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Maclaurin Coefficient Estimates for a New General Subclasses of m-Fold Symmetric Holomorphic Bi-Univalent Functions

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Abstract

The purpose of the present paper is to introduce and investigate two new general subclasses $\mathcal{MA}_{\Sigma_m}(\delta,\lambda;\alpha)$ and $\mathcal{MA}_{\Sigma_m}(\delta,\lambda;\beta)$ of Σ_m consisting of holomorphic and m-fold symmetric bi-univalent functions defined in the open unit disk U. For functions belonging to the two classes introduced here, we derive estimates on the initial coefficients $|d_{m+1}|$ and $|d_{2m+1}|$. We get new special cases for our results. In addition, Several related classes are also investigated and connections to earlier known outcomes are made.

1 Introduction

Let $\mathcal{A}$ denote the class of functions of the form:

$$k(s) = s + \sum_{n=2}^{\infty} d_n s^n, \tag{1.1}$$

which are holomorphic in the open unit disk $U = \{s : |s| < 1\}$, and let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of the form (1.1) which are also univalent in U. The

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Koebe one-quarter theorem [4] states that the image of U under every function k from $\mathcal{S}$ contains a disk of radius 1/4. Thus every such univalent function has an inverse k^{-1} which satisfies

$$k^{-1}(k(s)) = s \quad (s \in U)$$

and

$$k\left(k^{-1}(r)\right) = r \quad \left(|r| < r_0(k);\ \ r_0 \geq \frac{1}{4}\right)$$

where

$$k^{-1}(r) = h(r) = r - d_2 r^2 + \left(2d_2^2 - d_3\right) r^3 - \left(5d_2^3 - 5d_2 d_3 + d_4\right) r^4 + \cdots . \quad (1.2)$$

A function $k \in \mathcal{A}$ is said to be bi-univalent in U, if both $k(s)$ and $k^{-1}(s)$ are univalent in U. We denote by Σ the class of all bi-univalent functions in U given by the Taylor-Maclaurin series expansion (1.1). Lewin [7] discussed the class of bi-univalent functions Σ and proved that the bound for the second coefficients of every $k \in \Sigma$ satisfies the inequality $|b_2| \leq 1.51$. Motivated by the work of Lewin [7], Brannan and Clunie [3] hypothesised that $|b_2| \leq \sqrt{2}$. Some examples of bi-univalent functions are $\frac{s}{1-s}, -\log(1-s)$ and $\frac{1}{2}\log\left(\frac{1+s}{1-s}\right)$ (see also Srivastava et al. [13]). The coefficient estimate problem involving the bound of $|d_n|\,(n \in \mathbb{N}\backslash\{1,2\})$ for every $k \in \Sigma$ is still an open problem [13].

For each function $k \in \mathcal{S}$, the function

$$h(s) = \sqrt[m]{k\left(s^m\right)} \quad (s \in U, m \in \mathbb{N}) \quad (1.3)$$

is univalent and maps the unit disk U into a region with m-fold symmetry. A function is said to be m-fold symmetric (see [5], [9]) if it has the following normalized form:

$$k(s) = s + \sum_{n=1}^{\infty} d_{mn+1} s^{mn+1} \quad (s \in \Delta, m \in \mathbb{N}). \quad (1.4)$$

We denote by $\mathcal{S}_m$ the class of m-fold symmetric univalent functions in U, which are normalized by the series expansion (1.4). In fact, the functions in the class $\mathcal{S}$

are one-fold symmetric. Analogous to the concept of m-fold symmetric univalent functions, we here introduced the concept of m-fold symmetric bi-univalent functions. Each function $k \in \Sigma$ generates an m-fold symmetric bi-univalent function for each integer $m \in \mathbb{N}$. The normalized form of k is given as in (1.4) and the series expansion for k^{-1}, which has been recently proven by Srivastava et al. [14], is given as follows:

$$h(r) = r - d_{m+1}r^{m+1} + \left[(m+1)d_{m+1}^2 - d_{2m+1}\right] r^{2m+1} - \left[\frac{1}{2}(m+1)(3m+2)d_{m+1}^3 - (3m+2)d_{m+1}d_{2m+1} + d_{3m+1}\right] r^{3m+1} + \cdots \tag{1.5}$$

where $k^{-1} = h$. We denote by Σ_m the class of m-fold symmetric bi-univalent functions in U. For $m = 1$, formula (1.5) coincides with formula (1.2) of the class Σ.

Recently, many penmen inverstigated bounds for various subclasses of m-fold symmetric holomorphic bi-univalent functions (see [1, 2, 6, 10, 11, 12, 14, 15]).

