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The Weibull Distribution with Estimable Shift Parameter

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Abstract

In this paper, a new lifetime distribution known as the Shifted Weibull (SHW) distribution with a shift parameter that does not necessarily determine the lower boundary of the support variable is proposed and studied. The study is motivated by the Shifted Exponential (SHE), Shifted Exponential-G (SHE-G) family of distributions and centred on shift parameter that is estimable. Some properties were derived. Estimation techniques namely; the maximum likelihood, least squares, weighted least squares, maximum product spacing, Cramer-von-Mises, Anderson-Darling and the right-tailed Anderson-Darling estimations are used. Two real data sets were deployed to show the usefulness and superiority of the proposed distribution relative to the parent distribution and other competing distributions. The weighted least squares estimator gave the best classical estimates of the parameters compared to other methods considered.

1 Introduction

In probability modeling of life phenomena, researchers often define the range of the values of the support variable in the interval $(0, \infty)$. However, this situation

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do not really hold as portrayed in the literature. Therefore, such models are not usually the exact representation of realities which consequently increase the degree of uncertainty in the inference made based on such models. The primary purpose of shifted distributions therefore is to take into consideration the uniqueness of every data set encountered in modeling. To this end, the shift parameter is used to adequately represent the actual initial boundary of the support variable in probability modeling. This dimension in modeling is recently receiving great attention in the statistical literature.

Many useful generalizations of existing distributions have advanced the field of probability distribution in the recent times. Notable among them are Klakattawi et al. [1], Aljarrah et al. [2], Alzaghal et al. [3], Shah et al. [4], Zubair et al. [5], Torrisi [6]. Few decades ago, generating new distributions by extensions, modification and other innovative approaches have dominated the literature. A number of works are worthy of mention.

Eghwerido et al. [7] proposed the Shifted Exponential-G family of distribution using Exponential distribution as the parent distribution. Eghwerido and Agu [8] proposed the shifted-Gompertz-G family of distribution using Gopertz distribution as the parent distribution. Torrisi [6] studied the Coulomb-Boltzmann-Shifted distribution. Onyekwere and Obulezi [9] introduced Chris-Jerry distribution. Onyekwere et al. [10] modified Shanker distribution using quadratic rank transmutation map. Anabike et al. [11] studied inference on the parameters of Zubair-Exponential distribution using the survival times of guinea pigs. Ashour and Eltehiwy [12] developed the Exponentiated power Lindley distribution. Warahena-Liyanage and Pararai [13] proposed a generalized power Lindley distribution with applications. MirMostafaee et al. [14] studied the exponentiated generalized power Lindley distribution: Properties and applications. Jan et al. [15] worked on the Exponentiated inverse power lindley distribution and its applications. Crow and Shimizu [16] introduced the Lognormal distributions. Heyde [17] further investigated a property of the lognormal distribution. Cohen and Whitten [18] explored the estimation in the three-parameter lognormal distribution. Zeghdoudi et al. [19] developed the

Lindley pareto distribution. Lazri and Zeghdoudi [20] further investigated the Lindley-Pareto distribution with its properties and application. Asgharzadeh et al. [21] developed the Pareto Poisson-Lindley distribution with applications. Cakmakyapan and Gamze [22] proposed the Lindley family of distributions with the properties and applications. Other relevant studies include Musa et al. [23], Musa et al. [24], Innocent et al. [25] and Obulezi et al. [26].

The remainder of this paper is organized in the following order. In Section 2, we derived the shifted distribution with visualization of the density function, distribution function, reliability function and hazard function. In Section 3, some useful properties which includes the moment and quantile function, entropy and the asymptotic behaviour of the shifted weibull distribution was also studied. In Section 4, we estimate the parameters of the proposed distribution using seven non-Bayesian approaches. In Section 5, we demonstrated the usefulness and superiority of the proposed distribution using two real life data sets and the paper is concluded in Section 6.

2 The Shifted Weibull (SHW) Distribution

Let $X \sim Weibull(\alpha, \beta)$, then $X \sim SHW(\alpha, \beta, \gamma)$ where γ is the shift parameter if the probability density function (p.d.f) and the cumulative distribution function (c.d.f) are defined respectively as

$$f(x) = \alpha\beta(x-\gamma)^{\beta-1}e^{-\alpha(x-\gamma)^{\beta}}; \quad x \geq \gamma \quad \alpha, \beta, \gamma > 0 \tag{1}$$

and

$$F(x) = 1 - e^{-\alpha(x-\gamma)^{\beta}}. \tag{2}$$

The survival and hazard rate function are respectively

$$S(x) = e^{-\alpha(x-\gamma)^{\beta}} \tag{3}$$

and

$$h(x) = \alpha\beta(x-\gamma)^{\beta-1}. \tag{4}$$

The hazard function is such that

$$\lim_{x\to\infty} h(x) = \infty; \quad \lim_{x\to\gamma} h(x) = 0.$$

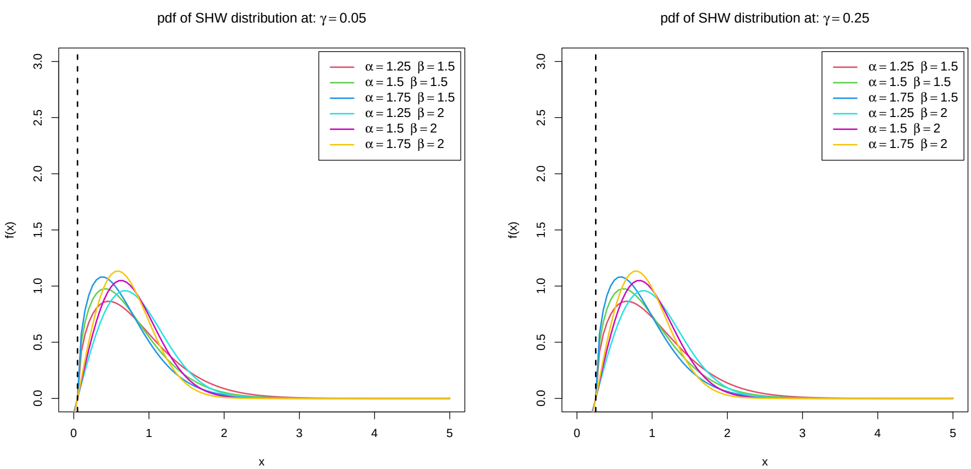
This implies that the hazard function can be either a monotone non-decreasing or a monotone non-increasing function. Similarly

