Earthline Journal of Mathematical Sciences
E-ISSN: 2581-8147
Volume 13, Number 1, 2023, Pages 235-249
https://doi.org/10.34198/ejms.13123.235249

Initial Coefficient Estimates for New Families of m-Fold Symmetric Bi-univalent Functions

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Abstract

In the present work, we define two new families of analytic and m-fold symmetric biunivalent functions in the open unit disk Δ. Also, for functions in each of the classes introduced here, we prove upper bounds for the initial coefficients $|b_{m+1}|$ and $|b_{2m+1}|$. Furthermore, we get new special cases for our results.

1 Introduction

Let $\mathcal{A}$ be the class of analytic functions in the open unit disk $\Delta = \{t : t \in \mathbb{C}, |t| < 1\}$ and normalized by the conditions $g(0) = 0 = g'(0) - 1$ and having the form shown below:

$$g(t) = t + \sum_{n=2}^{\infty} b_n t^n \tag{1.1}$$

Also, the class of all functions in $\mathcal{A}$ that are univalent in Δ is denoted by $\mathcal{S}$.

The Koebe one-quarter theorem [4] ensures that the image of Δ under every univalent function $g(t) \in \mathcal{A}$ contains the disk of radius $1/4$. Thus every univalent

Received: June 2, 2023; Accepted: June 18, 2023; Published: July 1, 2023
2020 Mathematics Subject Classification: 30C45, 30C50.
Keywords and phrases: analytic function, m-fold symmetric analytic function, bi-univalent function, m-fold symmetric analytic bi-univalent function, coefficient bounds.
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function $g(t)$ has an inverse g^{-1} satisfying

$$g^{-1}(g(t)) = t \quad (t \in \Delta)$$

and

$$g\left(g^{-1}(w)\right) = w \quad \left(|w| < r_0(g);\ r_0(g) \geq \frac{1}{4}\right),$$

where

$$g^{-1}(w) = h(w) = w - b_2 w^2 + \left(2b_2^2 - b_3\right) w^3 - \left(5b_2^3 - 5b_2 b_3 + b_4\right) w^4 + \cdots. \quad (1.2)$$

A function $g \in \mathcal{A}$ is said to be bi-univalent in Δ, if both $g(t)$ and $g^{-1}(t)$ are univalent in Δ. We denote by Σ the class of all bi-univalent functions in Δ given by the Taylor-Maclaurin series expansion (1.1). Lewin [7] discussed the class of bi-univalent functions Σ and proved that the bound for the second coefficients of every $g \in \Sigma$ satisfies the inequality $|b_2| \leq 1.51$. Motivated by the work of Lewin [7], Brannan and Clunie [3] hypothesised that $|b_2| \leq \sqrt{2}$. Some examples of bi-univalent functions are $\frac{t}{1-t}, -\log(1-t)$ and $\frac{1}{2}\log\left(\frac{1+t}{1-t}\right)$ (see also Srivastava et al. [15]). The coefficient estimate problem involving the bound of $|b_n|\,(n \in \mathbb{N}\backslash\{1,2\})$ for every $g \in \Sigma$ is still an open problem [15].

For each function $g \in \mathcal{S}$, the function

$$k(t) = \sqrt[m]{g\left(t^m\right)} \quad (t \in \Delta,\ \ m \in \mathbb{N}) \quad (1.3)$$

is univalent and maps the unit disk Δ into a region with m-fold symmetry. A function is told to be m-fold symmetric (see [5, 10]) if it has the following normalized form:

$$g(t) = t + \sum_{n=1}^{\infty} b_{mn+1} t^{mn+1} \quad (t \in \Delta,\ \ m \in \mathbb{N}). \quad (1.4)$$

We symbolize by $\mathcal{S}_m$ the class of m-fold symmetric analytic univalent functions in Δ, which are normalized by the series expansion (1.4). In fact, the functions in the class $\mathcal{S}$ are one-fold symmetric (that is, $m = 1$).