The aim of this paper is to derive estimates on the initial coefficients $|d_{m+1}|$ and $|d_{2m+1}|$ for functions belonging to the new general subclasses $\mathcal{MA}_{\Sigma_m}(\delta, \lambda; \alpha)$ and $\mathcal{MA}_{\Sigma_m}(\delta, \lambda; \beta)$ of Σ_m. We get new special cases for our results. In addition, Several related classes are also investigated and connections to earlier known outcomes are made.

In order to derive our main results, we have to recall here the following lemma [4].

Lemma 1.1. *If $p \in \mathcal{P}$, then $|c_n| \leq 2$ for each $n \in \mathbb{N}$, where $\mathcal{P}$ is the family of all Functions p, holomorphic in U, for which*

$$\mathrm{Re}(p(s)) > 0 \quad (s \in U)$$

where

$$p(s) = 1 + c_1 s + c_2 s^2 + \cdots .$$

2 Coefficient Bounds for the Function Class $\mathcal{MA}_{\Sigma_m}(\delta, \lambda; \alpha)$

Definition 2.1. *A function $k(s) \in \Sigma_m$ given by (1.4) is told to be in the class $\mathcal{MA}_{\Sigma_m}(\delta, \lambda; \alpha)$ if the following conditions are fulfilled:*

$$\left|\arg\left[(1-\delta)\left((1-\lambda)k'(s) + \lambda\frac{sk'(s)}{k(s)}\right) + \delta\left(\frac{sk'(s)}{(1-\lambda)s + \lambda k(s)}\right)\right]\right| < \frac{\alpha\pi}{2} \quad (s \in U) \tag{2.1}$$

and

$$\left|\arg\left[(1-\delta)\left((1-\lambda)h'(r) + \lambda\frac{rh'(r)}{h(r)}\right) + \delta\left(\frac{rh'(r)}{(1-\lambda)r + \lambda h(r)}\right)\right]\right| < \frac{\alpha\pi}{2} \quad (r \in U), \tag{2.2}$$

where the function $h = k^{-1}$ is given by (1.5) and $(0 \le \delta \le 1; 0 \le \lambda \le 1; 0 < \alpha \le 1)$.

Theorem 2.1. *Let the function $k(s)$, given by (1.4), be in the class $\mathcal{MA}_{\Sigma_m}(\delta, \lambda; \alpha)$. Then*

$$|d_{m+1}| \le \frac{2\alpha}{\sqrt{|\alpha m(m-\lambda+1) + \alpha\lambda(2\lambda\delta - 2\delta - \lambda + 1) + m(m-2\lambda+2) + (1-\lambda)^2|}} \tag{2.3}$$

and

$$|d_{2m+1}| \le \frac{2\alpha^2(m+1)}{(m-\lambda+1)^2} + \frac{2\alpha}{(2m-\lambda+1)}. \tag{2.4}$$

Proof. Let $k(s) \in \mathcal{MA}_{\Sigma_m}(\delta, \lambda; \alpha)$. Then

$$(1-\delta)\left((1-\lambda)k'(s) + \lambda\frac{sk'(s)}{k(s)}\right) + \delta\left(\frac{sk'(s)}{(1-\lambda)s + \lambda k(s)}\right) = [p(s)]^\alpha \tag{2.5}$$

and

$$(1-\delta)\left((1-\lambda)h'(r) + \lambda\frac{rh'(r)}{h(r)}\right) + \delta\left(\frac{rh'(r)}{(1-\lambda)r + \lambda h(r)}\right) = [q(r)]^\alpha \tag{2.6}$$

where $h = k^{-1}$ and $p(s), q(r)$ in $\mathcal{P}$ and have the forms

$$p(s) = 1 + p_m s^m + p_{2m}s^{2m} + p_{3m}s^{3m} + \cdots \tag{2.7}$$

and

$$q(r) = 1 + q_m r^m + q_{2m} r^{2m} + q_{3m} r^{3m} + \cdots . \tag{2.8}$$

It follows from (2.5) and (2.6) that

$$(m - \lambda + 1) d_{m+1} = \alpha p_m \tag{2.9}$$

$$(2m - \lambda + 1) d_{2m+1} - \lambda(m - \delta\lambda + \delta) d_{m+1}^2 = \alpha p_{2m} + \frac{\alpha(\alpha-1)}{2} p_m^2 \tag{2.10}$$

$$-(m - \lambda + 1) d_{m+1} = \alpha q_m \tag{2.11}$$

$$\begin{aligned}[m(2m - 2\lambda + 3) &+ (1-\lambda)(1-\delta\lambda)] d_{m+1}^2 - (2m - \lambda + 1) d_{2m+1} \\ &= \alpha q_{2m} + \frac{\alpha(\alpha-1)}{2} q_m^2 .\end{aligned} \tag{2.12}$$