$$\lim_{x\to\infty} F(x) = 1; \quad \lim_{x\to\gamma} F(x) = 0$$

and

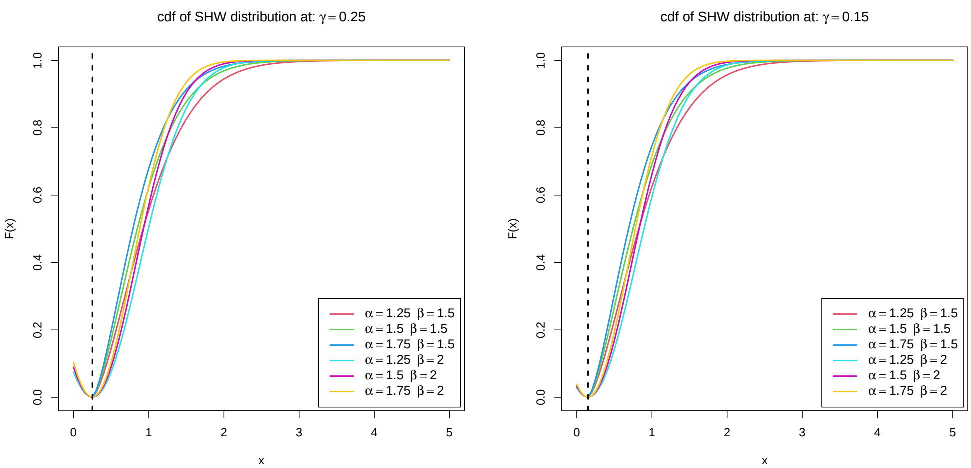
$$\lim_{x\to\infty} S(x) = 0; \quad \lim_{x\to\gamma} S(x) = 1.$$

See graphs below:



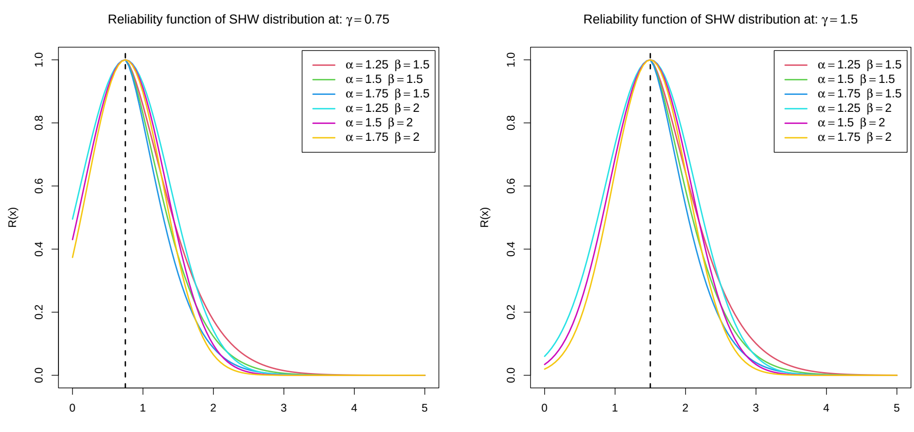
(a) (b)

Figure 1



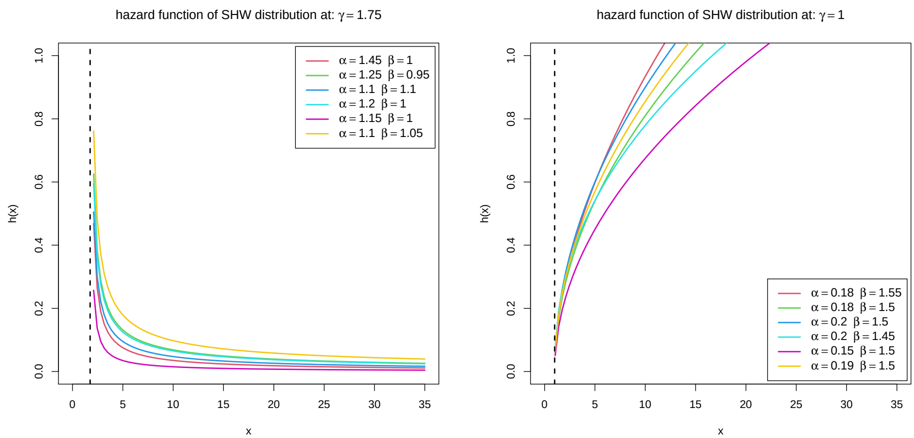
(a) (b)

Figure 2



(a) (b)

Figure 3



(a) (b)

Figure 4

The hazard rate of the Shifted Weibull distribution is a montone non-increasing function as well as montone non-decreasing function. This is obvious from the hazard function plots in Figure 4a and 4b.

3 Properties of the Shifted Weibull Distribution

Definition 3.1. Let $X \sim SHW(\alpha, \beta, \gamma)$, then the r^{th} crude moment is

$$\mu_r' = \sum_{k=0}^{\beta-1}\sum_{j=0}^{r+k}\binom{\beta-1}{k}\binom{r+k}{j}\gamma^{\beta+r-j-1}\alpha^{\frac{\beta-j-1}{\beta}}\Gamma\left(\frac{j+1}{\beta}\right). \tag{5}$$

Proof. The r^{th} crude moment of a distribution with p.d.f $f(x)$ is defined as

$$\mu_r' = E(x^r) = \int_{\gamma}^{\infty} x^r f(x)dx \tag{6}$$

substituting the p.d.f of the proposed SHW distribution in equation (1), we have

$$\mu_r^{'} = \alpha\beta \int_{\gamma}^{\infty} x^r \left(x-\gamma\right)^{\beta-1} e^{-\alpha(x-\gamma)^{\beta}} dx. \tag{7}$$

Applying binomial expansion yields

$$\mu_r^{'} = \sum_{k=0}^{\beta-1} \binom{\beta-1}{k} \gamma^{\beta-k} \int_{\gamma}^{\infty} x^{k+r} e^{-\alpha(x-\gamma)^{\beta}} dx \tag{8}$$

let $u = \alpha\left(x-\gamma\right)^{\beta}; \quad du = \alpha\beta\left(x-\gamma\right)^{\beta-1} dx$

$$\mu_r^{'} = \frac{1}{\gamma\alpha^{\frac{r+k-\beta+1}{\beta}}} \sum_{k=0}^{\beta-1} \binom{\beta-1}{k} \gamma^{\beta-k} \int_{0}^{\infty} u^{\frac{1}{\beta}} \left(u^{\frac{1}{\beta}} + \alpha^{\frac{1}{\beta}}\gamma\right)^{r+k} e^{-u} du. \tag{9}$$

