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Generalized Tribonacci Polynomials

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Abstract

In this paper, we investigate the generalized Tribonacci polynomials and we deal with, in detail, two special cases which we call them $(r,s,t)-$Tribonacci and $(r,s,t)-$Tribonacci-Lucas polynomials. We also introduce and investigate a new sequence and its two special cases namely the generalized co-Tribonacci, $(r,s,t)-$co-Tribonacci and $(r,s,t)-$co-Tribonacci-Lucas polynomials, respectively. We present Binet's formulas, generating functions, Simson formulas, and the summation formulas for these polynomial sequences. Moreover, we give some identities and matrices related to these polynomials. Furthermore, we evaluate the infinite sums of special cases of $(r,s,t)-$Tribonacci and $(r,s,t)-$Tribonacci-Lucas polynomials.

1 Introduction: Generalized Tribonacci Polynomials

Recently, there have been so many studies of the sequences of numbers and polynomials in the literature and they were widely used in many research areas, such as architecture, nature, art, physics and engineering. The sequence of Fibonacci numbers $\{F_n\}$ is defined by the second-order linear recurrence sequence

$$F_n = F_{n-1} + F_{n-2}, \qquad n \geq 2,$$

with initial conditions $F_0 = 0$, $F_1 = 1$. A generalization of the sequence $\{F_n\}$ is sequence of Fibonacci polynomials which are defined by the second-order linear recurrence sequence

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x),\ F_0(x) = 0,\ F_1(x) = 1, \quad n \geq 2.$$

The Fibonacci numbers are recovered by evaluating the polynomials $F_n(x)$ at $x = 1$. The Fibonacci numbers and polynomials and their generalizations have many interesting

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properties and applications to almost every field. For some references on special cases of second-order linear recurrence sequences of polynomials and numbers, see for instance [4,5,6,14,18,19] for papers and [1,3,8,9,10,15,17] for books.

In this paper, we investigate the third order generalization of Fibonacci numbers and polynomials.

The generalized Tribonacci polynomials (or generalized $(r(x), s(x), t(x))-$Tribonacci polynomials or x-Tribonacci numbers or generalized $(r(x), s(x), t(x))$-polynomials or 3-step Fibonacci polynomials)

$$\{W_n(W_0(x), W_1(x), W_2(x); r(x), s(x), t(x))\}_{n\geq 0}$$

(or $\{W_n(x)\}_{n\geq 0}$ or shortly $\{W_n\}_{n\geq 0}$) is defined as follows:

$$\begin{aligned} W_n(x) &= r(x)W_{n-1}(x) + s(x)W_{n-2}(x) + t(x)W_{n-3}(x), \\ W_0(x) &= a(x), W_1(x) = b(x), W_2(x) = c(x), \quad n \geq 3 \end{aligned} \tag{1.1}$$

where $W_0(x), W_1(x), W_2(x)$ are arbitrary complex (or real) polynomials with real coefficients and $r(x)$, $s(x)$ and $t(x)$ are polynomials with real coefficients and $t(x) \neq 0$.

Special cases of this sequence has been studied by many authors. For some references on special cases of generalized Tribonacci polynomials, see for example [2,11,12,13].

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n}(x) = -\frac{s(x)}{t(x)}W_{-(n-1)}(x) - \frac{r(x)}{t(x)}W_{-(n-2)}(x) + \frac{1}{t(x)}W_{-(n-3)}(x)$$

for $n = 1, 2, 3, ...$ when $t(x) \neq 0$. Therefore, recurrence (1.1) holds for all integers n. Note that for $n \geq 1$, $W_{-n}(x)$ need not to be a polynomial in the ordinary sense.

Binet's formula of generalized Tribonacci polynomials, as $\{W_n\}$ is a third-order recurrence sequence (difference equation), can be calculated using its characteristic equation (the cubic equation, auxiliary equation, polynomial) which is given as

$$z^3 - r(x)z^2 - s(x)z - t(x) = 0. \tag{1.2}$$

The roots of characteristic equation of $\{W_n\}$ will be denoted as $\alpha(x) = \alpha(x, r, s, t), \beta(x) = \beta(x, r, s, t), \gamma(x) = \gamma(x, r, s, t)$.

Remark 1. *For the sake of simplicity throughout the rest of the paper, we use*

$$W_n, r, s, t, W_0, W_1, W_2, \alpha, \beta, \gamma,$$

instead of

$$W_n(x), r(x), s(x), t(x), W_0(x), W_1(x), W_2(x), \alpha(x), \beta(x), \gamma(x),$$

respectively, unless otherwise stated. For example, we write

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3}, \qquad W_0 = a, W_1 = b, W_2 = c, \quad n \geq 3$$

for the equation (1.1). Also we write U_n, U_0, U_1, U_2 *instead of* $U_n(x)$ *with initial conditions* $U_0(x), U_1(x), U_2(x)$ *for any subsequence* $\{U_n(x)\}$ *of* $\{W_n\}$.

The roots of characteristic equation of $\{W_n\}$ are

$$\alpha = \frac{r}{3} + C_1 + C_2, \tag{1.3}$$

$$\beta = \frac{r}{3} + \omega C_1 + \omega^2 C_2, \tag{1.4}$$

$$\gamma = \frac{r}{3} + \omega^2 C_1 + \omega C_2, \tag{1.5}$$

where

$$\begin{aligned} C_1 &= \left(\frac{r^3}{27} + \frac{rs}{6} + \frac{t}{2} + \sqrt{\Delta_1}\right)^{1/3}, \; C_2 = \left(\frac{r^3}{27} + \frac{rs}{6} + \frac{t}{2} - \sqrt{\Delta_1}\right)^{1/3}, \\ \Delta_1 &= \Delta_1(r, s, t) = \frac{r^3 t}{27} - \frac{r^2 s^2}{108} + \frac{rst}{6} - \frac{s^3}{27} + \frac{t^2}{4}, \\ \omega &= \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3). \end{aligned}$$

Note that (in the three distinct roots case: $\alpha \neq \beta \neq \gamma$) we have the following formula: for $n = 1, 2, 3, ...$ we have

$$W_{-n} = \frac{1}{t^n} \frac{\beta^n \gamma^n (\beta - \gamma) p_1 - \alpha^n \gamma^n (\alpha - \gamma) p_2 + \alpha^n \beta^n (\alpha - \beta) p_3}{\alpha^n (\beta - \gamma) p_1 - \beta^n (\alpha - \gamma) p_2 + \gamma^n (\alpha - \beta) p_3} W_n. \tag{1.6}$$

We have the following identities between α, β, γ and r, s, t.

Lemma 2. *There are close relations between the roots of characteristic equation (1.2) and* r, s, t *as follows.*

(a) *Arbitrary Roots Case* (α, β, γ *are arbitrary).*

$$\left\{ \begin{array}{c} \alpha + \beta + \gamma = r, \\ \alpha\beta + \alpha\gamma + \beta\gamma = -s, \\ \alpha\beta\gamma = t. \end{array} \right. \tag{1.7}$$

(b) *Two Distinct Roots Case* ($\alpha \neq \beta = \gamma$).

$$\begin{aligned} \alpha &\neq \beta = \gamma \\ &\Leftrightarrow \\ r &= \alpha + 2\beta, \\ s &= -\beta\left(2\alpha + \beta\right), \\ t &= \alpha\beta^2. \end{aligned}$$

(c) *Single Root Case* ($\alpha = \beta = \gamma = \frac{r}{3}$).

$$\begin{aligned} \alpha &= \beta = \gamma \\ &\Leftrightarrow r = 3\alpha,\ s = -3\alpha^2,\ t = \alpha^3 \\ &\Leftrightarrow \frac{r}{3} = \alpha,\ s = -\frac{1}{3}r^2,\ t = \frac{1}{27}r^3. \end{aligned}$$

and

$$\alpha = \beta = \gamma \Leftrightarrow \Delta_1 = C_1 = C_2 = 0.$$

Proof. The identities in (1.7) are well known. In fact, just compare

$$z^3 - rz^2 - sz - t = 0$$

and

$$(z-\alpha)\left(z-\beta\right)(z-\gamma) = z^3 - (\alpha+\beta+\gamma)z^2 + (\alpha\beta + \alpha\gamma + \beta\gamma)z - \alpha\beta\gamma = 0.$$

Use (1.2), (1.3)-(1.5) and (1.7) to prove (b) and (c). □

Using the roots of characteristic equation and the recurrence relation of W_n, Binet's formula can be given as follows:

Theorem 3. *For all integers n, Binet's formula of generalized Tribonacci polynomials is given as follows.*

(a) *(Three Distinct Roots Case: $\alpha \neq \beta \neq \gamma$) Binet's formula of generalized Tribonacci polynomials is*

$$\begin{aligned} W_n &= \frac{p_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{p_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{p_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)} \qquad (1.8) \\ &= A_1\alpha^n + A_2\beta^n + A_3\gamma^n, \end{aligned}$$

where

$$p_1 = W_2-(\beta+\gamma)W_1+\beta\gamma W_0,\ p_2 = W_2-(\alpha+\gamma)W_1+\alpha\gamma W_0,\ p_3 = W_2-(\alpha+\beta)W_1+\alpha\beta W_0,$$

and

$$\begin{aligned}
A_1 &= \frac{p_1}{(\alpha-\beta)(\alpha-\gamma)} = \frac{W_2-(\beta+\gamma)W_1+\beta\gamma W_0}{(\alpha-\beta)(\alpha-\gamma)} \\
&= \frac{(\alpha W_2+\alpha(-r+\alpha)W_1+tW_0)}{r\alpha^2+2s\alpha+3t}, \\
A_2 &= \frac{p_2}{(\beta-\alpha)(\beta-\gamma)} = \frac{W_2-(\alpha+\gamma)W_1+\alpha\gamma W_0}{(\beta-\alpha)(\beta-\gamma)} \\
&= \frac{(\beta W_2+\beta(-r+\beta)W_1+tW_0)}{r\beta^2+2s\beta+3t}, \\
A_3 &= \frac{p_3}{(\gamma-\alpha)(\gamma-\beta)} = \frac{W_2-(\alpha+\beta)W_1+\alpha\beta W_0}{(\gamma-\alpha)(\gamma-\beta)} \\
&= \frac{(\gamma W_2+\gamma(-r+\gamma)W_1+tW_0)}{r\gamma^2+2s\gamma+3t}.
\end{aligned}$$

