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$(LCS)_n$ −Manifolds Admitting Almost η−Ricci Solitons on Some Special Curvature Tensors

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Abstract

In this paper, we consider $(LCS)_n$ manifold admitting almost η−Ricci solitons by means of curvature tensors. Ricci pseudosymmetry concepts of $(LCS)_n$ manifold admitting η−Ricci soliton have introduced according to the choice of some special curvature tensors such as pseudo-projective, W_1, W_1^* and W_2. Then, again according to the choice of the curvature tensor, necessary conditions are searched for $(LCS)_n$ manifold admitting η−Ricci soliton to be Ricci semisymmetric. Then some characterizations are obtained and some classifications have made.

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1 Introduction

In [1], the notion of Lorentzian concircular structure $(LCS)_n-$manifolds has been initiated by A.A. Shaikh by giving an example that generalizes the concept of LP-Sasakian manifolds. A.A. Shaikh and H. Ahmad discussed certain transformations on an $(LCS)_n$ manifold and showed that the $(LCS)_n$ manifold remained invariant under $D-$homothetic deformation [2]. Later, it was shown by C.A. Mantica and L.G. Molinari that Lorentz concircular manifolds coincide with generalized Robertson-Walker spacetimes [3]. Again, weakly symmetric $(LCS)_n$ manifolds in [4], Φ-recurrent $(LCS)_n$ manifolds in [5], generalized $\Phi-$recurrent $(LCS)_n$ manifolds in [6], invariant submanifolds of $(LCS)_n$ manifolds in [7], Ricci solitons for $(LCS)_n$ manifolds in [8] and $\eta-$Ricci solitons for $(LCS)_n$ maifolds in [9] have also been studied by various mathematicians. Again, M. Atçeken et al. discussed pseudoparallel invariant submanifolds of $(LCS)_n$ manifolds in [10] and S.K. Hui et al. discussed Ricci solitons on Ricci pseudosymmetric $(LCS)_n$ manifolds [11].

An $n-$dimensional Lorentzian manifold M is a smoothly connected paracompact Hausdorff manifold with the Lorentzian metric g, that is, M contains a smooth symmetric tensor field of type $(0,2)$ such that for each point $p \in M$, $g_p : T_{M(p)} \times T_M(p) \to \mathbb{R}$ is a non-degenerate inner product of signature $(-,+,...,+)$, where $T_M(p)$ denotes the tangent vector space of M at p and $\mathbb{R}$ is the real number space. A non-zero vector $X_p \in T_M(p)$ is said to be timelike (resp., non-spacelike, null, spacelike) if it satisfies $g_p(X_p, X_p) < 0$ (resp., $\leq 0, = 0, > 0)$. Its causal character is the category into which a given vector fields that is, the category into which a given vector falls is called its causal character. Let M^n be a Lorentzian manifold that admits the unit timelike concircular vector field ξ, often known as the manifold's characteristic vector field. Then we have

$$g(\xi, \xi) = -1. \quad (1)$$

Since ξ is a unit concircular vector field, there exists a non-zero η such that

$$g(\vartheta_1, \xi) = \eta(\vartheta_1) \quad (2)$$

the equation of the following form holds

$$(\nabla_{\vartheta_1}\eta)\,\vartheta_2 = \alpha\left[g\left(\vartheta_1,\vartheta_2\right)+\eta\left(\vartheta_1\right)\eta\left(\vartheta_2\right)\right], \tag{3}$$

for all $\vartheta_1, \vartheta_2 \in \Gamma(TM)$, where $\Gamma(TM)$ and ∇ denote the set differentiable vector fields set and the Levi-Civita connection on M, respectively. α is a non-zero function that satisfies

$$\nabla_{\vartheta_1}\alpha = \vartheta_1(\alpha) = d\alpha(\vartheta_1) = \rho\eta(\vartheta_1), \tag{4}$$

ρ being a certain scalar function. If setting

$$\nabla_{\vartheta_1}\xi = \alpha\phi\vartheta_1, \tag{5}$$

then from (3) and (5) , it can be seen

$$\phi\vartheta_1 = \vartheta_1 + \eta(\vartheta_1)\,\xi. \tag{6}$$

It follows that ϕ is a $(1,1)-$type symmetric tensor. Hence we

$$\eta(\xi) = -1, \phi\xi = 0, \eta(\phi\vartheta_1) = 0, \tag{7}$$

and

$$g(\phi\vartheta_1,\phi\vartheta_2) = g(\vartheta_1,\vartheta_2)+\eta(\vartheta_1)\,\eta(\vartheta_2)\,, \phi^2\vartheta_1 = \vartheta_1+\eta(\vartheta_1)\,\xi. \tag{8}$$

As a result, the Lorentzian manifold M, along with the unit timelike concircular vector field ξ and its associated $1-$form-η and $(1,1)-$tensor field ϕ, is said to be an almost paracontact Lorentzian manifold with a concircular structure, or simply $(LCS)_n$ manifold [11]. In a $(LCS)_n$ manifolds, we have

$$(\nabla_{\vartheta_1}\phi)\,\vartheta_2 = \alpha\left[g(\vartheta_1,\vartheta_2)\,\xi+\eta(\vartheta_2)\,\vartheta_1+2\eta(\vartheta_1)\,\eta(\vartheta_2)\,\xi\right], \tag{9}$$

$$R(\vartheta_1,\vartheta_2)\,\xi = \left(\alpha^2-\rho\right)\left[\eta(\vartheta_2)\,\vartheta_1-\eta(\vartheta_1)\,\vartheta_2\right], \tag{10}$$

$$R(\xi,\vartheta_1)\,\vartheta_2 = \left(\alpha^2-\rho\right)\left[g(\vartheta_1,\vartheta_2)\,\xi-\eta(\vartheta_2)\,\vartheta_1\right], \tag{11}$$

$$\eta\left(R(\vartheta_1,\vartheta_2)\,\vartheta_3\right) = \left(\alpha^2-\rho\right)g\left(\eta(\vartheta_1)\,\vartheta_2-\eta(\vartheta_2)\,\vartheta_1,\vartheta_3\right) \tag{12}$$

$$S(\vartheta_1,\xi) = \left(\alpha^2-\rho\right)(n-1)\,\eta(\vartheta_1)\,, \tag{13}$$

$$\vartheta_1(\rho) = d\rho(\vartheta_1) = \beta\eta(\vartheta_1), \tag{14}$$

where R and S denote the Riemannian curvature and Ricci tensors of M.