For (2.9) and (2.11), we get

$$p_m = -q_m \tag{2.13}$$

and

$$2(m - \lambda + 1)^2 d_{m+1}^2 = \alpha^2 \left(p_m^2 + q_m^2\right). \tag{2.14}$$

From (2.10), (2.12) and (2.14) we find

$$\begin{aligned}&[m(2m - 2\lambda + 3) + (1-\lambda)(1-\delta\lambda) - \lambda(m - \delta\lambda + \delta)] d_{m+1}^2 \\ &\qquad = \alpha \left(p_{2m} + q_{2m}\right) + \frac{\alpha(\alpha-1)}{2} \left(p_m^2 + q_m^2\right) \qquad (2.15) \\ &\qquad = \alpha \left(p_{2m} + q_{2m}\right) + \frac{(\alpha-1)}{\alpha} (m - \lambda + 1)^2 d_{m+1}^2 .\end{aligned}$$

Therefore, we have

$$d_{m+1}^2 = \frac{\alpha^2 \left(p_{2m} + q_{2m}\right)}{m\alpha(m - \lambda + 1) + \lambda\alpha(2\delta\lambda - 2\delta - \lambda + 1) + m(m - 2\lambda + 2) + (1-\lambda)^2} . \tag{2.16}$$

Stratifying Lemma (1.1) for the coefficients p_{2m} and q_{2m}, we get

$$|d_{m+1}| \leq \frac{2\alpha}{\sqrt{|m\alpha(m - \lambda + 1) + \alpha\lambda(2\delta\lambda - 2\delta - \lambda + 1) + m(m - 2\lambda + 2) + (1-\lambda)^2|}} \tag{2.17}$$

The last inequality gives the desired estimate on $|d_{m+1}|$ given in (2.3).

Next, the bound on $|d_{2m+1}|$ is then found by subtracting (2.12) from (2.10).

$$2(2m-\lambda+1)d_{2m+1} - \left(2m^2+3m-m\lambda-\lambda+1\right)d_{m+1}^2 = \alpha\left(p_{2m}-q_{2m}\right) + \frac{\alpha(\alpha-1)}{2}\left(p_m^2-q_m^2\right). \tag{2.18}$$

By using (2.13), (2.14) and (2.18), we get

$$d_{2m+1} = \frac{\alpha^2(m+1)\left(p_m^2+q_m^2\right)}{4(m-\lambda+1)^2} + \frac{\alpha\left(p_{2m}-q_{2m}\right)}{2(2m-\lambda+1)}. \tag{2.19}$$

Stratifying Lemma (1.1) once again for the coefficients p_m, p_{2m} and q_m, q_{2m}, we get

$$|d_{2m+1}| \leq \frac{2\alpha^2(m+1)}{(m-\lambda+1)^2} + \frac{2\alpha}{(2m-\lambda+1)}.$$

This proves Theorem (2.1). □

3 Coefficient Bounds for the Function Class $\mathcal{MA}_{\Sigma_m}(\delta,\lambda;\beta)$

Definition 3.1. *A function $k(s) \in \Sigma_m$ given by (1.4) is told to be in the class $\mathcal{MA}_{\Sigma_m}(\delta,\lambda;\beta)$ if the following conditions are fulfilled:*

$$\operatorname{Re}\left((1-\delta)\left((1-\lambda)k'(s)+\lambda\frac{sk'(s)}{k(s)}\right)+\delta\left(\frac{sk'(s)}{(1-\lambda)s+\lambda k(s)}\right)\right) > \beta \quad (s \in U) \tag{3.1}$$

and

$$\operatorname{Re}\left((1-\delta)\left((1-\lambda)h'(r)+\lambda\frac{rh'(r)}{h(r)}\right)+\delta\left(\frac{rh'(r)}{(1-\lambda)r+\lambda h(r)}\right)\right) > \beta \quad (r \in U) \tag{3.2}$$

where the function $h = k^{-1}$ is given by (1.5) and $(0 \leq \delta \leq 1; 0 \leq \lambda \leq 1; 0 \leq \beta < 1)$.