(b) *(Two Distinct Roots Case: $\alpha\neq\beta=\gamma$) Binet's formula of generalized Tribonacci polynomials is*

$$W_n = A_1\alpha^n+(A_2+A_3n)\beta^n \tag{1.9}$$

where

$$\begin{aligned}
A_1 &= \frac{W_2-2\beta W_1+\beta^2W_0}{(\beta-\alpha)^2} = \frac{4W_2-4(r-\alpha)W_1+(r-\alpha)^2W_0}{(r-3\alpha)^2}, \\
A_2 &= \frac{-W_2+2\beta W_1-\alpha(2\beta-\alpha)W_0}{(\beta-\alpha)^2} = \frac{4(-W_2+(r-\alpha)W_1+\alpha(-r+2\alpha)W_0)}{(r-3\alpha)^2}, \\
A_3 &= \frac{W_2-(\beta+\alpha)W_1+\beta\alpha W_0}{\beta(\beta-\alpha)} = \frac{2(2W_2-(r+\alpha)W_1+\alpha(r-\alpha)W_0)}{(r-3\alpha)(r-\alpha)}.
\end{aligned}$$

(c) *(Single Root Case: $\alpha=\beta=\gamma=\frac{r}{3}$) Binet's formula of generalized Tribonacci polynomials is*

$$W_n = (A_1+A_2n+A_3n^2)\times\alpha^n \tag{1.10}$$

where

$$\begin{aligned}
A_1 &= W_0, \\
A_2 &= \frac{-W_2+4W_1\alpha-3W_0\alpha^2}{2\alpha^2} = \frac{3}{2r^2}(-3W_2+4rW_1-r^2W_0), \\
A_3 &= \frac{W_2-2W_1\alpha+W_0\alpha^2}{2\alpha^2} = \frac{1}{2r^2}(9W_2-6rW_1+r^2W_0).
\end{aligned}$$

Proof. **(a)** If the roots α, β, γ of (1.8) are distinct, then (the sequences $(\alpha^n)_{n\geq 0}$, $(\beta^n)_{n\geq 0}$ and $(\gamma^n)_{n\geq 0}$ are solutions of (1.1) and) the general formula of W_n is in the following form:

$$W_n = A_1\alpha^n + A_2\beta^n + A_3\gamma^n$$

where the coefficients A_1, A_2 and A_3 are determined by the system of linear equations

$$\begin{aligned} W_0 &= A_1 + A_2 + A_3 \\ W_1 &= A_1\alpha + A_2\beta + A_3\gamma \\ W_2 &= A_1\alpha^2 + A_2\beta^2 + A_3\gamma^2. \end{aligned}$$

Solving these three simultaneous equations for W_0, W_1 and W_2, we obtain the required result.

(b) If (1.8) have two distinct roots i.e., $\alpha \neq \beta = \gamma$, then W_n is in the following form:

$$W_n = A_1\alpha^n + (A_2 + A_3 n) \times \beta^n = A_1\alpha^n + A_2\beta^n + A_3 n\beta^n$$

where A_2, A_3 and A_1 are the polynomials whose values are determined by the values W_0, W_1 and any other known value of the sequence. By using the values W_0, W_1 and W_2, we obtain

$$\begin{aligned} W_0 &= A_1 \times \alpha^0 + A_2 \times \beta^0 + A_3 \times 0 \times \beta^0 \\ W_1 &= A_1 \times \alpha^1 + A_2 \times \beta^1 + A_3 \times 1 \times \beta^1 \\ W_2 &= A_1 \times \alpha^2 + A_2 \times \beta^2 + A_3 \times 2 \times \beta^2 \end{aligned}$$

i.e.,

$$\begin{aligned} W_0 &= A_1 + A_2 \\ W_1 &= A_1\alpha + (A_2 + A_3)\beta = A_1\alpha + A_2\beta + A_3\beta \\ W_2 &= A_1\alpha^2 + (A_2 + 2A_3)\beta^2 = A_1\alpha^2 + A_2\beta^2 + 2A_3\beta^2 \end{aligned}$$

Solving these three simultaneous equations for W_0, W_1 and W_2, we get the required result.

(c) If the roots α, β, γ of (1.8) are equal, i.e., $\alpha = \beta = \gamma$, then W_n is in the following form:

$$W_n = (A_1 + A_2 n + A_3 n^2) \times \alpha^n$$

where A_1, A_2 and A_3 are the polynomials whose values are determined by the values W_0, W_1 and any other known value of the sequence. By using the values W_0, W_1 and W_2, we obtain

$$\begin{aligned} W_0 &= A_1 + 0 \times A_2 + 0 \times A_3 \\ W_1 &= \alpha A_1 + \alpha A_2 + \alpha A_3 \\ W_2 &= \alpha^2 A_1 + 2\alpha^2 A_2 + 4\alpha^2 A_3 \end{aligned}$$

Solving these three simultaneous equations for W_0, W_1 and W_2, we get the required result.

□

Remark 4. *Note that the Binet form of a sequence satisfying Theorem (3) for non-negative integers is valid for all integers n (see [7], this result of Howard and Saidak [7] is even true in the case of higher-order recurrence relations).*

Note that (a), (b) and (c) of the above theorem can be given as follows:

$$W_n = \begin{cases} A_1\alpha^n + A_2\beta^n + A_3\gamma^n & , \quad \text{if } \alpha \neq \beta \neq \gamma \text{ (Three Distinct Roots Case)} \\ A_1\alpha^n + (A_2 + A_3 n)\beta^n & , \quad \text{if } \alpha \neq \beta = \gamma \text{ (Two Distinct Roots Case)} \\ (A_1 + A_2 n + A_3 n^2)\alpha^n & , \quad \text{if } \alpha = \beta = \gamma \text{ (Single Root Case)} \end{cases} \tag{1.11}$$

where each triple A_1, A_2, A_3 are given as in Theorem 3 (a), (b) and (c), respectively.

Lemma 5.

(a) *(Three Distinct Roots Case: $\alpha \neq \beta \neq \gamma$) If A_1, A_2 and A_3 are as in Theorem 3 (a), then we have*

$$A_1 + A_2 + A_3 = W_0.$$

(b) *(Two Distinct Roots Case: $\alpha \neq \beta = \gamma$) If A_1, A_2 and A_3 are as in Theorem 3 (b), then we have*

$$A_1 + A_2 + A_3 = \frac{-W_2 + (\alpha + \beta)W_1 - \beta^2 W_0}{\beta(\alpha - \beta)}.$$

(c) *(Single Root Case: $\alpha = \beta = \gamma = \frac{r}{3}$) If A_1, A_2 and A_3 are as in Theorem 3 (c), then we have*

$$A_1 + A_2 + A_3 = \frac{W_1}{\alpha} = \frac{3}{r}W_1.$$

Proof. Use Theorem 3. □

Theorem 3 can be given in the following form:

Theorem 6. *Binet's formula of generalized Tribonacci polynomials is given as follows according to the roots of characteristic equation (1.2):*

(a) *(Three Distinct Roots Case: $\alpha \neq \beta \neq \gamma$)*

$$\begin{aligned} W_n &= \frac{W_2-(\beta+\gamma)W_1+\beta\gamma W_0}{(\alpha-\beta)(\alpha-\gamma)}\alpha^n+\frac{W_2-(\alpha+\gamma)W_1+\alpha\gamma W_0}{(\beta-\alpha)(\beta-\gamma)}\beta^n \\ &\quad +\frac{W_2-(\alpha+\beta)W_1+\alpha\beta W_0}{(\gamma-\alpha)(\gamma-\beta)}\gamma^n, \end{aligned}$$

i.e.,

$$\begin{aligned} W_n &= \frac{(\alpha W_2+\alpha(-r+\alpha)W_1+tW_0)}{r\alpha^2+2s\alpha+3t}\alpha^n+\frac{(\beta W_2+\beta(-r+\beta)W_1+tW_0)}{r\beta^2+2s\beta+3t}\beta^n \\ &\quad +\frac{(\gamma W_2+\gamma(-r+\gamma)W_1+tW_0)}{r\gamma^2+2s\gamma+3t}\gamma^n. \end{aligned}$$