Let us now give the $(LCS)_n$ manifold example given by A. Shaikh in [12].

Example 1. *We consider* $3-$*dimensional manifold* $M = \left\{(x,y,z) \in \mathbb{R}^3\right\}$, *where* (x,y,z) *is the standard coordinates in* $\mathbb{R}^3$. *Let* $\{e_1, e_2, e_3\}$ *be linearly independent global frame on* M *given by*

$$e_1 = e^{-z}\left(\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right),$$

$$e_2 = e^{-z}\frac{\partial}{\partial y},\ e_3 = e^{-2z}\frac{\partial}{\partial z}.$$

Let g *be the Lorentzian metric defined by*

$$g(e_1, e_3) = g(e_2, e_3) = g(e_1, e_2) = 0,$$

$$g(e_1, e_1) = g(e_2, e_2) = 1, g(e_3, e_3) = -1.$$

Let η *be the* $1-$*form defined by*

$$\eta(\vartheta_4) = g(\vartheta_4, e_3)$$

for any $\vartheta_4 \in \chi(M)$. *Let* ϕ *be the* $(1,1)-$*tensor field defined by*

$$\phi e_1 = e_1, \phi e_2 = e_2, \phi e_3 = 0.$$

Then using the linearity of ϕ *and* g, *we have*

$$\eta(e_3) = -1, \phi^2\vartheta_4 = \vartheta_4 + \eta(\vartheta_4)e_3,$$

and

$$g(\phi\vartheta_4, \phi W) = g(\vartheta_4, W) + \eta(\vartheta_4)\eta(W),$$

for any $\vartheta_4, W \in \chi(M)$. *Thus for* $e_3 = \xi, (\phi, \xi, \eta, g)$ *defines a Lorentzian paracontact structure on* M.

Let ∇ *be the Levi-Civita connection concerning the Lorentzian metric* g *and* R *be the curvature tensor of* g. *Then we have*

$$[e_1, e_2] = -e^{-z}e_2, [e_1, e_3] = e^{-2z}e_1, [e_2, e_3] = e^{-2z}e_2.$$

Taking $e_3 = \xi$ *and using Kozsul formula for the Lorentzian metric* g, *we can easily calculate*

$$\nabla_{e_1}e_3 = e^{-2z}e_1, \nabla_{e_1}e_2 = 0, \nabla_{e_1}e_1 = e^{-2z}e_3,$$

$$\nabla_{e_2}e_3 = e^{-2z}e_2, \nabla_{e_2}e_2 = e^{-2z}e_3 - e^{-z}e_1, \nabla_{e_3}e_3 = 0,$$

$$\nabla_{e_2}e_1 = e^{-2z}e_2, \nabla_{e_3}e_2 = 0, \nabla_{e_3}e_1 = 0.$$

From the above t can be easily seen that (ϕ, ξ, η, g) *is an* $(LCS)_3$ −*structure on* M. *Consequently* $M^3(\phi, \xi, \eta, g)$ *is an* $(LCS)_3$ *manifold with* $\alpha = e^{-2z} \neq 0$ *such that*

$$\vartheta_1(\alpha) = \rho\eta(\vartheta_1),$$

where $\rho = 2e^{-4z}$. *Using the above relations, we can easily calculate the non-vanishing components of the curvature tensor as follows:*

$$R(e_2, e_3)e_3 = e^{-4z}e_2, R(e_1, e_3)e_3 = e^{-4z}e_1,$$

$$R(e_1, e_2)e_2 = e^{-4z}e_1 - e^{-2z}e_1, R(e_2, e_3)e_2 = e^{-4z}e_3,$$

$$R(e_1, e_3)e_1 = e^{-4z}e_3, R(e_1, e_2)e_1 = -e^{-4z}e_2 + e^{-2z}e_2,$$

and the components which can easily calculate the non-vanishing components of the Ricci tensor S *as follows:*

$$S(e_1, e_1) = 2e^{-4z} - e^{-2z}, S(e_2, e_2) = 2e^{-4z} - e^{-2z}, S(e_3, e_3) = 2e^{-4z}.$$

On the other hand, Ricci solitons and $\eta-$Ricci solitons are natural generalizations of Einstein metrics. In [13], R.S. Hamilton introduced on a Riemannian manifold (M, g) an evolution equation for metrics, called the Ricci flow

$$\frac{\partial}{\partial t}g(t) = -2S(g(t)),$$

which is used to deform a metric by smoothing out its singularities. Hence Ricci solitons may be regarded as generalized fixed points of the Ricci flow modeling the formation of singularities. Precisely, a Ricci soliton on a Riemannian manifold (M, g) is defined as a triple (g, ξ, λ) on M satisfying

$$L_\xi g + 2S + 2\lambda g = 0, \tag{15}$$

where L_ξ is the Lie derivative operator along the vector field ξ and λ is a real constant. We note that if ξ is a Killing vector field, then the Ricci soliton reduces to an Einstein metric (g, λ). Furthermore, in [14], generalization is the notion of $\eta-$Ricci soliton defined by J.T. Cho and M. Kimura as a quadruple (g, ξ, λ, μ) satisfying

$$L_\xi g + 2S + 2\lambda g + 2\mu\eta \oplus \eta = 0, \tag{16}$$

where λ and μ are real constants and η is the dual of ξ and S denotes the Ricci tensor of M. Furthermore if λ and μ are smooth functions on M, then it called almost $\eta-$Ricci soliton on M [14].

An almost $\eta-$Ricci soliton (g, ξ, λ, μ) is called steady if $\lambda = 0$, if shrinking $\lambda < 0$, and if expanding $\lambda > 0$.

A more general notion is that of $\eta-$Ricci soliton introduced in [14] and it was applied on Hopf hypersurfaces in complex space forms in [15].

Many mathematicians have been interested in the geometry of Ricci solitons over the past 20 years. It has gained more significance especially since Perelman used Ricci solitons to answer the long-standing Poincare conjecture. Furthermore, in [16], the Ricci solitons in contact geometry were studied by R. Sharma.

The notion of Ricci pseudosymmetric manifold was introduced by Deszcz in [17]. Ricci pseudosymmetric $(LCS)_n-$manifolds admitting Ricci solitons were studied in [11]. They searched the conditions concircular Ricci pseudosymmetric, projective Ricci pseudosymmetric, conharmonic Ricci pseudosymmetric, and W_3-Ricci pseudosymmetric $(LCS)_n-$manifolds whose metric tensor admit Ricci soliton from a point of view.