Analogous to the concept of m-fold symmetric univalent functions, we here introduced the concept of m-fold symmetric analytic bi-univalent function. Each function $g \in \Sigma$ generates an m-fold symmetric analytic bi-univalent function for each integer $m \in \mathbb{N}$. The normalized form of g is given as in (1.4) and the series expansion for g^{-1}, which has been recently proven by Srivastava et al. [16], is given as follows:

$$g^{-1}(w) = h(w) = w - b_{m+1}w^{m+1} + \left[(m+1)b_{m+1}^2 - b_{2m+1}\right] w^{2m+1} - \left[\frac{1}{2}(m+1)(3m+2)b_{m+1}^3 - (3m+2)b_{m+1}b_{2m+1} + b_{3m+1}\right] w^{3m+1} + \cdots . \tag{1.5}$$

We symbolize by Σ_m the class of m-fold symmetric analytic bi-univalent functions in Δ. For $m = 1$, formulation (1.5) synchronizes with formulation (1.2) of the class Σ. Some examples of m-fold symmetric analytic bi-univalent functions are listed below [16]:

$$\left(\frac{t^m}{1-t^m}\right)^{\frac{1}{m}}, \quad \left[-\log\left(1-t^m\right)\right]^{\frac{1}{m}} \quad \text{and} \quad \left[\frac{1}{2}\log\left(\frac{1+t^m}{1-t^m}\right)\right]^{\frac{1}{m}}$$

with the corresponding inverse functions

$$\left(\frac{w^m}{1+w^m}\right)^{\frac{1}{m}}, \quad \left(\frac{e^{w^m}-1}{e^{w^m}}\right)^{\frac{1}{m}} \quad \text{and} \quad \left(\frac{e^{2w^m}-1}{e^{2w^m}+1}\right)^{\frac{1}{m}},$$

respectively. Recently, many authors investigated bounds for various subclasses of m-fold bi-univalent functions (see $[1, 2, 6, 11, 12, 13, 14, 17]$).

The purpose of this work is to introduce two new subclasses of function class Σ_m and derive estimates on initial coefficients $|b_{m+1}|$ and $|b_{2m+1}|$ for functions in these new subclasses. Many related classes are also found out and connections to earlier known results are made. We have to remember the following lemma here so as to derive our basic results.

Lemma 1.1. *([4]) If $p \in \mathcal{P}$, then $|c_n| \leq 2$ for each $n \in \mathbb{N}$, where $\mathcal{P}$ is the family of all functions p, analytic in Δ, for which*

$$R(p(t)) > 0 \quad \textit{where} \quad p(t) = 1 + c_1 t + c_2 t^2 + \cdots \quad (t \in \Delta).$$

2 Coefficient Bounds for the Function Class $\mathcal{LH}_{\Sigma_m}(\tau, \lambda, \delta; \alpha)$

Definition 2.1. *A function $g(t) \in \Sigma_m$ given by (1.4) is told to be in the class $\mathcal{LH}_{\Sigma_m}(\tau, \lambda, \delta; \alpha)$ if the following conditions are fulfilled:*

$$\left|\arg\left(1 + \frac{1}{\tau}\left[(1-\lambda)\frac{tg'(t)}{g(t)}\left(\frac{g(t)}{t}\right)^{\delta} + \lambda\left(1 + \frac{tg''(t)}{g'(t)}\right)\left(g'(t)\right)^{\delta} - 1\right]\right)\right| < \frac{\alpha\pi}{2} \quad (t \in \Delta) \tag{2.1}$$

and

$$\left|\arg\left(1 + \frac{1}{\tau}\left[(1-\lambda)\frac{wh'(w)}{h(w)}\left(\frac{h(w)}{w}\right)^{\delta} + \lambda\left(1 + \frac{wh''(w)}{h'(w)}\right)\left(h'(w)\right)^{\delta} - 1\right]\right)\right| < \frac{\alpha\pi}{2} \quad (w \in \Delta), \tag{2.2}$$

where the function $h = g^{-1}$ is given by (1.5) and $(0 \leq \lambda \leq 1; \delta \geq 0; \tau \in C\backslash\{0\}; 0 < \alpha \leq 1)$.

Theorem 2.1. *Let the function $g(t)$, given by (1.4), be in the class $\mathcal{LH}_{\Sigma_m}(\tau, \lambda, \delta; \alpha)$. Then*

$$|b_{m+1}| \leq \frac{2\alpha|\tau|}{\sqrt{\left|\tau\left[\alpha(2m+\delta)(2\lambda m+1)(m+1) + \alpha(\delta-1)\left(2\lambda m^2(m+2) + m(3\lambda\delta+2) + \delta\right)\right] - (\alpha-1)[(m+\delta)(\lambda m+1)]^2\right|}} \tag{2.3}$$

and

$$|b_{2m+1}| \leq \frac{2(m+1)|\tau|^2\alpha^2}{[(m+\delta)(\lambda m+1)]^2} + \frac{2|\tau|\alpha}{(2m+\delta)(2\lambda m+1)}. \tag{2.4}$$