(b) *(Two Distinct Roots Case: $\alpha \neq \beta = \gamma$)*

$$W_n=\frac{W_2-2\beta W_1+\beta^2W_0}{(\beta-\alpha)^2}\alpha^n+(\frac{-W_2+2\beta W_1-\alpha(2\beta-\alpha)W_0}{(\beta-\alpha)^2}+\frac{W_2-(\beta+\alpha)W_1+\beta\alpha W_0}{\beta(\beta-\alpha)}n)\beta^n$$

i.e.,

$$\begin{aligned} W_n &= \frac{4W_2-4(r-\alpha)W_1+(r-\alpha)^2W_0}{(r-3\alpha)^2}\alpha^n \\ &\quad +\frac{1}{\beta(r-3\beta)^2}((-\beta W_2+2\beta^2W_1+(2r\beta^2+(r^2+8s)\beta+8t)W_0) \\ &\quad +((3\beta-r)W_2-(r-3\beta)(\beta-r)W_1-(r\beta^2+(r^2+6s)\beta+6t)W_0)n)\beta^n. \end{aligned}$$

(c) *(Single Root Case: $\alpha=\beta=\gamma=\frac{r}{3}$)*

$$\begin{aligned} W_n &= \frac{1}{2}(2\alpha^2W_0+(-W_2+4W_1\alpha-3W_0\alpha^2)n+(W_2-2W_1\alpha+W_0\alpha^2)n^2)\alpha^{n-2} \\ &= \frac{1}{2}(n(n-1)W_2-2n(n-2)\alpha W_1+(n-1)(n-2)\alpha^2W_0)\alpha^{n-2} \\ &= \frac{1}{18}(9n(n-1)W_2-6n(n-2)rW_1+(n-1)(n-2)r^2W_0)\left(\frac{r}{3}\right)^{n-2}. \end{aligned}$$

Proof. Use (1.7) and Theorem 3. □

Note that each of Theorem 3, equations (1.11) and Theorem 6 have their advantages to present Binet's formula of generalized Tribonacci polynomials.

If some of the roots of characteristic equation is 1 then we get the following corollary as a special case of Theorem 6 .

Corollary 7. *Binet's formula of generalized Tribonacci polynomials is given as follows according to the roots of characteristic equation (1.2):*

(a) *(Three Distinct Roots Case: $\alpha \neq \beta \neq \gamma = 1$)*

$$W_n = \frac{W_2 - (\beta+1)W_1 + \beta W_0}{(\alpha-\beta)(\alpha-1)}\alpha^n + \frac{W_2 - (\alpha+1)W_1 + \alpha W_0}{(\beta-\alpha)(\beta-1)}\beta^n + \frac{W_2 + (-r+1)W_1 + tW_0}{t-r+2},$$

i.e.,

$$W_n = \frac{(\alpha W_2 + \alpha(-r+\alpha)W_1 + tW_0)}{r\alpha^2 + 2s\alpha + 3t}\alpha^n + \frac{(\beta W_2 + \beta(-r+\beta)W_1 + tW_0)}{r\beta^2 + 2s\beta + 3t}\beta^n + \frac{W_2 + (-r+1)W_1 + tW_0}{r+2s+3t}.$$

(b) *(Two Distinct Roots Case: $\alpha \neq \beta = \gamma = 1$)*

$$W_n = \frac{1}{(1-t)^2}((W_2 - 2W_1 + W_0)\alpha^n + (-W_2 + 2W_1 + t(t-2)W_0) + (1-t)(W_2 - (1+t)W_1 + tW_0)n).$$

(c) *(Single Root Case: $\alpha = \beta = \gamma = 1 = \frac{r}{3}$)*

$$W_n = \frac{1}{2}(n(n-1)W_2 - 2n(n-2)W_1 + (n-1)(n-2)W_0).$$

Proof. Use (1.7) and Theorem 6. Note that, in (a), since $\gamma = 1$, we get $r + s + t = 1$ and so $r + 2s + 3t - (t - r + 2) = 2r + 2s + 2t - 2 = 0$, i.e., $r + 2s + 3t = t - r + 2$. □

If some of the roots of characteristic equation is -1 then we get the following corollary as a special case of Theorem 6.

Corollary 8. *Binet's formula of generalized Tribonacci polynomials is given as follows according to the roots of characteristic equation (1.2):*

(a) *(Three Distinct Roots Case: $\alpha \neq \beta \neq \gamma = -1$)*

$$W_n = \frac{W_2 - (\beta-1)W_1 - \beta W_0}{(\alpha-\beta)(\alpha+1)}\alpha^n + \frac{W_2 - (\alpha-1)W_1 - \alpha W_0}{(\beta-\alpha)(\beta+1)}\beta^n + \frac{W_2 - (r+1)W_1 - tW_0}{r-t+2}(-1)^n,$$

i.e.,

$$W_n = \frac{(\alpha W_2 + \alpha(-r+\alpha)W_1 + tW_0)}{r\alpha^2 + 2s\alpha + 3t}\alpha^n + \frac{(\beta W_2 + \beta(-r+\beta)W_1 + tW_0)}{r\beta^2 + 2s\beta + 3t}\beta^n + \frac{W_2 - (r+1)W_1 - tW_0}{-r+2s-3t}(-1)^n.$$

(b) *(Two Distinct Roots Case: $\alpha \neq \beta = \gamma = -1$)*

$$\begin{aligned} W_n \;=\; & \frac{1}{(r+3)^2}((W_2 + 2W_1 + W_0)\alpha^n + ((W_2 + 2W_1 + (-r^2 + 2r - 8s + 8t)W_0) \\ & + ((-3-r)W_2 + (r+1)(r+3)W_1 + (r^2 - r + 6s - 6t)W_0)n)(-1)^{n+1}). \end{aligned}$$

(c) *(Single Root Case:* $\alpha=\beta=\gamma=-1=\dfrac{r}{3}$*)*

$$W_n=\frac{1}{2}(n(n-1)W_2+2n(n-2)W_1+(n-1)(n-2)W_0)\,(-1)^{n-2}.$$

Next, we give the ordinary generating function $\sum\limits_{n=0}^{\infty}W_nu^n$ of the sequence W_n.

Lemma 9. *Suppose that* $f_{W_n}(u)=\sum\limits_{n=0}^{\infty}W_nu^n$ *is the ordinary generating function of the generalized Tribonacci (sequence of) polynomials* $\{W_n\}_{n\geq 0}$. *Then,* $\sum\limits_{n=0}^{\infty}W_nu^n$ *is given by*

$$\sum_{n=0}^{\infty}W_nu^n=\frac{W_0+(W_1-rW_0)u+(W_2-rW_1-sW_0)u^2}{1-ru-su^2-tu^3}. \tag{1.12}$$

Proof. Using the definition of generalized Tribonacci polynomials, and substracting $ru\sum_{n=0}^{\infty}W_nu^n$, $su^2\sum_{n=0}^{\infty}W_nu^n$ and $tu^3\sum_{n=0}^{\infty}W_nu^n$ from $\sum_{n=0}^{\infty}W_nu^n$ we get

$$\begin{aligned}
(1-ru-su^2-tu^3)\sum_{n=0}^{\infty}W_nu^n &= \sum_{n=0}^{\infty}W_nu^n-ru\sum_{n=0}^{\infty}W_nu^n-su^2\sum_{n=0}^{\infty}W_nu^n-tu^3\sum_{n=0}^{\infty}W_nu^n\\
&= \sum_{n=0}^{\infty}W_nu^n-r\sum_{n=0}^{\infty}W_nu^{n+1}-s\sum_{n=0}^{\infty}W_nu^{n+2}-t\sum_{n=0}^{\infty}W_nu^{n+3}\\
&= \sum_{n=0}^{\infty}W_nu^n-r\sum_{n=1}^{\infty}W_{n-1}u^n-s\sum_{n=2}^{\infty}W_{n-2}u^n-t\sum_{n=3}^{\infty}W_{n-3}u^n\\
&= (W_0+W_1u+W_2u^2)-r(W_0u+W_1u^2)-sW_0u^2\\
&\quad +\sum_{n=3}^{\infty}(W_n-rW_{n-1}-sW_{n-2}-tW_{n-3})u^n\\
&= W_0+W_1u+W_2u^2-rW_0u-rW_1u^2-sW_0u^2\\
&= W_0+(W_1-rW_0)u+(W_2-rW_1-sW_0)u^2.
\end{aligned}$$

Rearranging above equation, we obtain

$$\sum_{n=0}^{\infty}W_nu^n=\frac{W_0+(W_1-rW_0)u+(W_2-rW_1-sW_0)u^2}{1-ru-su^2-tu^3}.$$

□

We next find Binet's formula of the generalized Tribonacci (sequence of) polynomials $\{W_n\}_{n\geq 0}$ by the use of generating function for W_n.