In the present paper, we tried to find conditions on almost $\eta-$Ricci pseudosymmetric, concircular almost $\eta-$Ricci pseudosymmetric, conharmonic almost $\eta-$Ricci pseudosymmetric and almost $M-$projectively almost $\eta-$Ricci pseudosymmetric $(LCS)_n-$manifolds whose metric tensors admit Ricci soliton.

2 Almost $\eta-$Ricci Solitons on Ricci Pseudosymmetric $(LCS)_n-$Manifolds

Now let (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on $(LCS)_n-$manifold. Then we have

$$\begin{aligned}(L_\xi g)(\vartheta_1,\vartheta_2) &= L_\xi g(\vartheta_1,\vartheta_2) - g(L_\xi\vartheta_1,\vartheta_2) - g(\vartheta_1,L_\xi\vartheta_2)\\ &= \xi g(\vartheta_1,\vartheta_2) - g([\xi,\vartheta_1],\vartheta_2) - g(\vartheta_1,[\xi,\vartheta_2])\\ &= g(\nabla_\xi\vartheta_1,\vartheta_2) - g(\vartheta_1,\nabla_\xi\vartheta_2) - g(\nabla_\xi\vartheta_1,\vartheta_2)\\ &\quad + g(\nabla_{\vartheta_1}\xi,\vartheta_2) - g(\nabla_\xi\vartheta_2,\vartheta_1) + g(\vartheta_1,\nabla_{\vartheta_2}\xi),\end{aligned}$$

for all $\vartheta_1,\vartheta_2\in\Gamma(TM)$. By using ϕ is symmetric and in view of (5), we have

$$(L_\xi g)(\vartheta_1,\vartheta_2) = 2\alpha g(\phi\vartheta_1,\vartheta_2). \tag{17}$$

Thus, in an $(LCS)_n-$manifolds, from (16) and (17), we have

$$\alpha g(\phi\vartheta_1,\vartheta_2) + S(\vartheta_1,\vartheta_2) + \lambda g(\vartheta_1,\vartheta_2) + \mu\eta(\vartheta_1)\eta(\vartheta_2) = 0. \tag{18}$$

For $\vartheta_2=\xi$, this implies that

$$S(\xi,\vartheta_1) = (\mu-\lambda)\eta(\vartheta_1). \tag{19}$$

Taking into account (13), we conclude that

$$\mu-\lambda = (n-1)\left(\alpha^2-\rho\right). \tag{20}$$

For an $n-$dimensional semi-Riemann manifold M, the W_1-curvature tensor is defined as

$$W_1(\vartheta_1,\vartheta_2)\vartheta_3 = R(\vartheta_1,\vartheta_2)\vartheta_3 + \frac{1}{n-1}\left[S(\vartheta_2,\vartheta_3)\vartheta_1 - S(\vartheta_1,\vartheta_3)\vartheta_2\right]. \tag{21}$$

For an $n-$dimensional $(LCS)_n$ manifold, if we choose $\vartheta_3 = \xi$ in (21), we can write

$$W_1(\vartheta_1, \vartheta_2)\xi = 2\left(\alpha^2 - \rho\right)[\eta(\vartheta_2)\vartheta_1 - \eta(\vartheta_1)\vartheta_2], \tag{22}$$

and similarly if we take the inner product of both sides of (21) by ξ, we get

$$\eta(W_1(\vartheta_1, \vartheta_2)\vartheta_3) = 2\left(\alpha^2 - \rho\right) g(\eta(\vartheta_1)\vartheta_2 - \eta(\vartheta_2)\vartheta_1, \vartheta_3). \tag{23}$$

Theorem 1. *Let M be a $n-$dimensional $(LCS)_n-$manifold. If M is W_1-flat, then M is an $\eta-$Einstein manifold.*

Proof. Let us assume that M is W_1-flat. So, we can write

$$W_1(\vartheta_1, \vartheta_2)\vartheta_3 = 0$$

for all $\vartheta_1, \vartheta_2, \vartheta_3 \in \chi(M)$, that is

$$R(\vartheta_1, \vartheta_2)\vartheta_3 = \frac{1}{n-1}[S(\vartheta_1, \vartheta_3)\vartheta_2 - S(\vartheta_2, \vartheta_3)\vartheta_1].$$

If we choose $\vartheta_1 = \xi$ in the last equality, we get

$$R(\xi, \vartheta_2)\vartheta_3 = \frac{1}{n-1}[S(\xi, \vartheta_3)\vartheta_2 - S(\vartheta_2, \vartheta_3)\xi].$$

If we use (11) and (13) in the last equation, we have

$$\frac{1}{n-1}S(\vartheta_2, \vartheta_3)\xi = \left(\alpha^2 - \rho\right)[g(\vartheta_2, \vartheta_3)\xi - 2\eta(\vartheta_3)\vartheta_2].$$

If we take the inner product of both sides of the last equality by $\xi \in \chi(M)$ and make the necessary adjustments, we obtain

$$S(\vartheta_2, \vartheta_3) = -(n-1)\left(\alpha^2 - \rho\right)[g(\vartheta_2, \vartheta_3) + 2\eta(\vartheta_2)\eta(\vartheta_3)].$$

This completes the proof. □

Definition 1. *Let M be an $n-$dimensional Riemannian manifold. If $W_1 \cdot S$ and $Q(g, S)$ are linearly dependent, then the manifold is said to be W_1-Ricci pseudosymmetric.*

In this case, there exists a function L_{W_1} on M such that

$$W_1 \cdot S = L_{W_1} Q\left(g, S\right).$$

Let us now investigate the W_1-Ricci pseudosymmetric case of the $n-$dimensional $(LCS)_n-$manifold admitting almost $\eta-$Ricci soliton.