Theorem 3.1. *Let the function $k(s)$, given by (1.4), be in the class $\mathcal{MA}_{\Sigma_m}(\delta,\lambda;\beta)$. Then*

$$|d_{m+1}| \leq 2\sqrt{\frac{(1-\beta)}{m(2m-3\lambda+3)+(1-\lambda)(1-2\delta\lambda)}} \tag{3.3}$$

and

$$|d_{2m+1}| \leq \frac{2(1-\beta)^2(m+1)}{(m-\lambda+1)^2} + \frac{2(1-\beta)}{(2m-\lambda+1)}. \tag{3.4}$$

Proof. It follows from (3.1) and (3.2) that there exists $p, q \in \mathcal{P}$ such that

$$(1-\delta)\left((1-\lambda)k'(s) + \lambda\frac{sk'(s)}{k(s)}\right) + \delta\left(\frac{sk'(s)}{(1-\lambda)s+\lambda k(s)}\right) = \beta + (1-\beta)p(s) \tag{3.5}$$

and

$$(1-\delta)\left((1-\lambda)h'(r) + \lambda\frac{rh'(r)}{h(r)}\right) + \delta\left(\frac{rh'(r)}{(1-\lambda)r+\lambda h(r)}\right) = \beta + (1-\beta)q(r) \tag{3.6}$$

where $p(s)$ and $q(r)$ have the forms (2.7) and (2.8). It follows from (3.5) and (3.6), we find

$$(m-\lambda+1)d_{m+1} = (1-\beta)p_m \tag{3.7}$$

$$(2m-\lambda+1)d_{2m+1} - \lambda(m-\delta\lambda+\delta)d_{m+1}^2 = (1-\beta)p_{2m} \tag{3.8}$$

$$-(m-\lambda+1)d_{m+1} = (1-\beta)q_m \tag{3.9}$$

and

$$[m(2m-2\lambda+3)+(1-\lambda)(1-\delta\lambda)]d_{m+1}^2 - (2m-\lambda+1)d_{2m+1} = (1-\beta)q_{2m}. \tag{3.10}$$

From (3.7) and (3.9), we get

$$p_m = -q_m \tag{3.11}$$

and

$$2(m-\lambda+1)^2 d_{m+1}^2 = (1-\beta)^2\left(p_m^2+q_m^2\right). \tag{3.12}$$

Adding (3.8) and (3.10), we get

$$\begin{aligned}\left[\left(2m^2+3m-2m\lambda\right)+(1-\lambda)(1-\delta\lambda)-\lambda(m-\delta\lambda+\delta)\right]d_{m+1}^2 \\ = (1-\beta)\left(p_{2m}+q_{2m}\right).\end{aligned} \tag{3.13}$$

Therefore, we obtain

$$d_{m+1}^2 = \frac{(1-\beta)\left(p_{2m}+q_{2m}\right)}{m(2m-3\lambda+3)+(1-\lambda)(1-2\delta\lambda)}. \tag{3.14}$$

Stratifying Lemma (1.1) for coefficients p_{2m} and q_{2m}, we readily get

$$|d_{m+1}| \leq 2\sqrt{\frac{(1-\beta)}{|m(2m-3\lambda+3)+(1-\lambda)(1-2\delta\lambda)|}}.$$

This gives the desired estimate on $|d_{m+1}|$ gives by (3.3).

Next, the bound on $|d_{2m+1}|$ is then found by subtracting (3.10) from (3.8).

$$2(2m-\lambda+1)d_{2m+1} - [m(2m-\lambda+3)+(1-\lambda)]d_{m+1}^2 \\ =(1-\beta)\left(p_{2m}-q_{2m}\right). \quad (3.15)$$

Or equivalently

$$d_{2m+1} = \frac{(m+1)d_{m+1}^2}{2} + \frac{(1-\beta)\left(p_{2m}-q_{2m}\right)}{2(2m-\lambda+1)}. \quad (3.16)$$

By substituting the value of d_{m+1}^2 from (3.12), we find

$$d_{2m+1} = \frac{(1-\beta)^2(m+1)\left(p_m^2+q_m^2\right)}{4(m-\lambda+1)^2} + \frac{(1-\beta)\left(p_{2m}-q_{2m}\right)}{2(2m-\lambda+1)}. \quad (3.17)$$

By using Lemma (1.1) once again for the coefficients p_m, p_{2m}, q_m and q_{2m} we get

$$|d_{2m+1}| \leq \frac{2(1-\beta)^2(m+1)}{(m-\lambda+1)^2} + \frac{2(1-\beta)}{(2m-\lambda+1)}.$$

This proves Theorem (3.1) □

4 Corollaries and Consequences

If we set $\delta = 1$ in definition (2.1) and definition (3.1), then the classes $\mathcal{MA}_{\Sigma_m}(\delta,\lambda;\alpha)$ and $\mathcal{MAA}_{\Sigma_m}(\delta,\lambda;\beta)$ shorten to the classes $\mathcal{MA}_{\Sigma_m}(\lambda;\alpha)$ and $\mathcal{MA}_{\Sigma_m}(\lambda;\beta)$ and thus, Theorem (2.1) and Theorem (3.1) shorten to Corollary (4.1) and Corollary (4.2), respectively.

