\rho,\mu}, \quad \forall u \in H. \tag{3.27}$$

Proof. Let $\overline{u} \in H$ be a solution of (2.1). Then, taking $v = u$ in (3.5), we have

$$\langle e^{T\overline{u}}, g(u) - g(\overline{u})\rangle \geq 0. \tag{3.28}$$

Also from (3.26), (3.28) and strongly exponentially monotonicity of T, we have

$$\begin{aligned} D_{\rho,\mu}(u) &\geq \langle e^{Tu}, g(u) - g(\overline{u})\rangle + (1/2\mu)\|g(u) - g(\overline{u})\|^2 - (1/2\rho)\|g(u) - g(\overline{u})\|^2 \\ &\geq \langle e^{T\overline{u}}, g(u) - g(\overline{u})\rangle + \alpha\|g(u) - g(\overline{u})\|^2 \\ &\quad +(1/2\mu)\|g(u) - g(\overline{u})\|^2 - (1/2\rho)\|g(u) - g(\overline{u})\|^2 \\ &\geq \sigma(\alpha + (1/2\mu) - (1/2\rho))\|u - \overline{u}\|^2, \end{aligned}$$

the required result (3.27) follows, where $\sigma > 0$ is strongly nonexpandicity constant of g. □

We now consider the dual merit functions for the exponentially general variational inequalities (2.1) and obtain some error bounds for the solution of the exponentially general variational inequalities (2.1). For simplicity and without loss of generality, we define

$$\varphi(u, v) := \langle e^{Tu}, g(u) - g(v)\rangle, \quad \forall u, v \in H. \tag{3.29}$$

Using this notation, regularized merit function M_ρ can be written as

$$\begin{aligned} M_\rho(u) &= \max_{v\in H:g(v)\in K}\{\langle e^{Tu}, g(u)-g(v)\rangle - (1/2\rho)\|g(u)-g(v)\|^2\} \\ &= \max_{v\in H:g(v)\in K}\{\varphi(u,v) - (1/2\rho)\|g(u)-g(v)\|^2\}. \end{aligned} \tag{3.30}$$

Clearly

$$M(u) = \max_{v\in H:g(v)\in K}\{\langle e^{Tu}, g(u)-g(v)\rangle\}, \tag{3.31}$$

which is also a merit function for the exponentially general variational inequalities (2.1). It is clear that the merit function $M(u)$ is not differentiable. The merit function $M(u)$ can be viewed as an extension of the merit function considered by for the exponentially variational inequalities. We now consider the dual regularized merit function associated with the exponentially general variational inequality, which is defined as

$$F_\mu(u) := \max_{v\in H:g(v)\in K}\{-\varphi(v,u) + (1/2\mu)\|g(u)-g(v)\|^2\}, \quad \forall u \in H : g(u) \in K, \tag{3.32}$$

where $\mu > 0$ is a constant. Clearly $F_\mu(u) \geq 0, \quad \forall u \in H$. Note that for $g = I$, the identity operator, the dual regularized merit function $F_\mu(u)$ reduces to

$$F(u) = \max_{v\in K}\{-\varphi(v,u)\} = \max_{v\in K}\{\langle e^{Tv}, u-v\rangle\}, \tag{3.33}$$

which is considered and studied for the classical exponentially variational inequalities. It is obvious that the dual merit function $F(u)$ is a convex function and is differentiable.

In particular, If T is an exponentially pseudomonotone, then the merit function $F(u)$ defined by (3.33) is nonnegative on K and vanishes at any solution of the exponentially general variational inequality (2.1). Following the technique of Noor [22, 23], one can easily prove the following result.

Theorem 3.7. *Let T be exponentially general pseudomonotone. Then $u \in H$: $g(u) \in K$ is a solution of problem (2.1), if and only if, $F_\mu(u) = 0, \forall u \in H : g(u) \in K$.*

We now show that the functions $F_\mu(u)$ defined by (3.33) is indeed a merit function for the exponentially general variational inequality (2.1) and this is the main motivation of our next result.