Theorem 2. *Let M be $(LCS)_n-$manifold and (g, ξ, λ, μ) be almost $\eta-$Ricci soliton on M. If M is a W_1-Ricci pseudosymmetric, then*

$$L_{W_1} = 2\left(\alpha^2 - \rho\right) \text{ or } \alpha = \mu - 2\lambda.$$

Proof. Let us assume that $(LCS)_n-$manifold be W_1-Ricci pseudosymmetric and (g, ξ, λ, μ) be almost $\eta-$Ricci soliton on $(LCS)_n-$manifold. That's mean

$$\left(W_1\left(\vartheta_1, \vartheta_2\right) \cdot S\right)\left(\vartheta_4, \vartheta_5\right) = L_{W_1} Q\left(g, S\right)\left(\vartheta_4, \vartheta_5; \vartheta_1, \vartheta_2\right),$$

for all $\vartheta_1, \vartheta_2, \vartheta_4, \vartheta_5 \in \Gamma\left(TM\right)$. From the last equation, we can easily write

$$\begin{aligned} &S\left(W_1\left(\vartheta_1, \vartheta_2\right)\vartheta_4, \vartheta_5\right) + S\left(\vartheta_4, W_1\left(\vartheta_1, \vartheta_2\right)\vartheta_5\right) \\ =&L_{W_1}\left\{S\left(\left(\vartheta_1 \wedge_g \vartheta_2\right)\vartheta_4, \vartheta_5\right) + S\left(\vartheta_4, \left(\vartheta_1 \wedge_g \vartheta_2\right)\vartheta_5\right)\right\}. \end{aligned} \tag{24}$$

If we choose $\vartheta_5 = \xi$ in (24), we get

$$\begin{aligned} &S\left(W_1\left(\vartheta_1, \vartheta_2\right)\vartheta_4, \xi\right) + S\left(\vartheta_4, W_1\left(\vartheta_1, \vartheta_2\right)\xi\right) \\ =&L_{W_1}\left\{S\left(g\left(\vartheta_2, \vartheta_4\right)\vartheta_1 - g\left(\vartheta_1, \vartheta_4\right)\vartheta_2, \xi\right)\right. \\ &\left.+S\left(\vartheta_4, \eta\left(\vartheta_2\right)\vartheta_1 - \eta\left(\vartheta_1\right)\vartheta_2\right)\right\}. \end{aligned} \tag{25}$$

If we make use of (13) and (22) in (25), we have

$$\begin{aligned} &\left(\alpha^2 - \rho\right)\left(n-1\right)\eta\left(W_1\left(\vartheta_1, \vartheta_2\right)\vartheta_4\right) \\ &+ 2\left(\alpha^2 - \rho\right)S\left(\vartheta_4, \eta\left(\vartheta_2\right)\vartheta_1 - \eta\left(\vartheta_1\right)\vartheta_2\right) \\ =&L_{W_1}\left\{\left(\alpha^2 - \rho\right)\left(n-1\right)g\left(\eta\left(\vartheta_1\right)\vartheta_2 - \eta\left(\vartheta_2\right)\vartheta_1, \vartheta_4\right)\right. \\ &\left.+S\left(\vartheta_4, \eta\left(\vartheta_2\right)\vartheta_1 - \eta\left(\vartheta_1\right)\vartheta_2\right)\right\}. \end{aligned} \tag{26}$$

If we use (23) in the (26), we get

$$\begin{aligned} &2\left(\alpha^2-\rho\right)^2(n-1)\,g\left(\eta\left(\vartheta_1\right)\vartheta_2-\eta\left(\vartheta_2\right)\vartheta_1,\vartheta_4\right)\\ &+2\left(\alpha^2-\rho\right)S\left(\eta\left(\vartheta_2\right)\vartheta_1-\eta\left(\vartheta_1\right)\vartheta_2,\vartheta_4\right)\\ =&L_{W_1}\left\{\left(\alpha^2-\rho\right)(n-1)\,g\left(\eta\left(\vartheta_1\right)\vartheta_2-\eta\left(\vartheta_2\right)\vartheta_1,\vartheta_4\right)\right.\\ &\left.+S\left(\vartheta_4,\eta\left(\vartheta_2\right)\vartheta_1-\eta\left(\vartheta_1\right)\vartheta_2\right)\right\}. \end{aligned} \tag{27}$$

If we use (18) and (6) in the (27), we can write

$$\left[\left(\alpha^2-\rho\right)(n-1)+\alpha+\lambda\right]\left[2\left(\alpha^2-\rho\right)-L_{W_1}\right]g\left(\eta\left(\vartheta_1\right)\vartheta_2-\eta\left(\vartheta_2\right)\vartheta_1,\vartheta_4\right)=0. \tag{28}$$

It is clear from (28),

$$L_{W_1}=2\left(\alpha^2-\rho\right),$$

or

$$\lambda=\left(\rho-\alpha^2\right)(n-1)-\alpha.$$

This completes the proof. □

Corollary 1. *Let M be $(LCS)_n-$manifold and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on M. If M is a W_1-Ricci semisymmetric, then*

$$\mu=-\alpha.$$

Corollary 2. *Let M be $(LCS)_n-$manifold and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on M. If M is a W_1-Ricci semisymmetric, then*
i) M is an expanding if $\left(\rho-\alpha^2\right)(n-1)>\alpha$,
ii) M is a shrinking if $\left(\rho-\alpha^2\right)(n-1)<\alpha$.

For an $n-$dimensional semi-Riemann manifold M, the W_1^*-curvature tensor is defined as

$$W_1^*\left(\vartheta_1,\vartheta_2\right)\vartheta_3=R\left(\vartheta_1,\vartheta_2\right)\vartheta_3-\frac{1}{n-1}\left[S\left(\vartheta_2,\vartheta_3\right)\vartheta_1-S\left(\vartheta_1,\vartheta_3\right)\vartheta_2\right]. \tag{29}$$

For an $n-$dimensional $(LCS)_n$ manifold, if we choose $\vartheta_3=\xi$ in (29), we can write

$$W_1^*\left(\vartheta_1,\vartheta_2\right)\xi=0, \tag{30}$$

and similarly, if we take the inner product of both sides of (30) by ξ, we get

$$\eta\left(W_1^*\left(\vartheta_1,\vartheta_2\right)\vartheta_3\right)=0. \tag{31}$$

Theorem 3. *Let M be a $n-$dimensional $(LCS)_n$ −manifold. If M is W_1^*-flat, then M is an Einstein manifold.*

Proof. Let's assume that M is W_1^*-flat. So, we can write

$$W_1^*\left(\vartheta_1,\vartheta_2\right)\vartheta_3=0$$

for all $\vartheta_1,\vartheta_2,\vartheta_3\in\chi(M)$. That is

$$R\left(\vartheta_1,\vartheta_2\right)\vartheta_3=\frac{1}{n-1}\left[S\left(\vartheta_2,\vartheta_3\right)\vartheta_1-S\left(\vartheta_1,\vartheta_3\right)\vartheta_2\right].$$

If we choose $\vartheta_1=\xi$ in the last equality, we get

$$R\left(\xi,\vartheta_2\right)\vartheta_3=\frac{1}{n-1}\left[S\left(\vartheta_2,\vartheta_3\right)\xi-S\left(\xi,\vartheta_3\right)\vartheta_2\right].$$