Applying binomial expansion, we have

$$\begin{aligned} &= \frac{1}{\gamma\alpha^{\frac{r+k-\beta+1}{\beta}}} \sum_{k=0}^{\beta-1} \binom{\beta-1}{k} \gamma^{\beta-k} \sum_{j=0}^{r+k} \binom{r+k}{j} \left(\alpha^{\frac{1}{\beta}}\gamma\right)^{r+k-j} \int_{0}^{\infty} u^{\frac{j}{\beta}+\frac{1}{\beta}-1} e^{-u} du \\ &= \frac{1}{\gamma\alpha^{\frac{r+k-\beta+1}{\beta}}} \sum_{k=0}^{\beta-1} \binom{\beta-1}{k} \gamma^{\beta-k} \sum_{j=0}^{r+k} \binom{r+k}{j} \left(\alpha^{\frac{1}{\beta}}\gamma\right)^{r+k-j} \Gamma\left(\frac{j+1}{\beta}\right). \end{aligned} \tag{10}$$

Therefore

$$\mu_r^{'} = \sum_{k=0}^{\beta-1} \sum_{j=0}^{r+k} \binom{\beta-1}{k} \binom{r+k}{j} \gamma^{\beta+r-j-1} \alpha^{\frac{\beta-j-1}{\beta}} \Gamma\left(\frac{j+1}{\beta}\right). \tag{11}$$

□

Definition 3.2. Let $X \sim SHW(\alpha, \beta, \gamma)$, then the mean of X is given as

$$\mu = \sum_{k=0}^{\beta-1} \sum_{j=0}^{k+1} \binom{\beta-1}{k} \binom{k+1}{j} \gamma^{\beta-j} \alpha^{\frac{\beta-j-1}{\beta}} \Gamma\left(\frac{j+1}{\beta}\right). \tag{12}$$

This is obtained by substituting 1 for r in equation (3.1)

The $2^{nd}, 3^{rd}$ and 4^{th} crude moments are obtained by replacing r in equation (3.1) with $2, 3$ and 4 respectively. Hence, we obtain

$$\mu_2^{'} = \sum_{k=0}^{\beta-1}\sum_{j=0}^{k+2}\binom{\beta-1}{k}\binom{k+2}{j}\gamma^{\beta+1-j}\alpha^{\frac{\beta-j-1}{\beta}}\Gamma\left(\frac{j+1}{\beta}\right) \tag{13}$$

$$\mu_3^{'} = \sum_{k=0}^{\beta-1}\sum_{j=0}^{k+3}\binom{\beta-1}{k}\binom{k+3}{j}\gamma^{\beta+2-j}\alpha^{\frac{\beta-j-1}{\beta}}\Gamma\left(\frac{j+1}{\beta}\right) \tag{14}$$

and

$$\mu_4^{'} = \sum_{k=0}^{\beta-1}\sum_{j=0}^{k+4}\binom{\beta-1}{k}\binom{k+4}{j}\gamma^{\beta+3-j}\alpha^{\frac{\beta-j-1}{\beta}}\Gamma\left(\frac{j+1}{\beta}\right). \tag{15}$$

Definition 3.3. Let $X \sim SHW(\alpha, \beta, \gamma)$, then the quantile function is

$$Q(u) = \left[-\frac{\ln(1-u)}{\alpha}\right]^{\frac{1}{\beta}} + \gamma; \quad \text{where} \quad u \sim U(0,1). \tag{16}$$

Proof. Substitute u for $F(x)$ and $Q(u)$ for x in equation (2) being the c.d.f of the proposed SHW distribution where $u \sim U(0,1)$, we obtain

$$u = 1 - e^{-\alpha(Q(u)-\gamma)^{\beta}}. \tag{17}$$

It is easy to see by simplifying (12) that

$$Q(u) = \left[-\frac{\ln(1-u)}{\alpha}\right]^{\frac{1}{\beta}} + \gamma. \tag{18}$$

□

Definition 3.4 (Entropy and Asymptotic Behaviour of SHW Distribution)**.** Entropy is a measure of the number of ways a system can be arranged, often taken to be a measure of "disorder". It is an information measure for non-negative $\omega \neq 1$. The Rény Entropy for a SHW distributed random variable X is

$$R_{\omega}(x) = \lim_{n\to\infty}\Big(I_{\omega}(f_n) - \log n\Big) = \frac{1}{1-\omega}\log\int_0^{\infty} f^{\infty}(x)dx. \tag{19}$$

For $\omega \to 1$, we have the special case of Shannon Entropy $R_s(x)$

$$\begin{aligned}
R_\omega(x) &= \frac{1}{1-\omega}\log\int_\gamma^\infty \left(\alpha\beta\,(x-\gamma)^{\beta-1}\,e^{-\alpha(x-\gamma)^\beta}\right)^w dx \\
&= \frac{1}{1-\omega}\log\left[\alpha^w\beta^w\int_\gamma^\infty (x-\gamma)^{\beta w-w}\,e^{-\alpha(x-\gamma)^\beta}dx\right] \\
&\text{Let } u=\alpha w(x-\gamma)^\beta \\
R_\omega(x) &= \frac{1}{1-\omega}\log\left[\frac{\alpha^{\frac{w+1}{\beta}}\beta^{w-1}}{w^{\frac{\beta w-w-1}{\beta}}}\int_0^\infty u^{w-1-\frac{w}{\beta}-\frac{1}{\beta}}e^{-u}du\right] \\
&= \frac{1}{1-\omega}\log\left[\frac{\alpha^{\frac{w+1}{\beta}}\beta^{w-1}}{w^{\frac{\beta w-w-1}{\beta}}}\gamma\left(\frac{\beta w-w-1}{\beta}\right)\right] \\
&= \frac{1}{1-\omega}\left\{\frac{w+1}{w}\log\alpha+(w-1)\log\beta+\right. \\
&\qquad\left.\log\left[\gamma\left(\frac{\beta w-w-1}{\beta}\right)\right]+\frac{\beta w-w-1}{\beta}\log w\right\}
\end{aligned} \tag{20}$$

The asymptotic behavior of the SHW distributed random variable is obtained by taking the limit of the pdf as $x \to 0$ and as $x \to \infty$.

$$\lim_{x\to 0}\alpha\beta(x-\gamma)^{\beta-1}e^{-\alpha(x-\gamma)^\beta}=0 \tag{21}$$

and

$$\lim_{x\to \infty}\alpha\beta(x-\gamma)^{\beta-1}e^{-\alpha(x-\gamma)^\beta}=0. \tag{22}$$