Theorem 10. *(Binet's formula of the generalized Tribonacci (sequence of) polynomials $\{W_n\}_{n\geq 0}$ for the three distinct roots case, i.e., $\alpha \neq \beta \neq \gamma$). For all integers n, we have*

$$W_n = \frac{q_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{q_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{q_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)} \tag{1.13}$$

where

$$\begin{aligned} q_1 &= W_0\alpha^2 + (W_1 - rW_0)\alpha + (W_2 - rW_1 - sW_0), \\ q_2 &= W_0\beta^2 + (W_1 - rW_0)\beta + (W_2 - rW_1 - sW_0), \\ q_3 &= W_0\gamma^2 + (W_1 - rW_0)\gamma + (W_2 - rW_1 - sW_0). \end{aligned}$$

Proof. Let

$$h(u) = 1 - ru - su^2 - tu^3.$$

Then for some α, β and γ we write

$$h(u) = (1-\alpha u)(1-\beta u)(1-\gamma u)$$

i.e.,

$$1 - ru - su^2 - tu^3 = (1-\alpha u)(1-\beta u)(1-\gamma u). \tag{1.14}$$

Hence $\frac{1}{\alpha}, \frac{1}{\beta}$ and $\frac{1}{\gamma}$ are the roots of $h(u)$. This gives α, β and γ as the roots of

$$h(\frac{1}{u}) = 1 - \frac{r}{u} - \frac{s}{u^2} - \frac{t}{u^3} = 0.$$

This implies $u^3 - ru^2 - su - t = 0$. Now, by (1.12) and (1.14), it follows that

$$\sum_{n=0}^{\infty} W_n u^n = \frac{W_0 + (W_1 - rW_0)u + (W_2 - rW_1 - sW_0)u^2}{(1-\alpha u)(1-\beta u)(1-\gamma u)}.$$

Then we write

$$\frac{W_0 + (W_1 - rW_0)u + (W_2 - rW_1 - sW_0)u^2}{(1-\alpha u)(1-\beta u)(1-\gamma u)} = \frac{B_1}{(1-\alpha u)} + \frac{B_2}{(1-\beta u)} + \frac{B_3}{(1-\gamma u)}. \tag{1.15}$$

So

$$W_0 + (W_1 - rW_0)u + (W_2 - rW_1 - sW_0)u^2 = B_1(1-\beta u)(1-\gamma u) + B_2(1-\alpha u)(1-\gamma u) + B_3(1-\alpha u)(1-\beta u).$$

If we consider $u = \frac{1}{\alpha}$, we get $W_0 + (W_1 - rW_0)\frac{1}{\alpha} + (W_2 - rW_1 - sW_0)\frac{1}{\alpha^2} = B_1(1-\frac{\beta}{\alpha})(1-\frac{\gamma}{\alpha})$. This gives

$$B_1 = \frac{\alpha^2(W_0 + (W_1 - rW_0)\frac{1}{\alpha} + (W_2 - rW_1 - sW_0)\frac{1}{\alpha^2})}{(\alpha-\beta)(\alpha-\gamma)} = \frac{W_0\alpha^2 + (W_1 - rW_0)\alpha + (W_2 - rW_1 - sW_0)}{(\alpha-\beta)(\alpha-\gamma)}.$$

Similarly, we obtain

$$B_2 = \frac{W_0\beta^2 + (W_1 - rW_0)\beta + (W_2 - rW_1 - sW_0)}{(\beta-\alpha)(\beta-\gamma)}, B_3 = \frac{W_0\gamma^2 + (W_1 - rW_0)\gamma + (W_2 - rW_1 - sW_0)}{(\gamma-\alpha)(\gamma-\beta)}.$$

Thus (1.15) can be written as

$$\sum_{n=0}^{\infty} W_n u^n = B_1(1-\alpha u)^{-1} + B_2(1-\beta u)^{-1} + B_3(1-\gamma u)^{-1}.$$

This gives

$$\sum_{n=0}^{\infty} W_n u^n = B_1 \sum_{n=0}^{\infty} \alpha^n u^n + B_2 \sum_{n=0}^{\infty} \beta^n u^n + B_3 \sum_{n=0}^{\infty} \gamma^n u^n = \sum_{n=0}^{\infty} (B_1\alpha^n + B_2\beta^n + B_3\gamma^n)u^n.$$

Therefore, comparing coefficients on both sides of the above equality, we obtain

$$W_n = B_1\alpha^n + B_2\beta^n + B_3\gamma^n$$

where

$$\begin{aligned} B_1 &= \frac{W_0\alpha^2 + (W_1 - rW_0)\alpha + (W_2 - rW_1 - sW_0)}{(\alpha-\beta)(\alpha-\gamma)}, \\ B_2 &= \frac{W_0\beta^2 + (W_1 - rW_0)\beta + (W_2 - rW_1 - sW_0)}{(\beta-\alpha)(\beta-\gamma)}, \\ B_3 &= \frac{W_0\gamma^2 + (W_1 - rW_0)\gamma + (W_2 - rW_1 - sW_0)}{(\gamma-\alpha)(\gamma-\beta)}, \end{aligned}$$

and then we get (1.13). □

Note that from (1.8) and (1.13) we have

$$\begin{aligned} W_2 - (\beta+\gamma)W_1 + \beta\gamma W_0 &= W_0\alpha^2 + (W_1 - rW_0)\alpha + (W_2 - rW_1 - sW_0), \\ W_2 - (\alpha+\gamma)W_1 + \alpha\gamma W_0 &= W_0\beta^2 + (W_1 - rW_0)\beta + (W_2 - rW_1 - sW_0), \\ W_2 - (\alpha+\beta)W_1 + \alpha\beta W_0 &= W_0\gamma^2 + (W_1 - rW_0)\gamma + (W_2 - rW_1 - sW_0), \end{aligned}$$

for the three distinct roots case, i.e., $\alpha \neq \beta \neq \gamma$.

In this paper, we define and investigate, in detail, two special cases of the generalized Tribonacci (sequences of) polynomials $\{W_n\}$ which we call them (r,s,t)-Tribonacci and (r,s,t)-Tribonacci-Lucas (sequences of) polynomials. (r,s,t)-Tribonacci (sequences of) polynomials $\{G_n\}_{n\geq 0}$ and (r,s,t)-Tribonacci-Lucas (sequences of) polynomials $\{H_n\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations

$$G_{n+3} = rG_{n+2} + sG_{n+1} + tG_n, \qquad G_0 = 0, G_1 = 1, G_2 = r, \tag{1.16}$$

$$H_{n+3} = rH_{n+2} + sH_{n+1} + tH_n, \qquad H_0 = 3, H_1 = r, H_2 = 2s + r^2. \tag{1.17}$$

The sequences $\{G_n\}_{n\geq 0}$ and $\{H_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} G_{-n} &= -\frac{s}{t}G_{-(n-1)} - \frac{r}{t}G_{-(n-2)} + \frac{1}{t}G_{-(n-3)}, \\ H_{-n} &= -\frac{s}{t}H_{-(n-1)} - \frac{r}{t}H_{-(n-2)} + \frac{1}{t}H_{-(n-3)}, \end{aligned}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (1.16) and (1.17) hold for all integers n.

Next, we present the first few values of the (r, s, t)-Tribonacci and (r, s, t)-Tribonacci-Lucas polynomials with positive and negative subscripts:

Table 1: The first few values of the special third-order numbers with positive and negative subscripts.

n	0	1	2	3	4
G_n	0	1	r	r^2+s	$r^3+2sr+t$
G_{-n}		0	$\frac{1}{t}$	$-\frac{s}{t^2}$	$-\frac{1}{t^4}(rt^2-s^2t)$
H_n	3	r	$2s+r^2$	$r^3+3sr+3t$	$r^4+4r^2s+4tr+2s^2$
H_{-n}		$-\frac{s}{t}$	$\frac{1}{t^2}(s^2-2rt)$	$\frac{1}{t^3}(-s^3+3rst+3t^2)$	$\frac{1}{t^4}(2r^2t^2-4rs^2t+s^4-4st^2)$

Note that for all n we have

$$G_{-n} = \frac{(\gamma-\beta)\alpha^{1-n} + (\alpha-\gamma)\beta^{1-n} + (\beta-\alpha)\gamma^{1-n}}{(\gamma-\beta)\alpha^{n+1} + (\alpha-\gamma)\beta^{n+1} + (\beta-\alpha)\gamma^{n+1}}G_n.$$

Lemma 9 gives the following results as particular examples (generating functions of (r, s, t)-Tribonacci and (r, s, t)-Tribonacci-Lucas polynomials).

Corollary 11. *Generating functions of (r, s, t)-Tribonacci and (r, s, t)-Tribonacci-Lucas polynomials are*

$$\begin{aligned} \sum_{n=0}^{\infty} G_n u^n &= \frac{u}{1-ru-su^2-tu^3}, \\ \sum_{n=0}^{\infty} H_n u^n &= \frac{3-2ru-su^2}{1-ru-su^2-tu^3}, \end{aligned}$$

respectively.

Proof. In Lemma 9, take $W_n = G_n$ with $G_0 = 0, G_1 = 1, G_2 = r$ and $W_n = H_n$ with $H_0 = 3, H_1 = r, H_2 = 2s + r^2$, respectively. □

For all integers n, Binet's formula of (r,s,t)-Tribonacci and (r,s,t)-Tribonacci-Lucas polynomials (using initial conditions in (1.16) and (1.17)) can be expressed as follows:

Theorem 12.

(a) *(Three Distinct Roots Case: $\alpha \neq \beta \neq \gamma$) For all integers n, Binet's formulas of (r,s,t)-Tribonacci and (r,s,t)-Tribonacci-Lucas polynomials are*

$$\begin{aligned} G_n &= \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)} \\ &= \frac{\alpha^{n+2}}{r\alpha^2+2s\alpha+3t} + \frac{\beta^{n+2}}{r\beta^2+2s\beta+3t} + \frac{\gamma^{n+2}}{r\gamma^2+2s\gamma+3t}, \\ H_n &= \alpha^n+\beta^n+\gamma^n, \end{aligned}$$

respectively.

(b) *(Two Distinct Roots Case: $\alpha \neq \beta = \gamma$) For all integers n, Binet's formulas of (r,s,t)-Tribonacci and (r,s,t)-Tribonacci-Lucas polynomials are*

$$\begin{aligned} G_n &= \frac{\alpha^{n+1}-(\alpha+n(\alpha-\beta))\beta^n}{(\beta-\alpha)^2}, \\ H_n &= \alpha^n+\beta^n+\gamma^n = \alpha^n+2\beta^n, \end{aligned}$$

respectively.