If we use (11) and (13) in the last equation, we have

$$S\left(\vartheta_2,\vartheta_3\right)\xi=\left(\alpha^2-\rho\right)(n-1)\,g\left(\vartheta_2,\vartheta_3\right)\xi.$$

If we take the inner product of both sides of the last equality by $\xi\in\chi(M)$ and make the necessary adjustments, we obtain

$$S\left(\vartheta_2,\vartheta_3\right)=\left(\alpha^2-\rho\right)(n-1)\,g\left(\vartheta_2,\vartheta_3\right).$$

This completes the proof. □

Definition 2. *Let M be an $n-$dimensional Riemannian manifold. If $W_1^*\cdot S$ and $Q(g,S)$ are linearly dependent, then the manifold is said to be W_1^*-Ricci pseudosymmetric.*

In this case, there exists a function $L_{W_1^*}$ on M such that

$$W_1^*\cdot S=L_{W_1^*}Q\left(g,S\right).$$

Let us now investigate the W_1^*-Ricci pseudosymmetric case of the $n-$dimensional $(LCS)_n$ −manifold.

Theorem 4. *Let M be $(LCS)_n$ −manifold and (g,ξ,λ,μ) be almost η−Ricci soliton on M. If M is a W_1^*−Ricci pseudosymmetric, then M is either W_1^*−Ricci semisymmetric or $\mu=-\alpha$.*

Proof. Let's assume that $(LCS)_n$ −manifold be W_1^*−Ricci pseudosymmetric and (g,ξ,λ,μ) be almost η−Ricci soliton on $(LCS)_n$ −manifold. That's mean

$$\left(W_1^*\left(\vartheta_1,\vartheta_2\right)\cdot S\right)\left(\vartheta_4,\vartheta_5\right)=L_{W_1^*}Q\left(g,S\right)\left(\vartheta_4,\vartheta_5;\vartheta_1,\vartheta_2\right),$$

for all $\vartheta_1,\vartheta_2,\vartheta_4,\vartheta_5\in\Gamma\left(TM\right)$. From the last equation, we can easily write

$$\begin{aligned}&S\left(W_1^*\left(\vartheta_1,\vartheta_2\right)\vartheta_4,\vartheta_5\right)+S\left(\vartheta_4,W_1^*\left(\vartheta_1,\vartheta_2\right)\vartheta_5\right)\\&=L_{W_1^*}\left\{S\left(\left(\vartheta_1\wedge_g\vartheta_2\right)\vartheta_4,\vartheta_5\right)+S\left(\vartheta_4,\left(\vartheta_1\wedge_g\vartheta_2\right)\vartheta_5\right)\right\}.\end{aligned}\tag{32}$$

If we choose $\vartheta_5=\xi$ in (32), we get

$$\begin{aligned}&S\left(W_1^*\left(\vartheta_1,\vartheta_2\right)\vartheta_4,\xi\right)+S\left(\vartheta_4,W_1^*\left(\vartheta_1,\vartheta_2\right)\xi\right)\\&=L_{W_1^*}\left\{S\left(g\left(\vartheta_2,\vartheta_4\right)\vartheta_1-g\left(\vartheta_1,\vartheta_4\right)\vartheta_2,\xi\right)\right.\\&\quad\left.+S\left(\vartheta_4,\eta\left(\vartheta_2\right)\vartheta_1-\eta\left(\vartheta_1\right)\vartheta_2\right)\right\}.\end{aligned}\tag{33}$$

If we make use of (13) and (30) in (33), we have

$$\begin{aligned}&L_{W_1^*}\left\{\left(\alpha^2-\rho\right)(n-1)\,g\left(\eta\left(\vartheta_1\right)\vartheta_2-\eta\left(\vartheta_2\right)\vartheta_1,\vartheta_4\right)+S\left(\vartheta_4,\eta\left(\vartheta_2\right)\vartheta_1-\eta\left(\vartheta_1\right)\vartheta_2\right)\right\}\\&=\left(\alpha^2-\rho\right)(n-1)\,\eta\left(W_1^*\left(\vartheta_1,\vartheta_2\right)\vartheta_4\right).\end{aligned}\tag{34}$$

If we use (31) in the (34), we get

$$L_{W_1^*}\left[\left(\alpha^2-\rho\right)(n-1)\,g\left(\eta\left(\vartheta_1\right)\vartheta_2-\eta\left(\vartheta_2\right)\vartheta_1,\vartheta_4\right)+S\left(\vartheta_4,\eta\left(\vartheta_2\right)\vartheta_1-\eta\left(\vartheta_1\right)\vartheta_2\right)\right]=0.\tag{35}$$

If we use (18) and (6) in the (35), we can write

$$L_{W_1^*}\left[\left(\alpha^2-\rho\right)(n-1)+\alpha+\lambda\right]g\left(\eta\left(\vartheta_1\right)\vartheta_2-\eta\left(\vartheta_2\right)\vartheta_1,\vartheta_4\right)=0.\tag{36}$$

It is clear from (36),

$$L_{W_1^*}=0,$$

or

$$\lambda = \left(\rho - \alpha^2\right)(n-1) - \alpha.$$

This completes the proof. □

Corollary 3. *Let M be $(LCS)_n-$manifold and (g, ξ, λ, μ) be almost $\eta-$Ricci soliton on M. If M is a W_1^*-Ricci pseudosymmetric, then M is either W_1^*-Ricci semisymmetric or*
i) M is an expanding if $\left(\rho - \alpha^2\right)(n-1) - \alpha > 0$,
ii) M is a shrinking if $\left(\rho - \alpha^2\right)(n-1) - \alpha < 0$.