Definition 3.5 (Distribution of the Order Statistics)**.** Suppose $X_1, X_1, .., X_n$ is a random sample of $X_{(r)}; (r = 1, 2, ..., n)$ are the r^{th} order statistics obtained by arranging X_r in ascending order of magnitude $\ni X_1 \le X_2 \le ... \le X_r$ and $X_1 = min(X_1, X_2, ..., X_r), X_r = max(X_1, X_2, ..., X_r)$ then the probability density function of the r^{th} order statistics is given by

$$f_{r:n}(x;\lambda,\theta)=\frac{n!}{(r-1)!(n-r)!}f_{SHW}(x;\lambda,\theta)\left[F_{SHW}(x;\theta)\right]^{r-1}\left[1-F_{SHW}(x;\lambda,\theta)\right]^{n-r}. \tag{23}$$

where $f_{SHW}(x;\lambda,\theta)$ and $F_{SHW}(x;\lambda,\theta)$ are the pdf and cdf of SHW distribution respectively. Hence, we have

$$f_{r:n}(x;\lambda,\theta) = \frac{n!}{(r-1)!(n-r)!}\alpha\beta(x-\gamma)^{\beta-1}e^{-\alpha(x-\gamma)^{\beta}}\left\{1-e^{-\alpha(x-\gamma)^{\beta}}\right\}^{r-1}\left\{e^{-\alpha(x-\gamma)^{\beta}}\right\}^{n-r}. \tag{24}$$

The pdf of the largest order statistics is gotten by replacing r with n, that is $r = n$

$$f_{n:n}(x;\lambda,\theta) = \alpha\beta n(x-\gamma)^{\beta-1}e^{-\alpha(x-\gamma)^{\beta}}\left\{1-e^{-\alpha)(x-\gamma)^{\beta}}\right\}^{n-1}. \tag{25}$$

The pdf of the smallest order statistics is gotten by replacing r with 1, that is $r = 1$

$$f_{1:n}(x;\lambda,\theta) = \alpha\beta n(x-\gamma)^{\beta-1}e^{-\alpha(x-\gamma)^{\beta}}\left\{e^{-\alpha(n-1)(x-\gamma)^{\beta}}\right\}. \tag{26}$$

4 Classical Methods of Estimation

In this section, we derive the estimates of the parameters using Maximum Likelihood estimation, Least squares estimation, weighted least squares estimation, maximum product spacing estimation, cramer von mises estimation, Anderson Darling estimation and right-tailed Anderson Darling estimation.

Definition 4.1 (Maximum Likelihood Estimation)**.** Let $(x_1, x_2, ..., x_n)$ be n random samples drawn from SHW distribution. Then the likelihood function is given as

$$
\begin{aligned}
\ell\left(x;\alpha,\beta,\gamma\right) &= \prod_{i=1}^{n} f_{SHW}\left(x;\alpha,\beta,\gamma\right) \\
&= \prod_{i=1}^{n} \alpha\beta(x-\gamma)^{\beta-1} e^{-\alpha(x-\gamma)^{\beta}} \\
&= \alpha^2\beta^2 e^{-\alpha\sum_{i=1}^{n}(x-\gamma)^{\beta}} \prod_{i=1}^{n}(x-\gamma)^{\beta-1} \\
\ln\ell &= n\ln\alpha + n\ln\beta - \alpha\sum_{i=1}^{n}(x-\gamma)^{\beta} + (\beta-1)\sum_{i=1}^{n}\ln(x-\gamma) \\
\ln\ell\alpha &= \frac{n}{\beta} - \sum_{i=1}^{n}(x-\gamma)^{\beta} = 0 \\
\hat{\alpha} &= \frac{n}{\sum_{i=1}^{n}(x-\gamma)^{\beta}} \\
\ln\ell\beta &= \frac{n}{\beta} - \alpha\sum_{i=1}^{n}(x-\gamma)^{\beta}\sum_{i=1}^{n}\ln(x-\gamma) + \sum_{i=1}^{n}(x-\gamma) = 0 \\
\ln\ell\gamma &= \alpha\beta\sum_{i=1}^{n}(x-\gamma)^{\beta-1} - (\beta-1)\sum_{i=1}^{n}\frac{1}{x-\gamma} = 0
\end{aligned}
\tag{27}
$$

β and γ has no closed-form solution, hence will be solved iteratively in R using Newton-Raphson's iterative algorithm.

We obtain approximate confidence intervals of the parameters based on the asymptotic distribution of the MLEs of the unknown parameters $\Phi = (\alpha, \beta, \gamma)$. The asymptotic variances and covariances of the MLE for parameters α, β and γ are given by elements of the inverse of the Fisher information matrix. It is not easy to obtain the exact mathematical expressions for the above-mentioned equations. Therefore, we give the approximate (observed) asymptotic variance-covariance matrix for the MLE, which is obtained by dropping the expectation operator E

$$I_{ij}^{-1}(\alpha,\beta,\gamma) = \begin{bmatrix} [2]\psi\alpha & \psi\alpha\beta & \psi\alpha\gamma \\ \psi\beta\alpha & [2]\psi\beta & \psi\beta\gamma \\ \psi\gamma\alpha & \psi\gamma\beta & [2]\psi\gamma \end{bmatrix}^{-1} = \begin{bmatrix} var(\hat{\alpha}) & cov(\hat{\alpha},\hat{\beta}) & var(\hat{\alpha},\hat{\gamma}) \\ cov(\hat{\beta},\hat{\alpha}) & var(\hat{\beta}) & cov(\hat{\beta},\hat{\gamma}) \\ cov(\hat{\gamma},\hat{\alpha}) & cov(\hat{\gamma},\hat{\beta}) & var(\hat{\gamma}) \end{bmatrix} \tag{28}$$

where

$$\begin{aligned} [2]\psi\alpha &= -\frac{n}{\alpha^2} \\ [2]\psi\beta &= -\frac{n}{\beta^2} - \alpha\beta\sum_{i=1}^{n}\ln(x-\gamma)\sum_{i=1}^{n}(x-\gamma)^{\beta}\sum_{i=1}^{n}\ln(x-\gamma) \\ [2]\psi\gamma &= -\alpha\beta(\beta-1)\sum_{i=1}^{n}(x-\gamma)^{\beta-2} + (\beta-1)\sum_{i=1}^{n}\frac{1}{(x-\gamma)^2} \\ \psi\alpha\beta &= -\sum_{i=1}^{n}\ln(x-\gamma)^{\beta}\sum_{i=1}^{n}\ln(x-\gamma) \\ \psi\beta\gamma &= -\alpha\beta(x-\gamma)^{\beta-2} - \sum_{i=1}^{n}\frac{1}{(x-\gamma)^2} \\ \psi\gamma\alpha &= \beta\sum_{i=1}^{n}(x-\gamma)^{\beta-1}. \end{aligned} \tag{29}$$

Approximate confidence intervals for α, β and γ can be obtained. Hence, a $100(1-\tau)\%$ confidence intervals for the parameters α, β and γ are

$$\hat{\alpha} \pm Z_{\frac{\tau}{2}}\sqrt{var(\hat{\alpha})}; \quad \hat{\beta} \pm Z_{\frac{\tau}{2}}\sqrt{var(\hat{\beta})}; \quad \hat{\gamma} \pm Z_{\frac{\tau}{2}}\sqrt{var(\hat{\gamma})} \tag{30}$$

where $Z_{\frac{\tau}{2}}$ is the percentile standard normal distribution with right-tailed probability.