(c) *(Single Root Case: $\alpha = \beta = \gamma = \frac{r}{3}$) For all integers n, Binet's formulas of (r,s,t)-Tribonacci and (r,s,t)-Tribonacci-Lucas polynomials are*

$$\begin{aligned} G_n &= \frac{n(n+1)}{2}\alpha^{n-1} = \frac{n(n+1)}{2}\left(\frac{r}{3}\right)^{n-1}, \\ H_n &= \alpha^n+\beta^n+\gamma^n = 3\alpha^n, \end{aligned}$$

respectively.

(d) *$(\alpha,\beta,\gamma :$ arbitrary$)$*

$$H_n = \alpha^n+\beta^n+\gamma^n.$$

Proof. (a),(b),(c): Use the equations in (1.7) and Theorem 3 (or Theorem 6) and initial conditions in (1.16) and (1.17). (d) follows from (a), (b) and (c). □

Note that the above theorem can be given as follows:

$$G_n = \begin{cases} \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}, \\ \frac{\alpha^{n+2}}{r\alpha^2+2s\alpha+3t} + \frac{\beta^{n+2}}{r\beta^2+2s\beta+3t} + \frac{\gamma^{n+2}}{r\gamma^2+2s\gamma+3t}, & , \quad \text{if } \alpha \neq \beta \neq \gamma \text{ (Distinct Roots Case)} \\ \frac{\alpha^{n+1}-(\alpha+n(\alpha-\beta))\beta^n}{(\beta-\alpha)^2} & , \quad \text{if } \alpha \neq \beta = \gamma \text{ (Two Distinct Roots Case)} \\ \frac{n(n+1)}{2}\alpha^{n-1} = \frac{n(n+1)}{2}\left(\frac{r}{3}\right)^{n-1} & , \quad \text{if } \alpha = \beta = \gamma = \frac{r}{3} \text{ (Single Root Case)} \end{cases}$$

and

$$H_n = \alpha^n + \beta^n + \gamma^n$$

where α, β, γ are arbitrary, i.e.,

$$H_n = \begin{cases} \alpha^n + \beta^n + \gamma^n & , \quad \text{if } \alpha \neq \beta \neq \gamma \text{ (Distinct Roots Case)} \\ \alpha^n + 2\beta^n & , \quad \text{if } \alpha \neq \beta = \gamma \text{ (Two Distinct Roots Case)} \\ 3\alpha^n = 3(\frac{r}{3})^n & , \quad \text{if } \alpha = \beta = \gamma = \frac{r}{3} \text{ (Single Root Case)} \end{cases}$$

If some of the roots of characteristic equation is 1 then we get the following corollary as a special case of Theorem 12 .

Corollary 13. *For all integers n, Binet's formulas of (r, s, t)-Tribonacci and (r, s, t)-Tribonacci-Lucas polynomials are given as follows:*

(a) *(Three Distinct Roots Case: $\alpha \neq \beta \neq \gamma = 1$)*

$$\begin{aligned} G_n &= \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-1)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-1)} + \frac{1}{(1-\alpha)(1-\beta)} \\ &= \frac{\alpha^{n+2}}{r\alpha^2+2s\alpha+3t} + \frac{\beta^{n+2}}{r\beta^2+2s\beta+3t} + \frac{1}{r+2s+3t}, \\ H_n &= \alpha^n + \beta^n + 1. \end{aligned}$$

(b) *(Two Distinct Roots Case: $\alpha \neq \beta = \gamma = 1$)*

$$\begin{aligned} G_n &= \frac{\alpha^{n+1} + ((1-\alpha)n - \alpha)}{(1-\alpha)^2}, \\ H_n &= \alpha^n + 2. \end{aligned}$$

(c) *(Single Root Case: $\alpha = \beta = \gamma = 1 = \frac{r}{3}$)*

$$\begin{aligned} G_n &= \frac{n(n+1)}{2}, \\ H_n &= 3. \end{aligned}$$

If some of the roots of characteristic equation is -1 then we get the following corollary as a special case of Theorem 12.

Corollary 14. *For all integers n, Binet's formulas of (r,s,t)-Tribonacci and (r,s,t)-Tribonacci-Lucas polynomials are given as follows:*

(a) *(Three Distinct Roots Case: $\alpha \neq \beta \neq \gamma = -1$)*

$$\begin{aligned} G_n &= \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha+1)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta+1)} + \frac{(-1)^{n+1}}{(1+\alpha)(1+\beta)} \\ &= \frac{\alpha^{n+2}}{r\alpha^2+2s\alpha+3t} + \frac{\beta^{n+2}}{r\beta^2+2s\beta+3t} + \frac{(-1)^n}{r-2s+3t}, \\ H_n &= \alpha^n + \beta^n + (-1)^n . \end{aligned}$$

(b) *(Two Distinct Roots Case: $\alpha \neq \beta = \gamma = -1$)*

$$\begin{aligned} G_n &= \frac{\alpha^{n+1} - (\alpha + (\alpha+1)n)(-1)^n}{(1+\alpha)^2}, \\ H_n &= \alpha^n + 2(-1)^n. \end{aligned}$$

(c) *(Single Root Case: $\alpha = \beta = \gamma = -1 = \frac{r}{3}$)*

$$\begin{aligned} G_n &= \frac{1}{2}n(n+1)(-1)^{n+1}, \\ H_n &= 3(-1)^n . \end{aligned}$$

2 Generalized co-Tribonacci Polynomials

In this section, for r,s,t satisfying Eq. (1.1), we define and investigate a new sequence and its two special cases, namely the generalized co-Tribonacci, $(r,s,t)-$co-Tribonacci and $(r,s,t)-$co-Tribonacci-Lucas polynomials.

For r,s,t satisfying Eq. (1.1), the generalized co-Tribonacci polynomials (or generalized $(r,s,t)-$co-Tribonacci polynomials or generalized co-3-step Fibonacci polynomials)

$$\{Y_n(Y_0(x),Y_1(x),Y_2(x);-s,-rt,t^2)\}_{n\geq 0}$$

(or shortly $\{Y_n(x)\}_{n\geq 0}$) is defined as follows:

$$Y_n(x) = -sY_{n-1}(x) - rtY_{n-2}(x) + t^2Y_{n-3}(x), \qquad Y_0(x)=d(x), Y_1(x)=e(x), Y_2(x)=f(x), \quad n\geq 3 \tag{2.1}$$

where $Y_0(x), Y_1(x), Y_2(x)$ are arbitrary complex (or real) polynomials with real coefficients.

The sequence $\{Y_n(x)\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$Y_{-n}(x) = -\frac{-rt}{t^2}Y_{-(n-1)}(x) - \frac{-s}{t^2}Y_{-(n-2)}(x) + \frac{1}{t^2}Y_{-(n-3)}(x)$$

for $n = 1, 2, 3, ...$ when $t \neq 0$. Therefore, recurrence (2.1) holds for all integer n. Note that for $n \geq 1$, $Y_{-n}(x)$ need not to be a polynomial in the ordinary sense.

Remark 15. *For simplicity, throughout the rest of the paper we denote*

$$r_1 = -s, s_1 = -rt, t_1 = t^2$$

and write (2.1) as

$$Y_n(x) = r_1 Y_{n-1}(x) + s_1 Y_{n-2}(x) + t_1 Y_{n-3}(x), \qquad Y_0(x) = d(x), Y_1(x) = e(x), Y_2(x) = f(x), \quad n \geq 3 \tag{2.2}$$

unless otherwise stated. So, we can easily use and modify the results which are given in section Introduction and forthcoming sections, just by setting (substituting) r_1, s_1, t_1 *for* r, s, t.

As $\{Y_n(x)\}$ is a third-order recurrence sequence (difference equation), it's characteristic equation is

$$y^3 - r_1 y^2 - s_1 y - t_1 = 0, \tag{2.3}$$

i.e.,

$$y^3 + sy^2 + rty - t^2 = 0. \tag{2.4}$$

The roots of characteristic equation of $\{W_n\}$ will be denoted as

$$\begin{aligned}
\theta_1(x) &= \theta_1(x, r_1, s_1, t_1) = \theta_1(x, -s, -rt, t^2), \\
\theta_2(x) &= \theta_2(x, r_1, s_1, t_1) = \theta_2(x, -s, -rt, t^2), \\
\theta_3(x) &= \theta_3(x, r_1, s_1, t_1) = \theta_3(x, -s, -rt, t^2).
\end{aligned}$$

Remark 16. *As before, for the sake of simplicity throughout the rest of the paper, we use*

$$Y_n, Y_0, Y_1, Y_2, \theta_1, \theta_2, \theta_3$$

instead of

$$Y_n(x), Y_0(x), Y_1(x), Y_2(x), \theta_1(x), \theta_2(x), \theta_3(x),$$

respectively, unless otherwise stated. For example, we write

$$Y_n = -sY_{n-1} - rtY_{n-2} + t^2Y_{n-3}, \qquad Y_0 = d, Y_1 = e, Y_2 = f, \quad n \geq 3$$

and

$$Y_n = r_1Y_{n-1} + s_1Y_{n-2} + t_1Y_{n-3}, \qquad Y_0 = d, Y_1 = e, Y_2 = f, \quad n \geq 3$$

for the equations (2.1) and (2.2). Also we write V_n, V_0, V_1, V_2 *instead of* $V_n(x)$ *with initial conditions* $V_0(x), V_1(x), V_2(x)$ *for any subsequence* $\{V_n(x)\}$ *of* $\{Y_n\}$.