For an $n-$dimensional semi-Riemann manifold M, the W_2-curvature tensor is defined as

$$W_2(\vartheta_1, \vartheta_2)\vartheta_3 = R(\vartheta_1, \vartheta_2)\vartheta_3 - \frac{1}{n-1}\left[g(\vartheta_2, \vartheta_3)Q\vartheta_1 - g(\vartheta_1, \vartheta_3)Q\vartheta_2\right]. \quad (37)$$

For an $n-$dimensional $(LCS)_n$ manifold, if we choose $\vartheta_3 = \xi$ in (37), we can write

$$\begin{aligned} W_2(\vartheta_1, \vartheta_2)\xi = &\left(\alpha^2 - \rho\right)\left[\eta(\vartheta_2)\vartheta_1 - \eta(\vartheta_1)\vartheta_2\right] \\ &- \frac{1}{n-1}\left[\eta(\vartheta_2)Q\vartheta_1 - \eta(\vartheta_1)Q\vartheta_2\right], \end{aligned} \quad (38)$$

and similarly, if we take the inner product of both sides of (37) by ξ, we get

$$\begin{aligned} \eta\left(W_2(\vartheta_1, \vartheta_2)\vartheta_3\right) = &\frac{1}{n-1}S\left(\eta(\vartheta_2)\vartheta_1 - \eta(\vartheta_1)\vartheta_2, \vartheta_3\right) \\ &- \left(\alpha^2 - \rho\right)g\left(\eta(\vartheta_2)\vartheta_1 - \eta(\vartheta_1)\vartheta_2, \vartheta_3\right). \end{aligned} \quad (39)$$

Theorem 5. *Let M be a $n-$dimensional $(LCS)_n-$manifold. If M is W_2-flat, then M is an Einstein manifold.*

Proof. Let us assume that M is W_2-flat. So, we can write

$$W_2(\vartheta_1, \vartheta_2)\vartheta_3 = 0$$

for all $\vartheta_1, \vartheta_2, \vartheta_3 \in \chi(M)$. That is

$$R(\vartheta_1, \vartheta_2)\vartheta_3 = \frac{1}{n-1}\left[g(\vartheta_2, \vartheta_3)Q\vartheta_1 - g(\vartheta_1, \vartheta_3)Q\vartheta_2\right].$$

If we choose $\vartheta_1 = \xi$ in the last equality, we get

$$R(\xi, \vartheta_2)\vartheta_3 = \frac{1}{n-1}\left[g(\vartheta_2, \vartheta_3)Q\xi - S(\xi, \vartheta_3)Q\vartheta_2\right].$$

If we use (11) and (13) in the last equation, we have

$$-\frac{1}{n-1}\eta(\vartheta_3)Q\vartheta_2 = \left(\alpha^2 - \rho\right)\left[2g(\vartheta_2, \vartheta_3)\xi - \eta(\vartheta_3)\vartheta_2\right].$$

If we first choose $\vartheta_3 = \xi$ and then we take the inner product of both sides of the last equality by $\vartheta_1 \in \chi(M)$ and make the necessary adjustments, we obtain

$$S(\vartheta_1, \vartheta_2) = \left(\alpha^2 - \rho\right)(n-1)\left[g(\vartheta_1, \vartheta_2) + 2\eta(\vartheta_1)\eta(\vartheta_2)\right].$$

This completes the proof. □

Definition 3. *Let M be an $n-$dimensional Riemannian manifold. If $W_2 \cdot S$ and $Q(g, S)$ are linearly dependent, then the manifold is said to be W_2-Ricci pseudosymmetric.*

In this case, there exists a function L_{W_2} on M such that

$$W_2 \cdot S = L_{W_2} Q(g, S).$$

Let us now investigate the W_2-Ricci pseudosymmetric case of the $n-$dimensional $(LCS)_n-$manifold admitting almost $\eta-$Ricci soliton.

Theorem 6. *Let M be $(LCS)_n-$manifold and (g, ξ, λ, μ) be almost $\eta-$Ricci soliton on M. If M is a W_2-Ricci pseudosymmetric, then*

$$L_{W_2} = \frac{(\alpha+\lambda)^2 + (\mu-\lambda)\left[(\mu-\lambda) + 2(\alpha+\lambda)\right]}{(n-1)(\mu+\alpha)}.$$

Proof. Let us assume that $(LCS)_n-$manifold be W_2-Ricci pseudosymmetric and (g, ξ, λ, μ) be almost $\eta-$Ricci soliton on $(LCS)_n-$manifold. That means

$$(W_2(\vartheta_1, \vartheta_2) \cdot S)(\vartheta_4, \vartheta_5) = L_{W_2} Q(g, S)(\vartheta_4, \vartheta_5; \vartheta_1, \vartheta_2),$$

for all $\vartheta_1, \vartheta_2, \vartheta_4, \vartheta_5 \in \Gamma(TM)$. From the last equation, we can easily write

$$\begin{aligned} &S(W_2(\vartheta_1,\vartheta_2)\vartheta_4,\vartheta_5)+S(\vartheta_4,W_2(\vartheta_1,\vartheta_2)\vartheta_5) \\ =&L_{W_2}\{S((\vartheta_1\wedge_g\vartheta_2)\vartheta_4,\vartheta_5)+S(\vartheta_4,(\vartheta_1\wedge_g\vartheta_2)\vartheta_5)\}. \end{aligned} \tag{40}$$

If we choose $\vartheta_5=\xi$ in (40), we get

$$\begin{aligned} &S(W_2(\vartheta_1,\vartheta_2)\vartheta_4,\xi)+S(\vartheta_4,W_2(\vartheta_1,\vartheta_2)\xi) \\ =&L_{W_2}\{S(g(\vartheta_2,\vartheta_4)\vartheta_1-g(\vartheta_1,\vartheta_4)\vartheta_2,\xi) \\ &+S(\vartheta_4,\eta(\vartheta_2)\vartheta_1-\eta(\vartheta_1)\vartheta_2)\}. \end{aligned} \tag{41}$$

If we make use of (13) and (38) in (41), we have

$$\begin{aligned} &L_{W_2}\left\{\left(\alpha^2-\rho\right)(n-1)g(\eta(\vartheta_1)\vartheta_2-\eta(\vartheta_2)\vartheta_1,\vartheta_4)+S(\vartheta_4,\eta(\vartheta_2)\vartheta_1-\eta(\vartheta_1)\vartheta_2)\right\} \\ =&\left(\alpha^2-\rho\right)(n-1)\eta(W_2(\vartheta_1,\vartheta_2)\vartheta_4)+S\Big(\vartheta_4,\left(\alpha^2-\rho\right)[\eta(\vartheta_2)\vartheta_1-\eta(\vartheta_1)\vartheta_2] \\ &-\frac{1}{n-1}[\eta(\vartheta_2)Q\vartheta_1-\eta(\vartheta_1)Q\vartheta_2]\Big). \end{aligned} \tag{42}$$