The classes $\mathcal{MA}_{\Sigma_m}(\lambda;\alpha)$ and $\mathcal{MA}_{\Sigma_m}(\lambda;\beta)$ are respectively defined as follows:

Definition 4.1. *A function $k(s) \in \Sigma_m$ given by (1.4) is told to be in the class $\mathcal{MA}_{\Sigma_m}(\lambda;\alpha)$ if the following conditions are fulfilled:*

$$\left|\arg\left(\frac{sk'(s)}{(1-\lambda)s+\lambda k(s)}\right)\right| < \frac{\alpha\pi}{2} \quad \textit{and} \quad \left|\arg\left(\frac{rh'(r)}{(1-\lambda)r+\lambda h(r)}\right)\right| < \frac{\alpha\pi}{2} \quad (s,r \in U),$$

where the function $h = k^{-1}$ is given by (1.5) and $(0 \le \lambda \le 1; 0 < \alpha \le 1)$.

Definition 4.2. *A function $k(s) \in \Sigma_m$ given by (1.4) is told to be in the class $\mathcal{MA}_{\Sigma_m}(\lambda;\beta)$ if the following conditions are fulfilled:*

$$\operatorname{Re}\left(\frac{sk'(s)}{(1-\lambda)s+\lambda k(s)}\right) > \beta \quad \textit{and} \quad \operatorname{Re}\left(\frac{rh'(r)}{(1-\lambda)r+\lambda h(r)}\right) > \beta \quad (s,r \in U)$$

where the function $h = k^{-1}$ is given by (1.5) and $(0 \le \lambda \le 1; 0 \le \beta < 1)$.

Corollary 4.1. *Let $k(s)$ given by (1.4) be in the class $\mathcal{MAE}_m(\lambda;\alpha)$. Then*

$$|d_{m+1}| \le \frac{2\alpha}{\sqrt{|\alpha m(m-\lambda+1)+\alpha\lambda(\lambda-1)+m(m-2\lambda+2)+(1-\lambda)^2|}}$$

and

$$|d_{2m+1}| \le \frac{2\alpha^2(m+1)}{(m-\lambda+1)^2} + \frac{2\alpha}{(2m-\lambda+1)}.$$

Corollary 4.2. *Let $k(s)$ given by (1.4) be in the class $\mathcal{MA}_{\Sigma_m}(\lambda;\beta)$. Then*

$$|d_{m+1}| \le 2\sqrt{\frac{(1-\beta)}{m(2m-3\lambda+3)+(1-\lambda)(1-2\lambda)}}$$

and

$$|d_{2m+1}| \le \frac{2(1-\beta)^2(m+1)}{(m-\lambda+1)^2} + \frac{2(1-\beta)}{(2m-\lambda+1)}.$$

If we set $\delta = 0$ in definition (2.1) and definition (3.1), then the classes $\mathcal{M_C A}_{\Sigma_m}(\delta,\lambda;\alpha)$ and $\mathcal{MA}_{\Sigma_m}(\delta,\lambda;\beta)$ shorten to the classes $\mathcal{HAA}_{\Sigma_m}(\lambda;\alpha)$ and $\mathcal{HA}_{\Sigma_m}(\lambda;\beta)$ and thus, Theorem (2.1) and Theorem (3.1) shorten to Corollary (4.3) and Corollary (4.4), respectively. The classes $\mathcal{HA}_{\Sigma_m}(\lambda;\alpha)$ and $\mathcal{HA}_{\Sigma_m}(\lambda;\beta)$ are respectively defined as follows:

Definition 4.3. *A function $k(s) \in \Sigma_m$ given by (1.4) is told to be in the class $\mathcal{HA}_{\Sigma_m}(\lambda;\alpha)$ if the following conditions are fulfilled :*

$$\left|\arg\left((1-\lambda)k'(s)+\lambda\frac{sk'(s)}{k(s)}\right)\right|<\frac{\alpha\pi}{2} \quad (s\in U)$$

and

$$\left|\arg\left((1-\lambda)h'(r)+\lambda\frac{rh'(r)}{h(r)}\right)\right|<\frac{\alpha\pi}{2} \quad (r\in U)$$

where the function $h = k^{-1}$ is given by (1.5) and $(0 \leq \lambda \leq 1; 0 < \alpha \leq 1)$.