Proof. It follows from (2.1) and (2.2) that

$$1 + \frac{1}{\tau}\left[(1-\lambda)\frac{tg'(t)}{g(t)}\left(\frac{g(t)}{t}\right)^{\delta} + \lambda\left(1 + \frac{tg''(t)}{g'(t)}\right)\left(g'(t)\right)^{\delta} - 1\right] = [p(t)]^{\alpha} \tag{2.5}$$

and

$$1 + \frac{1}{\tau}\left[(1-\lambda)\frac{wh'(w)}{h(w)}\left(\frac{h(w)}{w}\right)^{\delta} + \lambda\left(1 + \frac{wh''(w)}{h'(w)}\right)\left(h'(w)\right)^{\delta} - 1\right] = [q(w)]^{\alpha}, \tag{2.6}$$

where the functions $p(t)$ and $q(w)$ are in $\mathcal{P}$ and have the following series representations:

$$p(t) = 1 + p_m t^m + p_{2m} t^{2m} + p_{3m} t^{3m} + \cdots \tag{2.7}$$

and

$$q(w) = 1 + q_m w^m + q_{2m} w^{2m} + q_{3m} w^{3m} + \cdots . \tag{2.8}$$

It follows from (2.5) and (2.6) that

$$\frac{1}{\tau}(m+\delta)(\lambda m+1) b_{m+1} = \alpha p_m, \tag{2.9}$$

$$\frac{1}{\tau}\left[(2m+\delta)(2\lambda m+1) b_{2m+1} + \frac{1}{2}(\delta-1)\left(2\lambda m^2(m+2) + m(3\lambda\delta+2)+\delta\right) b_{m+1}^2\right]$$
$$= \alpha p_{2m} + \frac{1}{2}\alpha(\alpha-1) p_m^2, \tag{2.10}$$

$$\frac{-1}{\tau}(m+\delta)(\lambda m+1) b_{m+1} = \alpha q_m \tag{2.11}$$

and

$$\begin{aligned}&\frac{1}{\tau}\left[(2m+\delta)(2\lambda m+1)\left((m+1)b_{m+1}^2 - b_{2m+1}\right)\right.\\ &\qquad\left.+\frac{1}{2}(\delta-1)\left(2\lambda m^2(m+2)+m(3\lambda\delta+2)+\delta\right) b_{m+1}^2\right]\\ &=\alpha q_{2m} + \frac{1}{2}\alpha(\alpha-1) q_m^2.\end{aligned} \tag{2.12}$$

From (2.9) and (2.11) we get

$$p_m = -q_m \tag{2.13}$$

and

$$\frac{2(m+\delta)^2(\lambda m+1)^2 b_{m+1}^2}{\tau^2} = \alpha^2\left(p_m^2 + q_m^2\right) \tag{2.14}$$

From (2.10), (2.12) and (2.14), we find

$$\begin{aligned}&\frac{\left[(m+1)(2m+\delta)(2\lambda m+1) + (\delta-1)\left(2\lambda m^2(m+2)+m(3\lambda\delta+2)+\delta\right)\right] b_{m+1}^2}{\tau}\\ &= \alpha\left(p_{2m}+q_{2m}\right) + \frac{\alpha(\alpha-1)}{2}\left(p_m^2+q_m^2\right)\\ &= \alpha\left(p_{2m}+q_{2m}\right) + \frac{\alpha(\alpha-1)}{2}\left(\frac{2((m+\delta)(\lambda m+1))^2 b_{m+1}^2}{\alpha^2\tau^2}\right).\end{aligned} \tag{2.15}$$

Therefore, we have

$$b_{m+1}^2 = \frac{\tau^2\alpha^2\left(p_{2m}+q_{2m}\right)}{\tau\left[\alpha(2m+\delta)(2\lambda m+1)(m+1)+\alpha(\delta-1)\left(2\lambda m^2(m+2)+m(3\lambda\delta+2)+\delta\right)\right]-(\alpha-1)[(m+\delta)(\lambda m+1)]^2}. \tag{2.16}$$

Stratifying Lemma (1.1) for coefficients p_{2m} and q_{2m}, we get

$$|b_{m+1}| \leq \frac{2\alpha|\tau|}{\sqrt{\left|\tau\left[\alpha(2m+\delta)(2\lambda m+1)(m+1)+\alpha(\delta-1)\left(2\lambda m^2(m+2)+m(3\lambda\delta+2)+\delta\right)\right]-(\alpha-1)[(m+\delta)(\lambda m+1)]^2\right|}}. \tag{2.17}$$

The final inequality provides the estimation for $|b_{m+1}|$ given in (2.3).