Theorem 3.8. *Let the operator T be strongly exponentially general monotone with constant $\alpha > 0$. Then $u \in H : g(u) \in K$ is a solution of (2.1), if and only if, $F_\mu(u) = 0$.*

Proof. Since T is strongly exponentially general monotone with constant $\alpha > 0$, we have

$$\begin{aligned}\varphi(u,v) &= \langle e^{Tu}, g(u)-g(v)\rangle \geq \langle e^{Tv}, g(v)-g(u)\rangle + \alpha\|g(v)-g(u)\|^2 \\ &= -\varphi(v,u) + \alpha\|g(v)-g(u)\|^2 \geq -\varphi(v,u) + (1/2\mu)\|g(v)-g(u)\|^2 \\ &\geq -\varphi(v,u), \quad \forall u,v \in H : g(u), g(v) \in K,\end{aligned}$$

which implies that

$$M(u) \geq F_\mu(u) \geq F(u), \quad \forall u \in H : g(u) \in K. \tag{3.34}$$

Let $\overline{u} \in H : g(\overline{u}) \in K$ be a solution of (2.1). Then, from Theorem 3.3 and Theorem 3.7, we have $M(\overline{u}) = 0$ and $F(\overline{u}) = 0$. Thus it follows from (3.34) that $F_\mu(\overline{u}) = 0$.

Conversely, let $F_\mu(u) = 0$. Clearly $F_\mu(u) \geq 0$ on K and it follows from (3.34) that $F(u) = 0$. Thus $u \in H : g(u) \in K$ is a solution of (2.1 by Lemma 3.1. □

We now obtain the upper error bound for the dual merit function $F_\mu(u)$.

Theorem 3.9. *Let T be strongly exponentially general monotone with a constant $\alpha > 0$ and let g be strongly nonexpanding with constant $\sigma > 0$. Then*

$$F_\mu(u) \geq \alpha\sigma\|u-\overline{u}\|^2, \quad \forall u \in H : g(u) \in K,$$

where $\overline{u} \in H$ is a solution of (2.1).

Proof. Let $\overline{u} \in H$ be a solution of (2.1). Then by Theorem 3.7, it follows that $F_\mu(\overline{u}) = 0$. Let $u \in H$ be an arbitrary. Then

$$\begin{aligned} F_\mu(u) &= \max_{v \in H: g(v) \in K} \{\langle e^{Tv}, g(u) - g(v)\rangle + (1/2\mu)\|g(v) - g(u)\|^2\} \\ &\geq \langle e^{T\overline{u}}, g(v) - g(\overline{u})\rangle + (1/2\mu)\|g(v) - g(u)\|^2 + \alpha\|g(u) - g(\overline{u})\|^2 \\ &\geq \alpha\|g(v) - g(\overline{u})\|^2 \geq \alpha\sigma\|u - \overline{u}\|^2, \quad \forall u, \overline{u} \in H. \end{aligned}$$

□

Using essentially the previous techniques and ideas, we can construct and analyze the dual difference for the exponentially general variational inequalities (2.1).

Conclusion: In this paper, we have shown that odd-order and non symmetric boundary value problems can be studied in the unified framework of exponentially general variational inequalities. Several special cases are also discussed. We have proved that the exponentially general variational inclusions are equivalent to fixed point problems. This alternative formulation is used to discuss the existence of a solution of the exponentially general variational inclusions. as well as to consider several new merit functions for exponentially general quasi variational inequalities. These gap functions are used to derive the error bounds for the approximate solutions. We have also shown that the the exponentially quasi variational inequalities are equivalent to the exponentially general variational inclusions for a suitable choice of the convex-valued set. The technique and ideas of this paper may be extended for exponentially random variational inequalities and related optimization problems. We expect that the ideas and techniques of this paper will motivate and inspire the interested readers to explore the applications of gap functions in various other related fields.

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