Definition 4.2 (Least Squares Estimation (LSE))**.** The Least Squares Estimation due to Swain et al. [27] to estimate the parameters of Beta distribution. Using the deductions from the work of Swain et al. [27], we write

$$E[F(x_{i:n}|\alpha,\beta\gamma)] = \frac{i}{n+1}.$$

$$V[F(x_{i:n}|\alpha,\beta\gamma)] = \frac{i(n-i+1)}{(n+1)^2(n+2)}.$$

The least squares estimates $\hat{\alpha}_{LSE}$, $\hat{\beta}_{LSE}$ and $\hat{\gamma}_{LSE}$ of the parameters α, β and γ are obtained by minimizing the function $L(\alpha,\beta,\gamma)$ with respect to α, β and γ

$$L(\alpha,\beta,\gamma) = \underset{(\alpha)}{\arg\min} \sum_{i=1}^{n} \Big[F(x_{i:n}|\alpha,\beta,\gamma) - \frac{i}{n+1}\Big]^2. \tag{31}$$

The estimates are obtained by solving the following non-linear equations

$$\begin{aligned}
&\sum_{i=1}^{n} \Big[F(x_{i:n}|\alpha,\beta,\gamma) - \frac{i}{n+1}\Big]^2 \Delta_1(x_{i:n}|\alpha,\beta,\gamma) = 0\\
&\sum_{i=1}^{n} \Big[F(x_{i:n}|\alpha,\beta,\gamma) - \frac{i}{n+1}\Big]^2 \Delta_2(x_{i:n}|\alpha,\beta,\gamma) = 0\\
&\sum_{i=1}^{n} \Big[F(x_{i:n}|\alpha,\beta,\gamma) - \frac{i}{n+1}\Big]^2 \Delta_3(x_{i:n}|\alpha,\beta,\gamma) = 0
\end{aligned} \tag{32}$$

where

$$\begin{aligned}
\Delta_1(x_{i:n}|\alpha,\beta,\gamma) &= (x-\gamma)^{\beta}\, e^{-\alpha(x-\gamma)^{\beta}}\\
\Delta_2(x_{i:n}|\alpha,\beta,\gamma) &= \alpha\,(x-\gamma)^{\beta} \ln{(x-\gamma)} e^{-\alpha(x-\gamma)^{\beta}}\\
\Delta_3(x_{i:n}|\alpha,\beta,\gamma) &= -\alpha\,(x-\gamma)^{\beta-1}\, e^{-\alpha(x-\gamma)^{\beta}}.
\end{aligned} \tag{33}$$

Definition 4.3 (Weighted Least Squares Estimation (WLSE))**.** The weighted least squares estimates $\hat{\alpha}_{WLSE}$, $\hat{\beta}_{WLSE}$ and $\hat{\gamma}_{WLSE}$ of SHW distribution parameters α, β and γ are obtained by minimizing the function $W(\alpha,\beta\gamma)$ with respect to α, β and γ

$$W(\alpha,\beta,\gamma) = \underset{(\alpha,\beta,\gamma)}{\arg\min} \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[F(x_{i:n}|\theta) - \frac{i}{n+1}\right]^2. \tag{34}$$

Solving the following non-linear equation yields the estimate

$$\begin{aligned}
&\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F(x_{i:n}|\theta)-\frac{i}{n+1}\right]\Delta_1(x_{i:n}|\theta)=0\\
&\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F(x_{i:n}|\theta)-\frac{i}{n+1}\right]\Delta_2(x_{i:n}|\theta)=0\\
&\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F(x_{i:n}|\theta)-\frac{i}{n+1}\right]\Delta_3(x_{i:n}|\theta)=0
\end{aligned}\tag{35}$$

where $\Delta_1(x.|\alpha,\beta,\gamma)$, $\Delta_2(x.|\alpha,\beta,\gamma)$ and $\Delta_3(x.|\alpha,\beta,\gamma)$ is as defined in (41) respectively.

Definition 4.4 (Maximum Product Spacing Estimation (MPSE))**.** A good substitute for the greatest likelihood approach is the maximum product spacing method, which approximates the Kullback-Leibler information measure. Let us now suppose that the data are ordered in an increasing manner. Then, the maximum product spacing for the SHW is given as follows

$$Gs(\alpha,\beta,\gamma|data)=\left(\prod_{i=1}^{n+1}D_l(x_i,\alpha,\beta,\gamma)\right)^{\frac{1}{n+1}},\tag{36}$$

where $D_l(x_i,\alpha,\beta,\gamma)=F(x_i;\alpha,\beta,\gamma)-F(x_{i-1};\alpha,\beta,\gamma)$, $i=1,2,3,...,n$.

Similarly, one can also choose to maximize the function

$$H(\alpha,\beta,\gamma)=\frac{1}{n+1}\sum_{i=1}^{n+1}\ln D_i(\alpha,\beta,\gamma).\tag{37}$$

By taking the first derivative of the function $H(\vartheta)$ with respect to α, β and γ, and solving the resulting nonlinear equations, at $\frac{\partial H(\phi)}{\partial\alpha}=0$, $\frac{\partial H(\phi)}{\partial\beta}=0$ and $\frac{\partial H(\phi)}{\partial\gamma}=0$, where $\phi=(\alpha,\beta,\gamma)$, we obtain the value of the parameter estimates.