If we use (39) in the (42), we get

$$\begin{aligned} &2\left(\alpha^2-\rho\right)S(\eta(\vartheta_2)\vartheta_1-\eta(\vartheta_1)\vartheta_2,\vartheta_4)-\left(\alpha^2-\rho\right)^2(n-1)g(\eta(\vartheta_2)\vartheta_1-\eta(\vartheta_1)\vartheta_2,\vartheta_4) \\ &\frac{1}{n-1}S(\vartheta_4,\eta(\vartheta_2)Q\vartheta_1-\eta(\vartheta_1)Q\vartheta_2) \\ =&L_{W_2}\left\{\left(\alpha^2-\rho\right)(n-1)g(\eta(\vartheta_1)\vartheta_2-\eta(\vartheta_2)\vartheta_1,\vartheta_4)+S(\vartheta_4,\eta(\vartheta_2)\vartheta_1-\eta(\vartheta_1)\vartheta_2)\right\}. \end{aligned} \tag{43}$$

If we use (18) and (6) in the (43), we can write

$$\begin{aligned} &-\left(\alpha^2-\rho\right)\left[2\alpha+2\lambda+\left(\alpha^2-\rho\right)(n-1)\right]g(\eta(\vartheta_2)\vartheta_1-\eta(\vartheta_1)\vartheta_2,\vartheta_4) \\ &+\frac{1}{n-1}(\alpha+\lambda)S(\vartheta_4,\eta(\vartheta_2)\vartheta_1-\eta(\vartheta_1)\vartheta_2) \\ =&L_{W_2}\left[\left(\alpha^2-\rho\right)(n-1)+\alpha+\lambda\right]g(\eta(\vartheta_1)\vartheta_2-\eta(\vartheta_2)\vartheta_1,\vartheta_4). \end{aligned} \tag{44}$$

If we use (18) again in (44), we have

$$\begin{aligned} &\left\{-\left(\alpha^2-\rho\right)\left[2\alpha+2\lambda+\left(\alpha^2-\rho\right)(n-1)\right]-\frac{(\alpha+\lambda)^2}{(n-1)}L_{W_2}\left[\left(\alpha^2-\rho\right)(n-1)+\alpha+\lambda\right]\right\} \\ &\times g(\eta(\vartheta_2)\vartheta_1-\eta(\vartheta_1)\vartheta_2,\vartheta_4)=0. \end{aligned} \tag{45}$$

It is clear from (45) ,

$$L_{W_2} = \frac{\mu + \alpha}{n-1}.$$

This completes the proof. □

Corollary 4. *Let M be $(LCS)_n$−manifold and (g, ξ, λ, μ) be almost η−Ricci soliton on M. If M is a W_2−Ricci semisymmetric, then*

$$\lambda = -\alpha - (n-1)\left[\alpha^2 - \rho\right].$$

For an n−dimensional semi-Riemann manifold M, the pseudo-projective curvature tensor is defined as

$$\begin{aligned} P_*(\vartheta_1, \vartheta_2)\vartheta_3 =& a_0 R(\vartheta_1, \vartheta_2)\vartheta_3 + a_1\left[S(\vartheta_2, \vartheta_3)\vartheta_1 - S(\vartheta_1, \vartheta_3)\vartheta_2\right] \\ & -\frac{r}{n}\left(\frac{a_0}{n-1} + a_1\right)\left[g(\vartheta_2, \vartheta_3)\vartheta_1 - g(\vartheta_1, \vartheta_3)\vartheta_2\right], \end{aligned} \tag{46}$$

where a_o, a_1 is the smooth function, and r is scalar curvature of manifold.

For an n−dimensional $(LCS)_n$ manifold, if we choose $\vartheta_3 = \xi$ in (46) , we can write

$$\begin{aligned} P_*(\vartheta_1, \vartheta_2)\xi = & \left\{\left(\alpha^2 - \rho\right)\left[a_0 + (n-1)a_1\right] - \frac{r}{n}\left(\frac{a_0}{n-1} + a_1\right)\right\} \\ & \times\left[\eta(\vartheta_2)\vartheta_1 - \eta(\vartheta_1)\vartheta_2\right], \end{aligned} \tag{47}$$

and similarly, if we take the inner product of both sides of (46) by ξ, we get

$$\begin{aligned} \eta\left(P_*(\vartheta_1, \vartheta_2)\vartheta_3\right) = & \left\{\left(\alpha^2 - \rho\right)\left[a_0 + (n-1)a_1\right] - \frac{r}{n}\left(\frac{a_0}{n-1} + a_1\right)\right\} \\ & \times g\left(\eta(\vartheta_1)\vartheta_2 - \eta(\vartheta_2)\vartheta_1, \vartheta_3\right). \end{aligned} \tag{48}$$

Theorem 7. *Let M be a n−dimensional $(LCS)_n$−manifold. If M is pseudo projective flat, then M is an η−Einstein manifold.*

Proof. Let us assume that M is pseudo-projective flat. So, we can write

$$P_*(\vartheta_1, \vartheta_2)\vartheta_3 = 0$$

for all $\vartheta_1, \vartheta_2, \vartheta_3 \in \chi(M)$. That is

$$a_0 R(\vartheta_1, \vartheta_2)\vartheta_3 = a_1 [S(\vartheta_1, \vartheta_3)\vartheta_2 - S(\vartheta_2, \vartheta_3)\vartheta_1] + \frac{r}{n}\left(\frac{a_0}{n-1} + a_1\right)[g(\vartheta_2, \vartheta_3)\vartheta_1 - g(\vartheta_1, \vartheta_3)\vartheta_2].$$

If we choose $\vartheta_1 = \xi$ in the last equality, we get

$$a_0 R(\xi, \vartheta_2)\vartheta_3 = a_1 [S(\xi, \vartheta_3)\vartheta_2 - S(\vartheta_2, \vartheta_3)\xi] + \frac{r}{n}\left(\frac{a_0}{n-1} + a_1\right)[g(\vartheta_2, \vartheta_3)\xi - g(\xi, \vartheta_3)\vartheta_2].$$

If we use (11) and (13) in the last equation, we have

$$a_1 S(\vartheta_2, \vartheta_3)\xi = \left[a_1\left(\alpha^2 - \rho\right)(n-1) - \frac{r}{n}\left(\frac{a_0}{n-1} + a_1\right) + a_0\left(\alpha^2 - \rho\right)\right]\eta(\vartheta_3)\vartheta_2 - a_0\left(\alpha^2 - \rho\right)\eta(\vartheta_2)\vartheta_3 + \frac{r}{n}\left(\frac{a_0}{n-1} + a_1\right)g(\vartheta_2, \vartheta_3)\xi.$$

If we take the inner product of both sides of the last equality by $\xi \in \chi(M)$ and make the necessary adjustments, we obtain

$$S(\vartheta_2, \vartheta_3) = \frac{r}{a_1 n}\left(\frac{a_0}{n-1} + a_1\right)g(\vartheta_2, \vartheta_3) - \left[\left(\alpha^2 - \rho\right)(n-1) - \frac{r}{a_1 n}\left(\frac{a_0}{n-1} + a_1\right)\right]\eta(\vartheta_2)\eta(\vartheta_3).$$

This completes the proof. □

Definition 4. *Let M be an $n-$dimensional Riemannian manifold. If $P_* \cdot S$ and $Q(g, S)$ are linearly dependent, then the manifold is said to be pseudo-projective Ricci pseudosymmetric.*

In this case, there exists a function L_{P_*} on M such that

$$P_* \cdot S = L_{P_*} Q(g, S).$$

Let us now investigate the pseudo-projective Ricci pseudosymmetric case of the $n-$dimensional $(LCS)_n-$manifold admitting almost $\eta-$Ricci soliton.