Next, in order to find the bound on $|b_{2m+1}|$, by subtracting (2.12) from (2.10), we get

$$\frac{[2(2m+\delta)(2\lambda m+1)]b_{2m+1}-[(m+1)(2m+\delta)(2\lambda m+1)]b_{m+1}^2}{\tau}$$
$$=\alpha\left(p_{2m}-q_{2m}\right)+\frac{\alpha(\alpha-1)}{2}\left(p_m^2-q_m^2\right). \tag{2.18}$$

By using (2.13), (2.14) and (2.18), we get

$$b_{2m+1}=\frac{(m+1)\tau^2\alpha^2\left(p_m^2+q_m^2\right)}{4[(m+\delta)(\lambda m+1)]^2}+\frac{\tau\alpha\left(p_{2m}-q_{2m}\right)}{2(2m+\delta)(2\lambda m+1)}. \tag{2.19}$$

Applying Lemma (1.1) once again for coefficients p_m, p_{2m} and q_m, q_{2m}, we obtain

$$|b_{2m+1}| \leq \frac{2(m+1)|\tau|^2\alpha^2}{[(m+\delta)(\lambda m+1)]^2}+\frac{2|\tau|\alpha}{(2m+\delta)(2\lambda m+1)}$$

This proves Theorem (2.1). □

3 Coefficient Bounds for the Function Class $\mathcal{LH}_{\Sigma_m}(\tau,\lambda,\delta;\beta)$

Definition 3.1. *A function $g(t)\in\Sigma_m$ given by (1.4) is told to be in the class $\mathcal{LH}_{\Sigma_m}(\tau,\lambda,\delta;\beta)$ if the following conditions are fulfilled:*

$$\operatorname{Re}\left(1+\frac{1}{\tau}\left[(1-\lambda)\frac{tg'(t)}{g(t)}\left(\frac{g(t)}{t}\right)^{\delta}+\lambda\left(1+\frac{tg''(t)}{g'(t)}\right)\left(g'(t)\right)^{\delta}-1\right]\right)>\beta \quad (t\in\Delta) \tag{3.1}$$

and

$$\mathrm{Re}\left(1+\frac{1}{\tau}\left[(1-\lambda)\frac{wh'(w)}{h(w)}\left(\frac{h(w)}{w}\right)^{\delta}+\lambda\left(1+\frac{wh''(w)}{h'(w)}\right)(h'(w))^{\delta}-1\right]\right)>\beta \quad (w\in\Delta) \tag{3.2}$$

where the function $h=g^{-1}$ *is given by (1.5) and* $(0\le\lambda\le 1;\delta\ge 0;\tau\in\mathbb{C}\backslash\{0\};0\le\beta<1)$.

Theorem 3.1. *Let the function* $g(t)$*, given by (1.4), be in the class* $\mathcal{LH}_{\Sigma_m}(\tau,\lambda,\delta;\beta)$*. Then*

$$|b_{m+1}|\le 2\sqrt{\frac{|\tau|(1-\beta)}{|(m+1)(2m+\delta)(2\lambda m+1)+(\delta-1)\left(2\lambda m^2(m+2)+m(3\lambda\delta+2)+\delta\right)|}} \tag{3.3}$$

and

$$|b_{2m+1}|\le\frac{2(m+1)|\tau|^2(1-\beta)^2}{[(m+\delta)(\lambda m+1)]^2}+\frac{2|\tau|(1-\beta)}{(2m+\delta)(2\lambda m+1)} \tag{3.4}$$

Proof. It follows from (3.1) and (3.2) that there exists $p,q\in\mathcal{P}$ such that

$$1+\frac{1}{\tau}\left[(1-\lambda)\frac{tg'(t)}{g(t)}\left(\frac{g(t)}{t}\right)^{\delta}+\lambda\left(1+\frac{tg''(t)}{g'(t)}\right)\left(g'(t)\right)^{\delta}-1\right]=\beta+(1-\beta)p(t) \tag{3.5}$$

and

$$1+\frac{1}{\tau}\left[(1-\lambda)\frac{wh'(w)}{h(w)}\left(\frac{h(w)}{w}\right)^{\delta}+\lambda\left(1+\frac{wh''(w)}{h'(w)}\right)\left(h'(w)\right)^{\delta}-1\right]=\beta+(1-\beta)q(w) \tag{3.6}$$

where $p(t)$ and $q(w)$ have the forms (2.7) and (2.8). It follows from (3.5) and (3.6), we get