Definition 4.4. *A function $k(s) \in \Sigma_m$ given by (1.4) is told to be in the class $\mathcal{HA}_{\Sigma_m}(\lambda;\beta)$ if the following conditions are fulfilled:*

$$\operatorname{Re}\left((1-\lambda)k'(s)+\lambda\frac{sk'(s)}{k(s)}\right)>\beta \ \text{ and } \ \operatorname{Re}\left((1-\lambda)h'(r)+\lambda\frac{rh'(r)}{h(r)}\right)>\beta \quad (s,r\in U)$$

where the function $h = k^{-1}$ is given by (1.5) and $(0 \leq \lambda \leq 1; 0 \leq \beta < 1)$.

Corollary 4.3. *Let $k(s)$ given by (1.4) be in the class $\mathcal{HA}_{\Sigma_m}(\lambda;\alpha)$. Then*

$$|d_{m+1}| \leq \frac{2\alpha}{\sqrt{|\alpha m(m-\lambda+1)+\alpha\lambda(1-\lambda)+m(m-2\lambda+2)+(1-\lambda)^2|}}$$

and

$$|d_{2m+1}| \leq \frac{2\alpha^2(m+1)}{(m-\lambda+1)^2}+\frac{2\alpha}{(2m-\lambda+1)}.$$

Corollary 4.4. *Let $k(s)$ given by (1.4) be in the class $\mathcal{HA}_{\Sigma_m}(\lambda;\beta)$. Then*

$$|d_{m+1}| \leq 2\sqrt{\frac{(1-\beta)}{m(2m-3\lambda+3)+(1-\lambda)}}$$

and

$$|d_{2m+1}| \leq \frac{2(1-\beta)^2(m+1)}{(m-\lambda+1)^2}+\frac{2(1-\beta)}{(2m-\lambda+1)}.$$

For one-fold symmetric holomorphic bi-univalent functions, the classes $\mathcal{MA}_{\Sigma_m}(\delta,\lambda;\alpha)$ and $\mathcal{MA}_{\Sigma_m}(\delta,\lambda;\beta)$ shorten to the classes $\mathcal{MA}_{\Sigma}(\delta,\lambda;\alpha)$ and $\mathcal{MAA}_{\Sigma}(\delta,\lambda;\beta)$ and thus, Theorem (2.1) and Theorem (3.1) shorten to Corollary (4.5) and Corollary (4.6), respectively.

The classes $\mathcal{MA}_{\Sigma}(\delta,\lambda;\alpha)$ and $\mathcal{MA}_{\Sigma}(\delta,\lambda;\beta)$ are defined in the following way:

Definition 4.5. *A function $k(s) \in \Sigma$ given by (1.1) is told to be in the class $\mathcal{MA}_{\Sigma}(\delta, \lambda; \alpha)$ if the following conditions are fulfilled:*

$$\left|\arg\left[(1-\delta)\left((1-\lambda)k'(s)+\lambda\frac{sk'(s)}{k(s)}\right)+\delta\left(\frac{sk'(s)}{(1-\lambda)s+\lambda k(s)}\right)\right]\right| < \frac{\alpha\pi}{2} \quad (s \in U)$$

and

$$\left|\arg\left[(1-\delta)\left((1-\lambda)h'(r)+\lambda\frac{rh'(r)}{h(r)}\right)+\delta\left(\frac{rh'(r)}{(1-\lambda)r+\lambda h(r)}\right)\right]\right| < \frac{\alpha\pi}{2} \quad (r \in U)$$

where the function $h = k^{-1}$ is given by (1.2) and $(0 \leq \delta \leq 1, 0 \leq \lambda \leq 1; 0 < \alpha \leq 1)$.