The roots of characteristic equation of $\{W_n\}$ are

$$\begin{aligned}
\theta_1 &= \frac{r_1}{3} + C_3 + C_4 = \frac{-s}{3} + C_3 + C_4, \\
\theta_2 &= \frac{r_1}{3} + \omega C_3 + \omega^2 C_4 = \frac{-s}{3} + \omega C_3 + \omega^2 C_4, \\
\theta_3 &= \frac{r_1}{3} + \omega^2 C_3 + \omega C_4 = \frac{-s}{3} + \omega^2 C_3 + \omega C_4,
\end{aligned}$$

where

$$\begin{aligned}
C_3 &= \left(\frac{r_1^3}{27} + \frac{r_1 s_1}{6} + \frac{t_1}{2} + \sqrt{\Delta_2}\right)^{1/3} = \left(-\frac{1}{27}s^3 + \frac{1}{6}rst + \frac{1}{2}t^2 + \sqrt{\Delta_2}\right)^{1/3}, \\
C_4 &= \left(\frac{r_1^3}{27} + \frac{r_1 s_1}{6} + \frac{t_1}{2} - \sqrt{\Delta_2}\right)^{1/3} = \left(-\frac{1}{27}s^3 + \frac{1}{6}rst + \frac{1}{2}t^2 - \sqrt{\Delta_2}\right)^{1/3}, \\
\Delta_2 &= \frac{r_1^3 t_1}{27} - \frac{r_1^2 s_1^2}{108} + \frac{r_1 s_1 t_1}{6} - \frac{s_1^3}{27} + \frac{t_1^2}{4} = \frac{1}{108}t^2\left(4r^3t - r^2s^2 - 4s^3 + 27t^2 + 18rst\right), \\
\omega &= \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3).
\end{aligned}$$

We have the following relations between the roots of characteristic equations of generalized Tribonacci and generalized co-Tribonacci polynomials:

Lemma 17. *There are the following relations between* α, β, γ *and* $\theta_1, \theta_2, \theta_3$*:*

(a)

$$\begin{aligned}
\theta_1 &= \beta\gamma, \\
\theta_2 &= \alpha\beta, \\
\theta_3 &= \alpha\gamma.
\end{aligned} \tag{2.5}$$

(b)

$$\theta_1 \neq \theta_2 \neq \theta_3 \Leftrightarrow \alpha \neq \beta \neq \gamma.$$

(c)

$$\theta_1 \neq \theta_2 = \theta_3 \Leftrightarrow \alpha \neq \beta = \gamma.$$

(d)

$$\theta_1 = \theta_2 = \theta_3 \Leftrightarrow \alpha = \beta = \gamma.$$

We have the following identities between $\theta_1, \theta_2, \theta_3$ and r_1, s_1, t_1.

Lemma 18. *There are close relations between the roots of characteristic equation (2.3) and r_1, s_1, t_1 as follows.*

(a) *Arbitrary Roots Case ($\theta_1, \theta_2, \theta_3$ are arbitrary).*

$$\begin{cases} \theta_1 + \theta_2 + \theta_3 = r_1 = -s, \\ \theta_1\theta_2 + \theta_1\theta_3 + \theta_2\theta_3 = -s_1 = rt, \\ \theta_1\theta_2\theta_3 = t_1 = t^2. \end{cases} \tag{2.6}$$

(b) *Two Distinct Roots Case ($\theta_1 \neq \theta_2 = \theta_3$).*

$$\begin{array}{rcl} \theta_1 & \neq & \theta_2 = \theta_3 \\ & \Leftrightarrow & \\ r_1 & = & \theta_1 + 2\theta_2, \\ s_1 & = & -\theta_2\left(2\theta_1 + \theta_2\right), \\ t_1 & = & \theta_1\theta_2^2. \end{array}$$

(c) *Single Root Case ($\theta_1 = \theta_2 = \theta_3$).*

$$\begin{array}{rcl} \theta_1 & = & \theta_2 = \theta_3 \\ & \Leftrightarrow & r_1 = 3\theta_1,\ s_1 = -3\theta_1^2,\ t_1 = \theta_1^3 \\ & \Leftrightarrow & \dfrac{r_1}{3} = \theta_1,\ s_1 = -\dfrac{1}{3}r_1^2,\ t_1 = \dfrac{1}{27}r_1^3. \end{array}$$

and

$$\theta_1 = \theta_2 = \theta_3 \Leftrightarrow \Delta_2 = C_3 = C_4 = 0.$$

Using the roots of characteristic equation and the recurrence relation of Y_n, Binet's formula can be given as follows:

Theorem 19. *For all integers n, Binet's formula of generalized co-Tribonacci polynomials is given as follows.*

(a) *(Three Distinct Roots Case: $\theta_1 \neq \theta_2 \neq \theta_3$) Binet's formula of generalized co-Tribonacci polynomials is*

$$\begin{aligned} Y_n &= \frac{p_4\theta_1^n}{(\theta_1-\theta_2)(\theta_1-\theta_3)} + \frac{p_5\theta_2^n}{(\theta_2-\theta_1)(\theta_2-\theta_3)} + \frac{p_6\theta_3^n}{(\theta_3-\theta_1)(\theta_3-\theta_2)} \\ &= B_1\theta_1^n + B_2\theta_2^n + B_3\theta_3^n, \end{aligned} \tag{2.7}$$

where

$$p_4 = Y_2-(\theta_2+\theta_3)Y_1+\theta_2\theta_3Y_0,\ p_5 = Y_2-(\theta_1+\theta_3)Y_1+\theta_1\theta_3Y_0,\ p_6 = Y_2-(\theta_1+\theta_2)Y_1+\theta_1\theta_2Y_0,$$

and

$$\begin{aligned} B_1 &= \frac{p_4}{(\theta_1-\theta_2)(\theta_1-\theta_3)} = \frac{Y_2-(\theta_2+\theta_3)Y_1+\theta_2\theta_3Y_0}{(\theta_1-\theta_2)(\theta_1-\theta_3)} \\ &= \frac{(\theta_1Y_2+\theta_1(-r_1+\theta_1)Y_1+t_1Y_0)}{r_1\theta_1^2+2s_1\theta_1+3t_1}, \\ B_2 &= \frac{p_5}{(\theta_2-\theta_1)(\theta_2-\theta_3)} = \frac{Y_2-(\theta_1+\theta_3)Y_1+\theta_1\theta_3Y_0}{(\theta_2-\theta_1)(\theta_2-\theta_3)} \\ &= \frac{(\theta_2Y_2+\theta_2(-r_1+\theta_2)Y_1+t_1Y_0)}{r_1\theta_2^2+2s_1\theta_2+3t_1}, \\ B_3 &= \frac{p_6}{(\theta_3-\theta_1)(\theta_3-\theta_2)} = \frac{Y_2-(\theta_1+\theta_2)Y_1+\theta_1\theta_2Y_0}{(\theta_3-\theta_1)(\theta_3-\theta_2)} \\ &= \frac{(\theta_3Y_2+\theta_3(-r_1+\theta_3)Y_1+t_1Y_0)}{r_1\theta_3^2+2s_1\theta_3+3t_1}. \end{aligned}$$

(b) *(Two Distinct Roots Case: $\theta_1 \neq \theta_2 = \theta_3$) Binet's formula of generalized co-Tribonacci polynomials is*

$$Y_n = B_1\theta_1^n + (B_2+B_3n)\theta_2^n \tag{2.8}$$

where

$$\begin{aligned} B_1 &= \frac{Y_2-2\theta_2Y_1+\theta_2^2Y_0}{(\theta_2-\theta_1)^2} = \frac{4Y_2-4(r_1-\theta_1)Y_1+(r_1-\theta_1)^2Y_0}{(r_1-3\theta_1)^2}, \\ B_2 &= \frac{-Y_2+2\theta_2Y_1-\theta_1(2\theta_2-\theta_1)Y_0}{(\theta_2-\theta_1)^2} = \frac{4(-Y_2+(r_1-\theta_1)Y_1+\theta_1(-r_1+2\theta_1)Y_0)}{(r_1-3\theta_1)^2}, \\ B_3 &= \frac{Y_2-(\theta_2+\theta_1)Y_1+\theta_2\theta_1Y_0}{\theta_2(\theta_2-\theta_1)} = \frac{2(2Y_2-(r_1+\theta_1)Y_1+\theta_1(r_1-\theta_1)Y_0)}{(r_1-3\theta_1)(r_1-\theta_1)}. \end{aligned}$$

(c) *(Single Root Case: $\theta_1 = \theta_2 = \theta_3 = \frac{r_1}{3}$) Binet's formula of generalized co-Tribonacci polynomials is*

$$Y_n = (B_1+B_2n+B_3n^2)\times\theta_1^n \tag{2.9}$$

where

$$\begin{aligned} B_1 &= Y_0, \\ B_2 &= \frac{-Y_2 + 4Y_1\theta_1 - 3Y_0\theta_1^2}{2\theta_1^2} = \frac{3}{2r_1^2}(-3Y_2 + 4r_1Y_1 - r_1^2Y_0), \\ B_3 &= \frac{Y_2 - 2Y_1\theta_1 + Y_0\theta_1^2}{2\theta_1^2} = \frac{1}{2r_1^2}(9Y_2 - 6r_1Y_1 + r_1^2Y_0). \end{aligned}$$

Note that (a), (b) and (c) of the above theorem can be given as follows:

$$Y_n = \begin{cases} B_1\theta_1^n + B_2\theta_2^n + B_3\theta_3^n & , \text{ if } \theta_1 \neq \theta_2 \neq \theta_3 \text{ (Three Distinct Roots Case)} \\ B_1\theta_1^n + (B_2 + B_3n)\theta_2^n & , \text{ if } \theta_1 \neq \theta_2 = \theta_3 \text{ (Two Distinct Roots Case)} \\ (B_1 + B_2n + B_3n^2)\theta_1^n & , \text{ if } \theta_1 = \theta_2 = \theta_3 \text{ (Single Root Case)} \end{cases} \tag{2.10}$$

where each triple B_1, B_2, B_3 given as in Theorem 19 (a), (b) and (c), respectively.