$$\frac{1}{\tau}(m+\delta)(\lambda m+1)b_{m+1}=(1-\beta)p_m \tag{3.7}$$

$$\frac{1}{\tau}\left[(2m+\delta)(2\lambda m+1)b_{2m+1}+\frac{1}{2}(\delta-1)\left(2\lambda m^2(m+2)+m(3\lambda\delta+2)+\delta\right)b_{m+1}^2\right]$$
$$=(1-\beta)p_{2m} \tag{3.8}$$

$$-\frac{1}{\tau}(m+\delta)(\lambda m+1)b_{m+1}=(1-\beta)q_m \tag{3.9}$$

and

$$\begin{aligned}&\frac{1}{\tau}\Big[(2m+\delta)(2\lambda m+1)\left((m+1)b_{m+1}^2-b_{2m+1}\right)\\&+\frac{1}{2}(\delta-1)\left(2\lambda m^2(m+2)+m(3\lambda\delta+2)+\delta\right)b_{m+1}^2\Big]\\=&(1-\beta)q_{2m}.\end{aligned} \tag{3.10}$$

From (3.7) and (3.9), we find

$$p_m=-q_m \tag{3.11}$$

and

$$\frac{2(m+\delta)^2(\lambda m+1)^2b_{m+1}^2}{\tau^2}=(1-\beta)^2\left(p_m^2+q_m^2\right) \tag{3.12}$$

Adding (3.8) and (3.10), we have

$$\begin{aligned}&\frac{\left[(m+1)(2m+\delta)(2\lambda m+1)+(\delta-1)\left(2\lambda m^2(m+2)+m(3\lambda\delta+2)+\delta\right)\right]b_{m+1}^2}{\tau}\\&=(1-\beta)\left(p_{2m}+q_{2m}\right).\end{aligned} \tag{3.13}$$

Therefore, we obtain

$$b_{m+1}^2=\frac{\tau(1-\beta)\left(p_{2m}+q_{2m}\right)}{(m+1)(2m+\delta)(2\lambda m+1)+(\delta-1)\left(2\lambda m^2(m+2)+m(3\lambda\delta+2)+\delta\right)}. \tag{3.14}$$

Applying Lemma (1.1) for coefficients p_{2m} and q_{2m}, we readily get

$$|b_{m+1}|\leq 2\sqrt{\frac{|\tau|(1-\beta)}{|(m+1)(2m+\delta)(2\lambda m+1)+(\delta-1)\left(2\lambda m^2(m+2)+m(3\lambda\delta+2)+\delta\right)|}}$$

The final inequality provides the estimation for $|b_{m+1}|$ given in (3.3).

Next, in order to find the bound on $|b_{2m+1}|$, by subtracting (3.10) from (3.8), we get

$$\begin{aligned}&\frac{2(2m+\delta)(2\lambda m+1)b_{2m+1}-(m+1)(2m+\delta)(2\lambda m+1)b_{m+1}^2}{\tau}\\&=(1-\beta)\left(p_{2m}-q_{2m}\right).\end{aligned} \tag{3.15}$$

Or equivalently

$$b_{2m+1} = \frac{(m+1)}{2} b_{m+1}^2 + \frac{\tau(1-\beta)\left(p_{2m}-q_{2m}\right)}{2(2m+\delta)(2\lambda m+1)} \tag{3.16}$$

By substituting the value of b_{m+1}^2 from (3.12), we find

$$b_{2m+1} = \frac{(m+1)\tau^2(1-\beta)^2\left(p_m^2+q_m^2\right)}{4[(m+\delta)(\lambda m+1)]^2} + \frac{\tau(1-\beta)\left(p_{2m}-q_{2m}\right)}{2(2m+\delta)(2\lambda m+1)} \tag{3.17}$$

Applying Lemma (1.1) once again for coefficients p_{2m}, p_m, q_{2m} and q_m, we easily obtain

$$|b_{2m+1}| \leq \frac{2(m+1)|\tau|^2(1-\beta)^2}{[(m+\delta)(\lambda m+1)]^2} + \frac{2|\tau|(1-\beta)}{(2m+\delta)(2\lambda m+1)}$$

This proves Theorem (3.1). □

4 Corollaries and Consequences

If we put $\lambda = 1$ in Definition (2.1) and Definition (3.1), subsequently the classes $\mathcal{LH}_{\Sigma_m}(\tau,\lambda,\delta;\alpha)$ and $\mathcal{LH}_{\Sigma_m}(\tau,\lambda,\delta;\beta)$ shorten to the classes $\mathcal{LH}_{\Sigma_m}(\tau,\delta;\alpha)$ and $\mathcal{LH}_{\Sigma_m}(\tau,\delta;\beta)$ and thus, Theorem (2.1) and Theorem (3.1) reduce to Corollary (4.1) and Corollary (4.2), respectively.