Definition 4.5 (Cramér-von-Mises Estimation (CVME))**.** The Cramér-von-Mises estimates $\hat{\alpha}_{CVME}$, $\hat{\beta}_{CVME}$ and $\hat{\gamma}_{CVME}$of the SHW distribution parameters α, β

and γ are obtained by minimizing the function $C(\alpha,\beta,\gamma)$ with respect to α, β and γ

$$C(\alpha,\beta,\gamma)=\underset{(\alpha,\beta,\gamma)}{\arg\min}\left\{\frac{1}{12n}+\sum_{i=1}^{n}\left[F(x_{i:n}|\alpha,\beta,\gamma)-\frac{2i-1}{2n}\right]^2\right\}. \tag{38}$$

The estimates are obtained by solving the following non-linear equations

$$\begin{aligned}\sum_{i=1}^{n}\left(F(x_{i:n}|\alpha,\beta,\gamma)-\frac{2i-1}{2n}\right)\Delta_1(x_{i:n}|\alpha,\beta,\gamma)=0\\ \sum_{i=1}^{n}\left(F(x_{i:n}|\alpha,\beta,\gamma)-\frac{2i-1}{2n}\right)\Delta_2(x_{i:n}|\alpha,\beta,\gamma)=0\\ \sum_{i=1}^{n}\left(F(x_{i:n}|\alpha,\beta,\gamma)-\frac{2i-1}{2n}\right)\Delta_3(x_{i:n}|\alpha,\beta,\gamma)=0\end{aligned} \tag{39}$$

where $\Delta_1(x.|\alpha,\beta,\gamma)$, $\Delta_2(x.|\alpha,\beta,\gamma)$ and $\Delta_3(x.|\alpha,\beta,\gamma)$ is as defined in (32) respectively.

Definition 4.6 (Anderson-Darling Estimation (ADE))**.** The Anderson-Darling estimates $\hat{\alpha}_{ADE}$, $\hat{\beta}_{ADE}$ and $\hat{\gamma}_{ADE}$ of the SHW distribution parameters α, β and γ are obtained by minimizing the function $A(\alpha,\beta,\gamma)$ with respect to α, β and γ

$$A(\alpha,\beta,\gamma)=\underset{(\alpha,\beta\gamma)}{\arg\min}\sum_{i=1}^{n}(2i-1)\Big\{\ln F(x_{i:n}|\alpha,\beta\gamma)+\ln\Big[1-F(x_{n+1-i:n}|\alpha,\beta\gamma)\Big]\Big\}. \tag{40}$$

The estimates are obtained by solving the following sets of non-linear equations

$$\begin{aligned}\sum_{i=1}^{n}(2i-1)\left[\frac{\Delta_1(x_{i:n}|\alpha,\beta,\gamma)}{F(x_{i:n}|\alpha,\beta,\gamma\alpha,\beta,\gamma)}-\frac{\Delta_1(x_{n+1-i:n}|\alpha,\beta,\gamma)}{1-F(x_{n+1-i:n}|\alpha,\beta,\gamma)}\right]=0\\ \sum_{i=1}^{n}(2i-1)\left[\frac{\Delta_2(x_{i:n}|\alpha,\beta,\gamma)}{F(x_{i:n}|\alpha,\beta,\gamma)}-\frac{\Delta_2(x_{n+1-i:n}|\alpha,\beta,\gamma)}{1-F(x_{n+1-i:n}|\alpha,\beta,\gamma)}\right]=0\\ \sum_{i=1}^{n}(2i-1)\left[\frac{\Delta_3(x_{i:n}|\alpha,\beta,\gamma)}{F(x_{i:n}|\alpha,\beta,\gamma)}-\frac{\Delta_3(x_{n+1-i:n}|\alpha,\beta,\gamma)}{1-F(x_{n+1-i:n}|\alpha,\beta,\gamma)}\right]=0\end{aligned} \tag{41}$$

where $\Delta_1(x.|\alpha,\beta,\gamma)$, $\Delta_2(x.|\alpha,\beta,\gamma)$ and $\Delta_3(x.|\alpha,\beta,\gamma)$ is as defined in (41) respectively.

Definition 4.7 (Right-Tailed Anderson-Darling Estimation (RTADE))**.** The Right-Tailed Anderson-Darling estimates $\hat{\alpha}_{RTADE}$, $\hat{\beta}_{RTADE}$ and $\hat{\gamma}_{RTADE}$ of the SHW distribution parameters α, β and γ are obtained by minimizing the function $R(\alpha, \beta, \gamma)$ with respect to α, β and γ

$$R(\alpha,\beta,\gamma) = \underset{(\alpha,\beta,\gamma)}{\arg\min}\left\{\frac{n}{2}-2\sum_{i=1}^{n} F(x_{i:n}|\alpha,\beta,\gamma)-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\ln\left[1-F(x_{n+1-i:n}|\alpha,\beta,\gamma)\right]\right\}. \tag{42}$$

The estimates can be obtained by solving the following set of non-linear equations

$$\begin{aligned}
-2\sum_{i=1}^{n}\frac{\Delta_1(x_{i:n}|\alpha,\beta,\gamma)}{F(x_{i:n}|\alpha,\beta,\gamma)} + \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\frac{\Delta_1(x_{n+1-i:n}|\alpha,\beta,\gamma)}{1-F(x_{n+1-i:n}|\alpha,\beta,\gamma)}\right] &= 0\\
-2\sum_{i=1}^{n}\frac{\Delta_2(x_{i:n}|\alpha,\beta,\gamma)}{F(x_{i:n}|\lambda,\theta)} + \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\frac{\Delta_2(x_{n+1-i:n}|\alpha,\beta,\gamma)}{1-F(x_{n+1-i:n}|\alpha,\beta,\gamma)}\right] &= 0\\
-2\sum_{i=1}^{n}\frac{\Delta_3(x_{i:n}|\alpha,\beta,\gamma)}{F(x_{i:n}|\alpha,\beta,\gamma)} + \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\frac{\Delta_3(x_{n+1-i:n}|\alpha,\beta,\gamma)}{1-F(x_{n+1-i:n}|\alpha,\beta,\gamma)}\right] &= 0
\end{aligned} \tag{43}$$

where $\Delta_1(x.|\alpha,\beta,\gamma)$ and $\Delta_2(x.|\alpha,\beta,\gamma)$ is as defined in (32) respectively. The estimates given in (26), (31), (34), (36), (38), (40) and (42) are obtained using **optim()** function in R with the Newton-Raphson iterative algorithm.

5 Application

In this section, we apply the proposed distribution to two real life data sets in order to determine its usefulness and fitness for use.