Definition 4.6. *A function $k(s) \in \Sigma$ given by (1.1) is told to be in the class $\mathcal{M}_{\mathcal{A}}\mathcal{A}_{\Sigma}(\delta, \lambda; \beta)$ if the following conditions are fulfilled:*

$$\operatorname{Re}\left((1-\delta)\left((1-\lambda)k'(s)+\lambda\frac{sk'(s)}{k(s)}\right)+\delta\left(\frac{sk'(s)}{(1-\lambda)s+\lambda k(s)}\right)\right) > \beta \quad (s \in U)$$

and

$$\operatorname{Re}\left((1-\delta)\left((1-\lambda)h'(r)+\lambda\frac{rh'(r)}{h(r)}\right)+\delta\left(\frac{rh'(r)}{(1-\lambda)r+\lambda h(r)}\right)\right) > \beta \quad (r \in U)$$

where the function $h = k^{-1}$ is given by (1.2) and $(0 \leq \delta \leq 1, 0 \leq \lambda \leq 1; 0 \leq \beta < 1)$.

Corollary 4.5. *Let $k(s)$ given by (1.1) be in the class $\mathcal{MA}_{\Sigma}(\delta, \lambda; \alpha)$. Then*

$$|d_2| \leq \frac{2\alpha}{\sqrt{\alpha(2-\lambda)+\alpha\lambda(2\delta\lambda-2\delta-\lambda+1)+(3-2\lambda)+(1-\lambda)^2}}$$

and

$$|d_3| \leq \frac{4\alpha^2}{(2-\lambda)} + \frac{2\alpha}{(3-\lambda)}.$$

Corollary 4.6. *Let $k(s)$ given by (1.1) be in the class $\mathcal{MA}_{\Sigma}(\delta, \lambda; \beta)$. Then*

$$|d_2| \leq 2\sqrt{\frac{(1-\beta)}{|(5-3\lambda)+(1-\lambda)(1-2\delta\lambda)|}} \quad \text{and} \quad |d_3| \leq \frac{4(1-\beta)^2}{(2-\lambda)^2} + \frac{2(1-\beta)}{(3-\lambda)}.$$

If we set $\delta = 1$ and $m = 1$ in definition(2.1) and definition(3.1), then the classes $\mathcal{MA}_{\Sigma_m}(\delta, \lambda; \alpha)$ and $\mathcal{MA}_{\Sigma_m}(\delta, \lambda; \beta)$ shorten to the classes $\mathcal{MAA}_{\Sigma}(\lambda; \alpha)$ and $\mathcal{MA}_{\Sigma}(\lambda; \beta)$ and thus, Theorem (2.1) and Theorem (3.1) shorten to Corollary (4.7) and Corollary (4.8), respectively.

The classes $\mathcal{M}_{\mathcal{A}}\mathcal{A}_{\Sigma}(\lambda; \alpha)$ and $\mathcal{M}_{\mathcal{D}}\mathcal{A}_{\Sigma}(\lambda; \beta)$, are respectively defined as follows:

Definition 4.7. *A function $k(s) \in \Sigma$ given by (1.1) is told to be in the class $\mathcal{MA}_{\Sigma}(\lambda; \alpha)$ if the following conditions are fulfilled:*

$$\left|\arg\left(\frac{sk'(s)}{(1-\lambda)s + \lambda k(s)}\right)\right| < \frac{\alpha\pi}{2} \quad \textit{and} \quad \left|\arg\left(\frac{rh'(r)}{(1-\lambda)r + \lambda h(r)}\right)\right| < \frac{\alpha\pi}{2} \quad (s, r \in U),$$

where the function $h = k^{-1}$ is given by (1.2) and $(0 \leq \lambda \leq 1; 0 < \alpha \leq 1)$.

Definition 4.8. *A function $k(s) \in \Sigma$ given by (1.1) is told to be in the class $\mathcal{MA}_{\Sigma}(\lambda; \beta)$ if the following conditions are fulfilled:*

$$\operatorname{Re}\left(\frac{sk'(s)}{(1-\lambda)s + \lambda k(s)}\right) > \beta \quad \textit{and} \quad \operatorname{Re}\left(\frac{rh'(r)}{(1-\lambda)r + \lambda h(r)}\right) > \beta \quad (s, r \in U),$$

where the function $h = k^{-1}$ is given by (1.2) and $(0 \leq \lambda \leq 1; 0 \leq \beta < 1)$.