The definitions for the classes $\mathcal{LH}_{\Sigma_m}(\tau,\delta;\alpha)$ and $\mathcal{LH}_{\Sigma_m}(\tau,\delta;\beta)$ are as follows:

Definition 4.1. *A function $g(t) \in \Sigma_m$ given by (1.4) is told to be in the class $\mathcal{LH}_{\Sigma_m}(\tau,\delta;\alpha)$ if the following conditions are fulfilled:*

$$\left|\arg\left(1+\frac{1}{\tau}\left[\left(1+\frac{tg''(t)}{g'(t)}\right)\left(g'(t)\right)^{\delta}-1\right]\right)\right| < \frac{\alpha\pi}{2} \quad (t \in \Delta)$$

and

$$\left|\arg\left(1+\frac{1}{\tau}\left[\left(1+\frac{wh''(w)}{h'(w)}\right)\left(h'(w)\right)^{\delta}-1\right]\right)\right| < \frac{\alpha\pi}{2} \quad (w \in \Delta),$$

where the function $h = g^{-1}$ is given by (1.5) and $(\delta \geq 0; \tau \in \mathbb{C}\backslash\{0\}; 0 < \alpha \leq 1)$.

Definition 4.2. *A function $g(t) \in \Sigma_m$ given by (1.4) is told to be in the class $\mathcal{LH}_{\Sigma_m}(\tau, \delta; \beta)$ if the following conditions are fulfilled:*

$$\operatorname{Re}\left(1+\frac{1}{\tau}\left[\left(1+\frac{tg''(t)}{g'(t)}\right)\left(g'(t)\right)^{\delta}-1\right]\right)>\beta \quad (t \in \Delta)$$

and

$$\operatorname{Re}\left(1+\frac{1}{\tau}\left[\left(1+\frac{wh''(w)}{h'(w)}\right)\left(h'(w)\right)^{\delta}-1\right]\right)>\beta \quad (w \in \Delta)$$

where the function $h = g^{-1}$ is given by (1.5) and $(\delta \geq 0; \tau \in \mathbb{C}\backslash\{0\}; 0 \leq \beta < 1)$.

Corollary 4.1. *Let $g(t)$ given by (1.4) be in the class $\mathcal{LH}_{\Sigma_m}(\tau, \delta; \alpha)$. Then*

$$|b_{m+1}| \leq \frac{2\alpha|\tau|}{\sqrt{|\tau[\alpha(2m+\delta)(2m+1)(m+1)+\alpha(\delta-1)(2m^2(m+2)+m(3\delta+2)+\delta)]-(\alpha-1)[(m+\delta)(m+1)]^2|}}$$

and

$$|b_{2m+1}| \leq \frac{2|\tau|^2\alpha^2(m+1)}{[(m+\delta)(m+1)]^2}+\frac{2\alpha|\tau|}{(2m+\delta)(2m+1)}$$

Corollary 4.2. *Let $g(t)$ given by (1.4) be in the class $\mathcal{LH}_{\Sigma_m}(\tau, \delta; \beta)$. Then*

$$|b_{m+1}| \leq 2\sqrt{\frac{|\tau|(1-\beta)}{|(m+1)(2m+\delta)(2m+1)+(\delta-1)\left(2m^2(m+2)+m(3\delta+2)+\delta\right)|}}$$

and

$$|b_{2m+1}| \leq \frac{2(m+1)|\tau|^2(1-\beta)^2}{[(m+\delta)(m+1)]^2}+\frac{2|\tau|(1-\beta)}{(2m+\delta)(2m+1)}.$$

For one-fold symmetric analytic biunivalent functions, the classes $\mathcal{LH}_{\Sigma_m}(\tau, \lambda, \delta; \alpha)$ and $\mathcal{LH}_{\Sigma_m}(\tau, \lambda, \delta; \beta)$ shorten to the classes $\mathcal{LH}_{\Sigma}(\tau, \lambda, \delta; \alpha)$ and $\mathcal{LH}_{\Sigma}(\tau, \lambda, \delta; \beta)$ and thus, Theorem (2.1) and Theorem (3.1) shorten to Corollary (4.2) and Corollary (4.3), respectively.