5.1 Data set on tensile strength of Carbon fibres

The following represent the tensile strength, measured in GPa, of 69 carbon fibers tested under tension at gauge lengths of 20mm studied by Shanker et al. [28]. We demonstrate that the proposed SHW distribution is superior by comparing its model performance and fitness with those of the Weibull distribution, Gamma

Table 1: Tensile strength, measured in GPa, of 69 carbon fibers tested under tension at gauge lengths of 20mm

1.312	1.314	1.479	1.552	1.7	1.803	1.861	1.865	1.944	1.958	1.966	1.997	2.006
2.021	2.027	2.055	2.063	2.098	2.14	2.179	2.224	2.24	2.253	2.27	2.272	2.274
2.301	2.301	2.359	2.382	2.382	2.426	2.434	2.435	2.478	2.49	2.511	2.514	2.535
2.554	2.566	2.57	2.586	2.629	2.633	2.642	2.648	2.684	2.697	2.726	2.77	2.773
2.8	2.809	2.818	2.821	2.848	2.88	2.954	3.012	3.067	3.084	3.09	3.096	3.128
3.233	3.433	3.585	3.858									

Distribution, Lindley Distribution (LD), Exponential Distribution (ED), Pareto (P) distribution, Lindley-Lomax (LL) distribution and Lindley-Pareto (LP) distribution using data on the tensile strength, measured in GPa, of 69 carbon fibers tested under tension at gauge lengths of 20mm as shown in Table 2. From the analytical measures of fitness, the model with the smaller values of log-likelihood (LL), the Akaike information criterion (AIC), the Bayesian information criterion (BIC), and KolmogorovâSmirnov (K-S) statistics, is best among others. See Uzoma et al. [29] for relevant modification on model performance criteria using Bayesian Information Criterion (BIC). From Table 2, the SHW distribution has a better fit to the data on the tensile strength, measured in GPa, of 69 carbon fibers tested under tension at gauge lengths of 20mm since its probability value is the largest among other probabilities that are greater than 0.05. Again, SHW distribution has the least values of the LL, AIC and BIC and hence performance better than the fitted distributions. From Table 3, the weighted least squares estimation (WLSE) gives the best estimates of the SHW parameters since the standard errors of the parameters are minimum. Therefore, the WLSE is the best non-Bayesian approach for estimating the parameters of the proposed distribution.

Table 2: The Analytical Measures of Model performance and MLE estimates for the fitted distributions using data on the tensile strength, measured in GPa, of 69 carbon fibers tested under tension at gauge lengths of 20mm

Distr.	Parameter	Estimate	Std Error	LL	AIC	BIC	K-S	P-Value
	α	0.1723	0.1710					
SHW	β	3.3162	0.7361	**-50.4328**	**106.8656**	**113.5679**	**0.0510**	**0.9899**
	γ	0.9283	0.2947					
Weibull	shape	5.2694	0.4687	-51.7166	107.4331	111.9013	0.6632	0.902
	scale	2.6583	0.0642					
Gamma	shape	22.8128	3.8558	-50.9856	105.9712	110.4394	0.0567	0.9703
	rate	9.2911	1.5878					
LD	scale	0.6536	0.0579	-119.311	240.622	242.8561	0.4004	1.688e-10
ED	scale	0.4073	0.0409	-130.9789	263.9578	266.1919	0.4477	3.916e-13
P	a	123.0462	168.432	-131.247	266.494	270.9622	0.4473	4.124e-13
	t	301.5539	413.904					
	a	69.2062	30.9224					
LL	b	0.0059	0.0026	-131.4559	268.9118	275.6141	0.4461	4.903e-13
	t	-0.0026	0.0478					
	a	0.4543	0.07724					
LP	k	3.6202	0.3196	-50.5038	107.0076	113.7099	0.0513	0.9892
	t	0.0037	0.0007					

5.2 Data on Failure times of mechanical components

The data are extracted from Murthy et al. [30] and studied by Mathew and Chesneau [31]. They represent the failure times of mechanical components.

30.94	18.51	16.62	51.56	22.85	22.38	19.08	49.56	17.12	10.67	25.43	10.24
27.47	14.70	14.10	29.93	27.98	36.02	19.40	14.97	22.57	12.26	18.14	18.84

We demonstrate that the proposed SHW distribution is superior by comparing its model performance and fitness with those of the Weibull distribution, Gamma Distribution, Log-Normal Distribution, Exponential Distribution (ED) and Lindley-Pareto (LP) distribution using data on the failure times of mechanical

Table 3: Classical Estimates for the SHW Distribution using data on the tensile strength, measured in GPa, of 69 carbon fibers tested under tension at gauge lengths of 20mm

		Parameters		
Method		α	β	γ
MLE	Estimate	0.1723	3.3162	0.9283
	Std. Error	0.1710	0.7362	0.2947
MPSE	Estimate	0.1362	3.3385	0.8225
	Std. Error	0.1511	0.7753	0.3396
LSE	Estimate	0.0891	3.9121	0.7684
	Std. Error	1.0580	7.9546	3.3213
WLSE	Estimate	0.0735	4.0013	0.7072
	Std. Error	0.0348	0.3124	0.1299
CVME	Estimate	0.1114	3.8198	0.8436
	Std. Error	1.6594	10.2348	4.1879
ADE	Estimate	0.1057	3.7614	0.8089
	Std. Error	0.3659	2.3750	0.9829
RTADE	Estimate	0.0093	5.0828	0.1196
	Std. Error	0.01251	0.9726	0.4763

components Table 4. From Table 4, the SHW distribution has a better fit to the data on data on the failure times of mechanical components since its probability value is the largest among other probabilities that are greater than 0.05. Again, SHW distribution has the least values of the LL, AIC and BIC and hence performance better than the fitted distributions. The fitted distributions in