Theorem 8. *Let M be $(LCS)_n$ −manifold and (g, ξ, λ, μ) be almost η−Ricci soliton on M. If M is a pseudo-projective Ricci pseudosymmetric, then*

$$L_{P_*} = \left(\alpha^2 - \rho\right)\left[a_0 + (n-1)\, a_1\right] - \frac{r}{n}\left(\frac{a_0}{n-1} + a_1\right)$$

or $\mu = -\alpha$.

Proof. Let's assume that $(LCS)_n$ −manifold is pseudo-projective Ricci pseudosymmetric and (g, ξ, λ, μ) be almost η−Ricci soliton on $(LCS)_n$ −manifold. That's mean

$$\left(P_*\left(\vartheta_1, \vartheta_2\right) \cdot S\right)\left(\vartheta_4, \vartheta_5\right) = L_{P_*} Q\left(g, S\right)\left(\vartheta_4, \vartheta_5; \vartheta_1, \vartheta_2\right),$$

for all $\vartheta_1, \vartheta_2, \vartheta_4, \vartheta_5 \in \Gamma\left(TM\right)$. From the last equation, we can easily write

$$\begin{aligned} &S\left(P_*\left(\vartheta_1, \vartheta_2\right)\vartheta_4, \vartheta_5\right) + S\left(\vartheta_4, P_*\left(\vartheta_1, \vartheta_2\right)\vartheta_5\right) \\ =&L_{P_*}\left\{S\left(\left(\vartheta_1 \wedge_g \vartheta_2\right)\vartheta_4, \vartheta_5\right) + S\left(\vartheta_4, \left(\vartheta_1 \wedge_g \vartheta_2\right)\vartheta_5\right)\right\}. \end{aligned} \tag{49}$$

If we choose $\vartheta_5 = \xi$ in (49), we get

$$\begin{aligned} &S\left(P_*\left(\vartheta_1, \vartheta_2\right)\vartheta_4, \xi\right) + S\left(\vartheta_4, P_*\left(\vartheta_1, \vartheta_2\right)\xi\right) \\ =&L_{P_*}\left\{S\left(g\left(\vartheta_2, \vartheta_4\right)\vartheta_1 - g\left(\vartheta_1, \vartheta_4\right)\vartheta_2, \xi\right)\right. \\ &\left. + S\left(\vartheta_4, \eta\left(\vartheta_2\right)\vartheta_1 - \eta\left(\vartheta_1\right)\vartheta_2\right)\right\}. \end{aligned} \tag{50}$$

If we make use of (13) and (47) in (50), we have

$$\begin{aligned} &\left(\alpha^2 - \rho\right)(n-1)\,\eta\left(P_*\left(\vartheta_1, \vartheta_2\right)\vartheta_4\right) + AS\left(\vartheta_4, \eta\left(\vartheta_2\right)\vartheta_1 - \eta\left(\vartheta_1\right)\vartheta_2\right) \\ =&L_{P_*}\left\{\left(\alpha^2 - \rho\right)(n-1)\, g\left(\eta\left(\vartheta_1\right)\vartheta_2 - \eta\left(\vartheta_2\right)\vartheta_1, \vartheta_4\right) + S\left(\vartheta_4, \eta\left(\vartheta_2\right)\vartheta_1 - \eta\left(\vartheta_1\right)\vartheta_2\right)\right\}, \end{aligned} \tag{51}$$

where $A = \left(\alpha^2 - \rho\right)\left[a_0 + (n-1)\, a_1\right] - \dfrac{r}{n}\left(\frac{a_0}{n-1} + a_1\right)$. If we use (48) in the (51), we get

$$\begin{aligned} &A\left(\alpha^2 - \rho\right)^2 (n-1)\, g\left(\eta\left(\vartheta_1\right)\vartheta_2 - \eta\left(\vartheta_2\right)\vartheta_1, \vartheta_4\right) + AS\left(\eta\left(\vartheta_2\right)\vartheta_1 - \eta\left(\vartheta_1\right)\vartheta_2, \vartheta_4\right) \\ =&L_{P_*}\left\{\left(\alpha^2 - \rho\right)(n-1)\, g\left(\eta\left(\vartheta_1\right)\vartheta_2 - \eta\left(\vartheta_2\right)\vartheta_1, \vartheta_4\right) + S\left(\vartheta_4, \eta\left(\vartheta_2\right)\vartheta_1 - \eta\left(\vartheta_1\right)\vartheta_2\right)\right\}. \end{aligned} \tag{52}$$

If we use (18) and (6) in the (52), we can write

$$\left[\left(\alpha^2-\rho\right)(n-1)+\alpha+\lambda\right]\left[A-L_{P_*}\right] g\left(\eta\left(\vartheta_1\right) \vartheta_2-\eta\left(\vartheta_2\right) \vartheta_1, \vartheta_4\right)=0. \tag{53}$$

It is clear from (53),

$$L_{P_*}=\left(\alpha^2-\rho\right)\left[a_0+(n-1) a_1\right]-\frac{r}{n}\left(\frac{a_0}{n-1}+a_1\right),$$

or

$$\lambda=\left(\rho-\alpha^2\right)(n-1)-\alpha.$$

This completes the proof. □

Corollary 5. *Let M be $(LCS)_n-$manifold and (g, ξ, λ, μ) be almost $\eta-$Ricci soliton on M. If M is a pseudo-projective Ricci semisymmetric, then M is either manifold with constant scalar curvature $r=n(n-1)\left(\alpha^2-\rho\right)$ or $\mu=-\alpha$.*

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Conflict of interest

The author declares no conflict of interest.

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