The definitions for the classes $\mathcal{LH}_{\Sigma}(\tau, \lambda, \delta; \alpha)$ and $\mathcal{LH}_{\Sigma}(\tau, \lambda, \delta; \beta)$ are as follows:

Definition 4.3. *A function $g(t) \in \Sigma$ given by (1.1) is told to be in the class $\mathcal{LH}_{\Sigma}(\tau,\lambda,\delta;\alpha)$ if the following conditions are fulfilled:*

$$\left|\arg\left(1+\frac{1}{\tau}\left[(1-\lambda)\frac{tg'(t)}{g(t)}\left(\frac{g(t)}{t}\right)^{\delta}+\lambda\left(1+\frac{tg''(t)}{g'(t)}\right)(g'(t))^{\delta}-1\right]\right)\right|<\frac{\alpha\pi}{2}\quad(t\in\Delta)$$

and

$$\left|\arg\left(1+\frac{1}{\tau}\left[(1-\lambda)\frac{wh'(w)}{h(w)}\left(\frac{h(w)}{w}\right)^{\delta}+\lambda\left(1+\frac{wh''(w)}{h'(w)}\right)(h'(w))^{\delta}-1\right]\right)\right|\frac{\alpha\pi}{2}\quad(w\in\Delta),$$

where the function $h=g^{-1}$ is given by (1.2) and $(0\leq\lambda\leq 1;\delta\geq 0,\tau\in\mathbb{C}\backslash\{0\};0<\alpha\leq 1)$.

Definition 4.4. *A function $g(t) \in \Sigma$ given by (1.1) is told to be in the class $\mathcal{LH}_{\Sigma}(\tau,\lambda,\delta;\beta)$ if the following conditions are fulfilled:*

$$\operatorname{Re}\left(1+\frac{1}{\tau}\left[(1-\lambda)\frac{\mathrm{t}g'(t)}{g(t)}\left(\frac{g(t)}{t}\right)^{\delta}+\lambda\left(1+\frac{\mathrm{t}g''(t)}{g'(t)}\right)\left(g'(t)\right)^{\delta}-1\right]\right)>\beta\quad(t\in\Delta)$$

and

$$\operatorname{Re}\left(1+\frac{1}{\tau}\left[(1-\lambda)\frac{wh'(w)}{h(w)}\left(\frac{h(w)}{w}\right)^{\delta}+\lambda\left(1+\frac{wh''(s)}{h'(s)}\right)(h'(w))^{\delta}-1\right]\right)>\beta\quad(w\in\Delta)$$

where the function $h=g^{-1}$ is given by (1.2) and $(0\leq\lambda\leq 1;\delta\geq 0;\tau\in\mathbb{C}\backslash\{0\};0\leq\beta<1)$.

Corollary 4.3. *Let $g(t)$ given by (1.1) be in the class $\mathcal{LH}_{\Sigma}(\tau,\lambda,\delta;\alpha)$. Then*

$$|b_2|\leq\frac{2\alpha|\tau|}{\sqrt{|\tau\left[\alpha(\delta+2)(2(2\lambda+1)+(\delta-1)(3\lambda+1))-(\alpha-1)(\delta+1)^2(\lambda+1)^2\right]|}}$$

and

$$|b_3|\leq\frac{4|\tau|^2\alpha^2}{[(\delta+1)(\lambda+1)]^2}+\frac{2\alpha|\tau|}{(\delta+2)(2\lambda+1)}.$$

Corollary 4.4. *Let $g(t)$ given by (1.1) be in the class $\mathcal{LH}_{\Sigma}(\tau,\lambda,\delta;\beta)$. Then*

$$|b_2|\leq 2\sqrt{\frac{|\tau|(1-\beta)}{|(\delta+2)((\lambda+1)+\delta(3\lambda+1))|}}.$$

and

$$|b_3|\leq\frac{4|\tau|^2(1-\beta)^2}{[(\delta+1)(\lambda+1)]^2}+\frac{2|\tau|(1-\beta)}{(\delta+2)(2\lambda+1)}.$$

If we set $\lambda = 1$ and $m = 1$ in Definition (2.1) and Definition (3.1), then the classes $\mathcal{LHH}_{\Sigma_m}(\tau,\lambda,\delta;\alpha)$ and $\mathcal{LHE}_{\Sigma_m}(\tau,\lambda,\delta;\beta)$ shorten to the classes $\mathcal{LHH}_{\Sigma}(\tau,\delta;\alpha)$ and $\mathcal{LHH}_{\Sigma}(\tau,\delta;\beta)$ and thus, Theorem (2.1) and Theorem (3.1) shorten to Corollary (4.5) and Corollary (4.6), respectively.