Lemma 20.

(a) *(Three Distinct Roots Case: $\theta_1 \neq \theta_2 \neq \theta_3$) If B_1, B_2 and B_3 are as in Theorem 19 (a), then we have*

$$B_1 + B_2 + B_3 = Y_0.$$

(b) *(Two Distinct Roots Case: $\theta_1 \neq \theta_2 = \theta_3$) If B_1, B_2 and B_3 are as in Theorem 19 (b), then we have*

$$B_1 + B_2 + B_3 = \frac{-Y_2 + (\theta_1 + \theta_2)Y_1 - \theta_2^2Y_0}{\theta_2(\theta_1 - \theta_2)}$$

(c) *(Single Root Case: $\theta_1 = \theta_2 = \theta_3 = \frac{r_1}{3}$) If B_1, B_2 and B_3 are as in Theorem 19 (c), then we have*

$$B_1 + B_2 + B_3 = \frac{Y_1}{\theta_1} = \frac{3}{r_1}Y_1$$

Theorem 19 can be given in the following form:

Theorem 21. *Binet's formula of generalized co-Tribonacci polynomials is given as follows according to the roots of characteristic equation (2.3):*

(a) *(Three Distinct Roots Case: $\theta_1 \neq \theta_2 \neq \theta_3$)*

$$\begin{aligned} Y_n &= \frac{Y_2-(\theta_2+\theta_3)Y_1+\theta_2\theta_3Y_0}{(\theta_1-\theta_2)(\theta_1-\theta_3)}\theta_1^n+\frac{Y_2-(\theta_1+\theta_3)Y_1+\theta_1\theta_3Y_0}{(\theta_2-\theta_1)(\theta_2-\theta_3)}\theta_2^n \\ &\quad +\frac{Y_2-(\theta_1+\theta_2)Y_1+\theta_1\theta_2Y_0}{(\theta_3-\theta_1)(\theta_3-\theta_2)}\theta_3^n \\ &= \frac{(\theta_1Y_2+\theta_1(-r_1+\theta_1)Y_1+t_1Y_0)}{r_1\theta_1^2+2s_1\theta_1+3t_1}\theta_1^n+\frac{(\theta_2Y_2+\theta_2(-r_1+\theta_2)Y_1+t_1Y_0)}{r_1\theta_2^2+2s_1\theta_2+3t_1}\theta_2^n \\ &\quad +\frac{(\theta_3Y_2+\theta_3(-r_1+\theta_3)Y_1+t_1Y_0)}{r_1\theta_3^2+2s_1\theta_3+3t_1}\theta_3^n. \end{aligned}$$

(b) *(Two Distinct Roots Case: $\theta_1 \neq \theta_2 = \theta_3$)*

$$Y_n=\frac{Y_2-2\theta_2Y_1+\theta_2^2Y_0}{(\theta_2-\theta_1)^2}\theta_1^n+\left(\frac{-Y_2+2\theta_2Y_1-\theta_1(2\theta_2-\theta_1)Y_0}{(\theta_2-\theta_1)^2}+\frac{Y_2-(\theta_2+\theta_1)Y_1+\theta_2\theta_1Y_0}{\theta_2(\theta_2-\theta_1)}n\right)\theta_2^n$$

and

$$\begin{aligned} Y_n &= \frac{4Y_2-4(r_1-\theta_1)Y_1+(r_1-\theta_1)^2Y_0}{(r_1-3\theta_1)^2}\theta_1^n \\ &\quad +\frac{1}{\theta_2(r_1-3\theta_2)^2}((-\theta_2Y_2+2\theta_2^2Y_1+(2r_1\theta_2^2+(r_1^2+8s_1)\theta_2+8t_1)Y_0) \\ &\quad +((3\theta_2-r_1)Y_2-(r_1-3\theta_2)(\theta_2-r_1)Y_1-(r_1\theta_2^2+(r_1^2+6s_1)\theta_2+6t_1)Y_0)n)\theta_2^n. \end{aligned}$$

(c) *(Single Root Case: $\theta_1=\theta_2=\theta_3=\frac{r_1}{3}$)*

$$\begin{aligned} Y_n &= \frac{1}{2}(2\theta_1^2Y_0+(-Y_2+4Y_1\theta_1-3Y_0\theta_1^2)n+(Y_2-2Y_1\theta_1+Y_0\theta_1^2)n^2)\theta_1^{n-2} \\ &= \frac{1}{2}(n(n-1)Y_2-2n(n-2)\theta_1Y_1+(n-1)(n-2)\theta_1^2Y_0)\theta_1^{n-2} \\ &= \frac{1}{18}(9n(n-1)Y_2-6n(n-2)r_1Y_1+(n-1)(n-2)r_1^2Y_0)\left(\frac{r_1}{3}\right)^{n-2}. \end{aligned}$$

Note that each of Theorem 19, equations (2.10) and Theorem 21 have their advantages to present Binet's formula of generalized co-Tribonacci numbers.

If some roots of characteristic equation is 1 then we get the following corollary as a special case of Theorem 21.

Corollary 22. *Binet's formula of generalized co-Tribonacci polynomials is given as follows according to the roots of characteristic equation (2.3):*

(a) *(Three Distinct Roots Case: $\theta_1 \neq \theta_2 \neq \theta_3 = 1$)*

$$\begin{aligned} Y_n &= \frac{Y_2-(\theta_2+1)Y_1+\theta_2 Y_0}{(\theta_1-\theta_2)(\theta_1-1)}\theta_1^n+\frac{Y_2-(\theta_1+1)Y_1+\theta_1 Y_0}{(\theta_2-\theta_1)(\theta_2-1)}\theta_2^n \\ &\quad +\frac{Y_2+(-r_1+1)Y_1+t_1 Y_0}{t_1-r_1+2} \\ &= \frac{(\theta_1 Y_2+\theta_1(-r_1+\theta_1)Y_1+t_1 Y_0)}{r_1\theta_1^2+2s_1\theta_1+3t_1}\theta_1^n+\frac{(\theta_2 Y_2+\theta_2(-r_1+\theta_2)Y_1+t_1 Y_0)}{r_1\theta_2^2+2s_1\theta_2+3t_1}\theta_2^n \\ &\quad +\frac{Y_2+(-r_1+1)Y_1+t_1 Y_0}{r_1+2s_1+3t_1} \end{aligned}$$

(b) *(Two Distinct Roots Case: $\theta_1 \neq \theta_2 = \theta_3 = 1$)* $\theta_2 = 1, \theta_3 = 1$

$$Y_n = \frac{1}{(1-t_1)^2}((Y_2-2Y_1+Y_0)\theta_1^n+(-Y_2+2Y_1+t_1(t_1-2)Y_0)+(1-t_1)(Y_2-(1+t_1)Y_1+t_1Y_0)n).$$

(c) *(Single Root Case: $\theta_1 = \theta_2 = \theta_3 = 1 = \frac{r_1}{3}$)*

$$Y_n = \frac{1}{2}(n(n-1)Y_2-2n(n-2)\times Y_1+(n-1)(n-2)Y_0).$$

If some roots of characteristic equation is -1 then we get the following corollary as a special case of Theorem 21 .

Corollary 23. *Binet's formula of generalized co-Tribonacci polynomials is given as follows according to the roots of characteristic equation (2.3):*

(a) *(Three Distinct Roots Case: $\theta_1 \neq \theta_2 \neq \theta_3 = -1$)*

$$\begin{aligned} Y_n &= \frac{Y_2-(\theta_2-1)Y_1-\theta_2 Y_0}{(\theta_1-\theta_2)(\theta_1+1)}\theta_1^n+\frac{Y_2-(\theta_1-1)Y_1-\theta_1 Y_0}{(\theta_2-\theta_1)(\theta_2+1)}\theta_2^n \\ &\quad +\frac{Y_2-(r_1+1)Y_1-t_1 Y_0}{r_1-t_1+2}(-1)^n \\ &= \frac{(\theta_1 Y_2+\theta_1(-r_1+\theta_1)Y_1+t_1 Y_0)}{r_1\theta_1^2+2s_1\theta_1+3t_1}\theta_1^n+\frac{(\theta_2 Y_2+\theta_2(-r_1+\theta_2)Y_1+t_1 Y_0)}{r_1\theta_2^2+2s_1\theta_2+3t_1}\theta_2^n \\ &\quad +\frac{(Y_2-(r_1+1)Y_1-t_1 Y_0)}{-r_1+2s_1-3t_1}(-1)^n. \end{aligned}$$

(b) *(Two Distinct Roots Case: $\theta_1 \neq \theta_2 = \theta_3 = -1$)*

$$\begin{aligned} Y_n &= \frac{1}{(r_1+3)^2}((Y_2+2Y_1+Y_0)\theta_1^n+((Y_2+2Y_1+(-r_1^2+2r_1-8s_1+8t_1)Y_0) \\ &\quad +((-3-r_1)Y_2+(r_1+1)(r_1+3)Y_1+(r_1^2-r_1+6s_1-6t_1)Y_0)n)(-1)^{n+1}). \end{aligned}$$

(c) *(Single Root Case: $\theta_1 = \theta_2 = \theta_3 = -1 = \frac{r_1}{3}$)*

$$Y_n = \frac{1}{2}(n(n-1)Y_2 + 2n(n-2)Y_1 + (n-1)(n-2)Y_0)(-1)^{n-2}.$$

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} Y_n u^n$ of the sequence Y_n.