Corollary 4.7. *Let $k(s)$ given by (1.1) be in the class $\mathcal{MA}_{\Sigma}(\lambda; \alpha)$. Then*

$$|d_2| \leq \frac{2\alpha}{\sqrt{2(1-\lambda)(\alpha+2) + \lambda^2(\alpha+1)}} \quad \textit{and} \quad |d_3| \leq \frac{4\alpha^2}{(2-\lambda)} + \frac{2\alpha}{(3-\lambda)}.$$

Corollary 4.8. *Let $k(s)$ given by (1.1) be in the class $\mathcal{MA}_{\Sigma}(\lambda; \beta)$. Then*

$$|d_2| \leq 2\sqrt{\frac{(1-\beta)}{(5-3\lambda) + (1-\lambda)(1-2\lambda)}} \quad \textit{and} \quad |d_3| \leq \frac{4(1-\beta)^2}{(2-\lambda)^2} + \frac{2(1-\beta)}{(3-\lambda)}.$$

If we set $\delta = 0$ and $m = 1$ in definition (2.1) and definition (3.1), then the classes $\mathcal{MA}_{\Sigma_m}(\delta, \lambda; \alpha)$ and $\mathcal{MA}_{\Sigma_m}(\delta, \lambda; \beta)$ shorten to the classes $\mathcal{HA}_{\Sigma}(\lambda; \alpha)$ and $\mathcal{HA}_{\Sigma}(\lambda; \beta)$ and thus, Theorem (2.1) and Theorem (3.1) shorten to Corollary (4.9) and Corollary (4.10), respectively.

The classes $\mathcal{HA}_{\Sigma}(\lambda; \alpha)$ and $\mathcal{HA}_{\Sigma}(\lambda; \beta)$ are respectively defined as follows:

Definition 4.9. *A function $k(s) \in \Sigma$ given by (1.1) is told to be in the class $\mathcal{HA}_\Sigma(\lambda;\alpha)$ if the following conditions are fulfilled:*

$$\left|\arg\left((1-\lambda)k'(s)+\lambda\frac{sk'(s)}{k(s)}\right)\right| < \frac{\alpha\pi}{2} \text{ and } \left|\arg\left((1-\lambda)h'(r)+\lambda\frac{rh'(r)}{h(r)}\right)\right| < \frac{\alpha\pi}{2}$$

$$(s, r \in U),$$

where the function $h = k^{-1}$ is given by (1.2) and $(0 \le \lambda \le 1; 0 < \alpha \le 1)$.

Definition 4.10. *A function $k(s) \in \Sigma$ given by (1.1) is told to be in the class $\mathcal{HA}_\Sigma(\lambda;\beta)$ if the following conditions are fulfilled:*

$$\operatorname{Re}\left((1-\lambda)k'(s)+\lambda\frac{sk'(s)}{k(s)}\right) > \beta \quad \text{and} \quad \operatorname{Re}\left((1-\lambda)h'(r)+\lambda\frac{rh'(r)}{h(r)}\right) > \beta$$

$$(s, r \in U),$$

where the function $h = k^{-1}$ is given by (1.2) and $(0 \le \lambda \le 1; 0 \le \beta < 1)$.

Corollary 4.9. *Let $k(s)$ given by (1.1) be in the class $\mathcal{HA}_\Sigma(\lambda;\alpha)$. Then*

$$|d_2| \le \frac{2\alpha}{\sqrt{\alpha(2-\lambda)+\alpha\lambda(1-\lambda)+(3-2\lambda)+(1-\lambda)^2}}$$

and

$$|d_3| \le \frac{4\alpha^2}{(2-\lambda)} + \frac{2\alpha}{(3-\lambda)}.$$

Corollary 4.10. *Let $k(s)$ given by (1.1) be in the class $\mathcal{HA}_\Sigma(\lambda;\beta)$. Then*

$$|d_2| \le 2\sqrt{\frac{(1-\beta)}{|(5-3\lambda)+(1-\lambda)|}} \quad \text{and} \quad |d_3| \le \frac{4(1-\beta)^2}{(2-\lambda)^2} + \frac{2(1-\beta)}{(3-\lambda)}.$$

Remark 4.1. *For m-fold symmetric holomorphic bi-univalent functions:*

1. *Putting $\delta = 0$ and $\lambda = 0$ in Theorem (2.1) and Theorem (3.1), we get the corresponding outcomes given by Srivastave et al. [14].*
2. *Putting $\delta = 0$ and $\lambda = 1$ in Theorem (2.1) and Theorem (3.1), we get the corresponding outcomes given by Altinkaya and Yalçin [1].*

Remark 4.2. *For one-fold symmetric holomorphic bi-univalent functions :*

1. *Putting $\delta = 0$ and $\lambda = 0$ in Theorem (2.1) and Theorem (3.1), we get the corresponding outcomes given by Srivastave et al. [13].*
2. *Putting $\delta = 0$ and $\lambda = 1$ in Theorem (2.1) and Theorem (3.1), we get the corresponding outcomes given by Murugusundaramoorthy et al. [8] .*

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