The classes $\mathcal{LH}_{\Sigma}(\tau,\delta;\alpha)$ and $\mathcal{LH}_{\Sigma}(\tau,\delta;\beta)$, are respectively defined as follows:

Definition 4.5. *A function $g(t) \in \Sigma$ given by (1.1) is told to be in the class $\mathcal{LH}_{\Sigma}(\tau,\delta;\alpha)$ if the following conditions are fulfilled:*

$$\left|\arg\left(1+\frac{1}{\tau}\left[\left(1+\frac{tg''(t)}{g'(t)}\right)\left(g'(t)\right)^{\delta}-1\right]\right)\right| < \frac{\alpha\pi}{2} \quad (t\in\Delta)$$

and

$$\left|\arg\left(1+\frac{1}{\tau}\left[\left(1+\frac{wh''(w)}{h'(w)}\right)\left(h'(w)\right)^{\delta}-1\right]\right)\right| < \frac{\alpha\pi}{2} \quad (w\in\Delta)$$

where the function $h = g^{-1}$ is given by (1.2) and $(\delta \geq 0; 0 < \alpha \leq 1)$.

Definition 4.6. *A function $g(t) \in \Sigma$ given by (1.1) is told to be in the class $\mathcal{LH}_{\Sigma}(\tau,\delta;\beta)$ if the following conditions are fulfilled:*

$$\operatorname{Re}\left(1+\frac{1}{\tau}\left[\left(1+\frac{tg''(t)}{g'(t)}\right)\left(g'(t)\right)^{\delta}-1\right]\right) > \beta \quad (t\in\Delta)$$

and

$$\operatorname{Re}\left(1+\frac{1}{\tau}\left[\left(1+\frac{wh''(w)}{h'(w)}\right)\left(h'(w)\right)^{\delta}-1\right]\right) > \beta \quad (w\in\Delta)$$

where the function $h = g^{-1}$ is given by (1.2) and $(\delta \geq 0; 0 \leq \beta < 1)$.

Corollary 4.5. *Let $g(t)$ given by (1.1) be in the class $\mathcal{LH}_{\Sigma}(\tau,\delta;\alpha)$. Then*

$$|b_2| \leq \frac{2\alpha|\tau|}{\sqrt{|\tau[2\alpha(\delta+2)(2\delta+1)]-4(\alpha-1)(\delta+1)^2|}} \quad \text{and} \quad |b_3| \leq \frac{\alpha^2|\tau|^2}{(\delta+1)^2}+\frac{2\alpha|\tau|}{3(\delta+2)}.$$

Corollary 4.6. *Let $g(t)$ given by (1.1) be in the class $\mathcal{LH}_{\Sigma}(\tau,\delta;\beta)$. Then*

$$|b_2| \leq \sqrt{\frac{2|\tau|(1-\beta)}{(\delta+2)(2\delta+1)}} \quad \text{and} \quad |b_3| \leq \frac{|\tau|^2(1-\beta)^2}{(\delta+1)^2}+\frac{2|\tau|(1-\beta)}{3(\delta+2)}.$$

Remark 4.1. *For m-fold symmetric analytic biunivalent functions:*

1. *For $\lambda = 0, \delta = 0$ and $\tau = 1$ in Theorems (2.1) and (3.1), we obtain the corresponding results given by Altinkaya and yalÃ§in [1].*

2. *For $\lambda = 1$ and $\delta = 0$ in Theorems (2.1) and (3.1), we obtain the corresponding results given by Kumar et al . [6].*

3. *For $\delta = 0$ and $\tau = 1$ in Theorems (2.1) and (3.1), we obtain the corresponding results given by Sivasubramanlan and Sivakumar [12].*

4. *For $\lambda = 0, \delta = 1$ and $\tau = 1$ in Theorems (2.1) and (3.1), we obtain the corresponding results given Srivastava et al. [16].*

Remark 4.2. *For one-fold symmetric analytic biunivalent functions:*

1. *For $\lambda = 0, \delta = 0$ and $\tau = 1$ in Theorems (2.1) and (3.1), we obtain the corresponding results given by Murugusundaramoorthy at el. [9].*

2. *For $\lambda = 1$ and $\delta = 0$ in Theorems (2.1) and (3.1), we obtain the corresponding results given by Kumar at el . [6].*

3. *For $\delta = 0$ and $\tau = 1$ in Theorems (2.1) and (3.1), we obtain the corresponding results given by Li and Wang [8].*

4. *For $\lambda = 0, \delta = 1$ and $\tau = 1$ in Theorems (2.1) and (3.1), we obtain the corresponding results given by Srivastava et al. [15].*

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