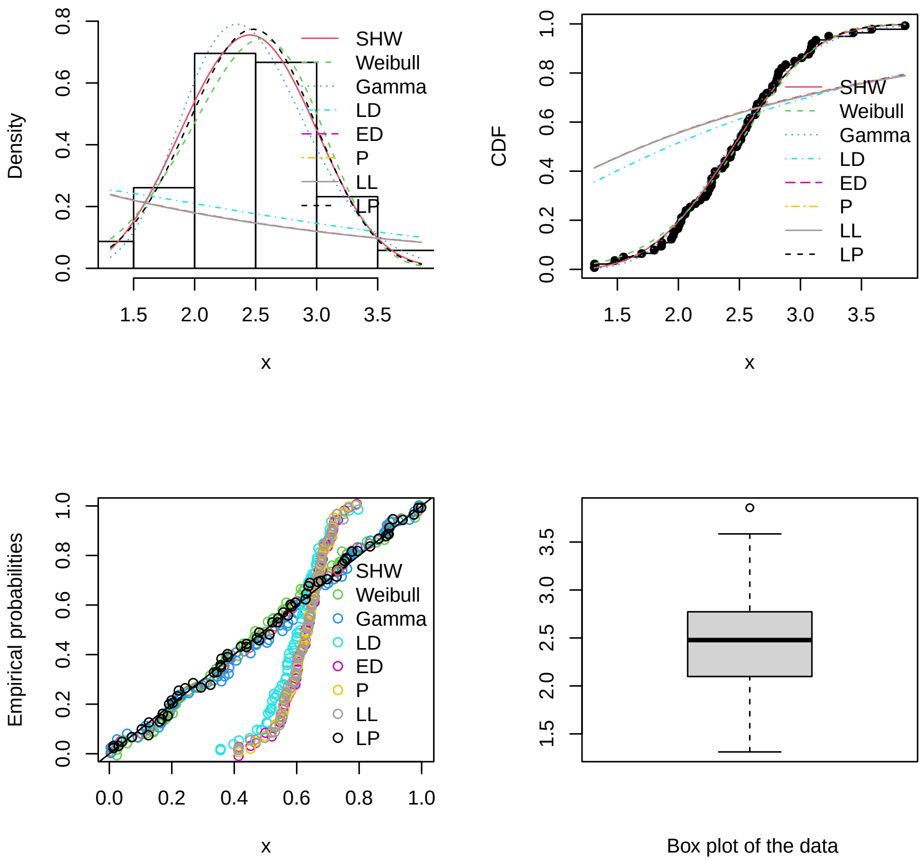


Figure 5: The estimated pdf, cdf, Kaplan-Meier and Box plots of the SHW and other fitted distributions using tensile strength, measured in GPa, of 69 carbon fibers tested under tension at gauge lengths of 20mm.

Figure 6 displays the density, CDF, empirical reliability, Kaplan-Meier and Box plots of the data on the failure times of mechanical components. This data set again shows that SHW is best fit compared to the parent distribution-Weibull and the popular lifetime distributions namely Weibull, Gamma and Lognormal among

Table 4: The Analytical Measures of Model performance and MLE estimates for the fitted distributions using data on the failure times of mechanical components

Distr.	Parameters	Estimate	Std. Error	LL	AIC	BIC	K-S	P-vale
	α	0.0472	0.0388					
SHW	β	1.1717	0.2616	**-84.8887**	**175.7773**	**179.3115**	**0.09598**	**0.9646**
	γ	10.0957	0.4669					
Weibull	shape	2.3067	0.3375	-88.891	181.782	184.1381	0.1433	0.656
	scale	26.0177	2.4471					
Gamma	shape	5.6915	1.5970	-86.9446	177.8892	180.2454	0.1404	0.6805
	rate	0.2478	0.0727					
Log-Normal	meanlog	3.0440	0.0849	-86.0300	176.06	178.4161	0.1167	0.8621
	sdlog	0.4158	0.0601					
ED	θ	0.0435	0.0089	-99.2232	200.4463	201.6244	0.3597	0.0028
	a	0.0491	0.0416					
LP	k	1.6310	0.2172	-87.8186	181.6373	185.1714	0.1418	0.6683
	t	0.0001	0					

others (see Table 4 for the estimates, standard error (Std. Err.), performance criteria and the p-value.

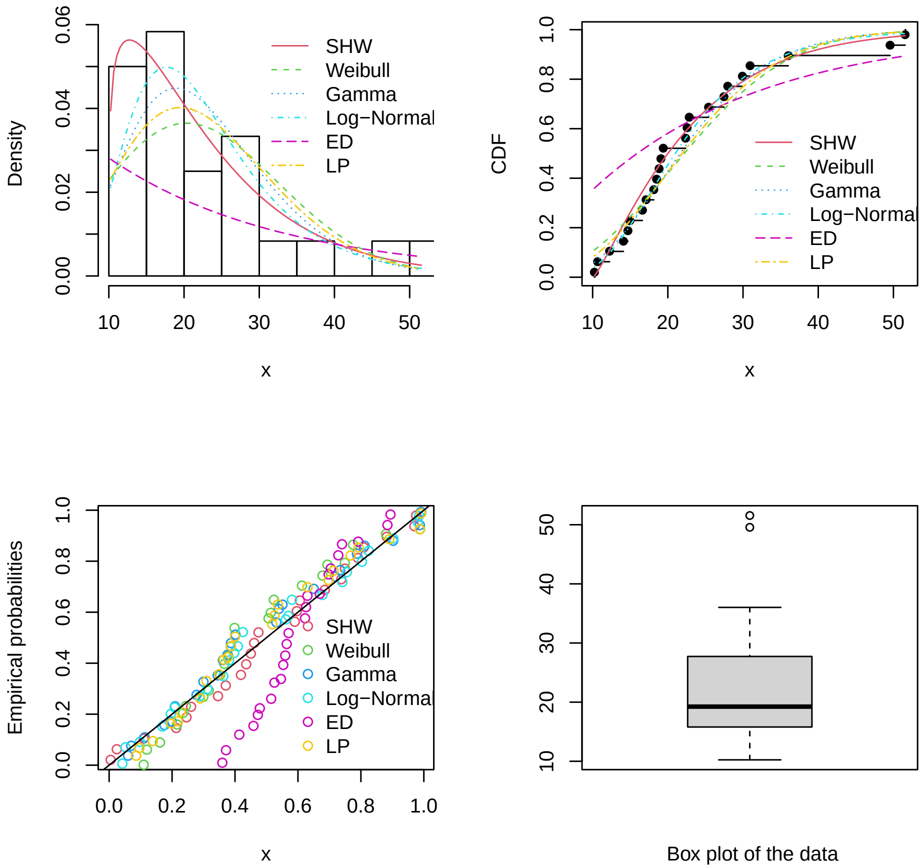


Figure 6: The estimated pdf, cdf, Kaplan-Meier and Box plots of the SHW and other fitted distributions using data on the failure times of mechanical components.

6 Conclusion

The Shifted Weibull distribution studied in this paper was for the purpose of taking into account the lower bound of the random variate distributed according to the proposed distribution. Some of the important properties are derived and examined, including moments and their measures, the moment generating function, the characteristic function, the hazard rate, Rény entropy, order statistics, and stochastic ordering. To estimate and study the parameters, seven methods are considered and comparisons are made. The techniques looked at include maximum likelihood estimation, maximum product spacing, least squares, weighted least squares, Cramer-von-Mises and the right-tailed Anderson-Darling. The application to data on the failure times of mechanical components reveals that demonstrates that the SHW distribution offered a strong fit to the data sets. The SHW distribution outperformed the competition when put up against other distributions based on the LL, AIC, BIC, KS, and probability values. Among the classical estimation methods, the weighted least squares estimation was the best using the data on the tensile strength, measured in GPa, of 69 carbon fibers tested under tension at gauge lengths of 20mm.

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