Lemma 24. *Suppose that $f_{Y_n}(u) = \sum_{n=0}^{\infty} Y_n u^n$ is the ordinary generating function of the generalized co-Tribonacci (sequence of) polynomials $\{Y_n\}_{n\geq 0}$. Then, $\sum_{n=0}^{\infty} Y_n u^n$ is given by*

$$\sum_{n=0}^{\infty} Y_n u^n = \frac{Y_0 + (Y_1 - r_1Y_0)u + (Y_2 - r_1Y_1 - s_1Y_0)u^2}{1 - r_1u - s_1u^2 - t_1u^3}.$$

We next present Binet's formula of the generalized co-Tribonacci (sequence of) polynomials $\{Y_n\}_{n\geq 0}$ by the use of generating function for Y_n.

Theorem 25. *(Binet's formula of the generalized co-Tribonacci polynomials $\{Y_n\}_{n\geq 0}$. For all integers n, we have*

$$Y_n = \frac{q_4\theta_1^n}{(\theta_1-\theta_2)(\theta_1-\theta_3)} + \frac{q_5\theta_2^n}{(\theta_2-\theta_1)(\theta_2-\theta_3)} + \frac{q_6\theta_3^n}{(\theta_3-\theta_1)(\theta_3-\theta_2)} \tag{2.11}$$

where

$$\begin{aligned} q_4 &= Y_0\theta_1^2 + (Y_1 - r_1Y_0)\theta_1 + (Y_2 - r_1Y_1 - s_1Y_0), \\ q_5 &= Y_0\theta_2^2 + (Y_1 - r_1Y_0)\theta_2 + (Y_2 - r_1Y_1 - s_1Y_0), \\ q_6 &= Y_0\theta_3^2 + (Y_1 - r_1Y_0)\theta_3 + (Y_2 - r_1Y_1 - s_1Y_0). \end{aligned}$$

Note that from (2.7) and (2.11) we have

$$\begin{aligned} Y_2 - (\theta_2+\theta_3)Y_1 + \theta_2\theta_3Y_0 &= Y_0\theta_1^2 + (Y_1 - r_1Y_0)\theta_1 + (Y_2 - r_1Y_1 - s_1Y_0), \\ Y_2 - (\theta_1+\theta_3)Y_1 + \theta_1\theta_3Y_0 &= Y_0\theta_2^2 + (Y_1 - r_1Y_0)\theta_2 + (Y_2 - r_1Y_1 - s_1Y_0), \\ Y_2 - (\theta_1+\theta_2)Y_1 + \theta_1\theta_2Y_0 &= Y_0\theta_3^2 + (Y_1 - r_1Y_0)\theta_3 + (Y_2 - r_1Y_1 - s_1Y_0). \end{aligned}$$

In this paper, we define and investigate, in detail, two special cases of the generalized co-Tribonacci polynomials $\{Y_n\}$ which we call them (r,s,t)-co-Tribonacci (or (r_1,s_1,t_1)-Tribonacci) and (r,s,t)-co-Tribonacci-Lucas (or (r_1,s_1,t_1)-Tribonacci-Lucas) polynomials. (r,s,t)-co-Tribonacci polynomials $\{U_n\}_{n\geq 0}$ and (r,s,t)-co-Tribonacci-Lucas polynomials $\{S_n\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations

$$\begin{aligned} U_{n+3} &= r_1U_{n+2}+s_1U_{n+1}+t_1U_n, \qquad U_0=0, U_1=1, U_2=r_1, &(2.12)\\ S_{n+3} &= r_1S_{n+2}+s_1S_{n+1}+t_1S_n, \qquad S_0=3, S_1=r_1, S_2=2s_1+r_1^2. &(2.13)\end{aligned}$$

The sequences $\{U_n\}_{n\geq 0}$ and $\{S_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} U_{-n} &= -\frac{s_1}{t_1}U_{-(n-1)}-\frac{r_1}{t_1}U_{-(n-2)}+\frac{1}{t_1}U_{-(n-3)},\\ S_{-n} &= -\frac{s_1}{t_1}S_{-(n-1)}-\frac{r_1}{t_1}S_{-(n-2)}+\frac{1}{t_1}S_{-(n-3)},\end{aligned}$$

for $n=1,2,3,...$ respectively. Therefore, recurrences (2.12) and (2.13) hold for all integers n.

The sequences $\{U_n\}$ and $\{S_n\}$ can be written in the following forms:

$$\begin{aligned} U_n &= -sU_{n-1}-rtU_{n-2}+t^2U_{n-3}, \qquad U_0=0, U_1=1, U_2=-s,\\ S_n &= -sS_{n-1}-rtS_{n-2}+t^2S_{n-3}, \qquad S_0=3, S_1=-s, S_2=-2rt+s^2.\end{aligned}$$

Next, we present the first few values of the (r,s,t)-co-Tribonacci and (r,s,t)-co-Tribonacci-Lucas polynomials with positive and negative subscripts (in terms of r_1,s_1,t_1):

Table 2: The first few values of the special third-order numbers with positive and negative subscripts.

n	0	1	2	3	4
U_n	0	1	r_1	$r_1^2+s_1$	$r_1^3+2s_1r_1+t_1$
U_{-n}		0	$\frac{1}{t_1}$	$-\frac{s_1}{t_1^2}$	$-\frac{1}{t_1^4}(r_1t_1^2-s_1^2t_1)$
S_n	3	r_1	$2s_1+r_1^2$	$r_1^3+3s_1r_1+3t_1$	$r_1^4+4r_1^2s_1+4t_1r_1+2s_1^2$
S_{-n}		$-\frac{s_1}{t_1}$	$\frac{1}{t_1^2}(s_1^2-2r_1t_1)$	$\frac{1}{t_1^3}(-s_1^3+3r_1s_1t_1+3t_1^2)$	$\frac{1}{t_1^4}(2r_1^2t_1^2-4r_1s_1^2t_1+s_1^4-4s_1t_1^2)$

We present the first few values of the (r,s,t)-co-Tribonacci and (r,s,t)-co-Tribonacci-Lucas polynomials with positive and negative subscripts (in terms of r,s,t):

Table 3: The first few values of the special third-order numbers with positive and negative subscripts.

n	0	1	2	3	4
U_n	0	1	$-s$	s^2-rt	$-s^3+2rst+t^2$
U_{-n}		0	$\frac{1}{t^2}$	$\frac{r}{t^3}$	$\frac{1}{t^4}\left(s+r^2\right)$
S_n	3	$-s$	$-2rt+s^2$	$-s^3+3rst+3t^2$	$2r^2t^2-4rs^2t+s^4-4st^2$
S_{-n}		$\frac{r}{t}$	$\frac{1}{t^2}\left(2s+r^2\right)$	$\frac{1}{t^3}\left(3t+3rs+r^3\right)$	$\frac{1}{t^4}\left(4r^2s+4rt+r^4+2s^2\right)$

For all integers n, Binet's formula of (r,s,t)-co-Tribonacci and (r,s,t)-co-Tribonacci-Lucas polynomials (using initial conditions in (1.16) and (1.17)) can be expressed as follows:

Theorem 26.

(a) *(Three Distinct Roots Case: $\theta_1 \neq \theta_2 \neq \theta_3$) For all integers n, Binet's formulas of (r,s,t)-co-Tribonacci and (r,s,t)-co-Tribonacci-Lucas polynomials are*

$$\begin{aligned} U_n &= \frac{\theta_1^{n+1}}{(\theta_1-\theta_2)(\theta_1-\theta_3)}+\frac{\theta_2^{n+1}}{(\theta_2-\theta_1)(\theta_2-\theta_3)}+\frac{\theta_3^{n+1}}{(\theta_3-\theta_1)(\theta_3-\theta_2)} \\ &= \frac{\theta_1^{n+2}}{r_1\theta_1^2+2s_1\theta_1+3t_1}+\frac{\theta_2^{n+2}}{r_1\theta_2^2+2s_1\theta_2+3t_1}+\frac{\theta_3^{n+2}}{r_1\theta_3^2+2s_1\theta_3+3t_1}, \\ S_n &= \theta_1^n+\theta_2^n+\theta_3^n, \end{aligned}$$

respectively.

(b) *(Two Distinct Roots Case: $\theta_1 \neq \theta_2 = \theta_3$) For all integers n, Binet's formulas of (r,s,t)-co-Tribonacci and (r,s,t)-co-Tribonacci-Lucas polynomials are*

$$\begin{aligned} U_n &= \frac{\theta_1^{n+1}-(\theta_1+n(\theta_1-\theta_2))\theta_2^n}{(\theta_2-\theta_1)^2}, \\ S_n &= \theta_1^n+\theta_2^n+\theta_3^n=\theta_1^n+2\theta_2^n, \end{aligned}$$

respectively.

(c) *(Single Root Case: $\theta_1=\theta_2=\theta_3=\frac{r_1}{3}$) For all integers n, Binet's formulas of (r,s,t)-co-Tribonacci and (r,s,t)-co-Tribonacci-Lucas polynomials are*

$$\begin{aligned} U_n &= \frac{n(n+1)}{2}\theta_1^{n-1}=\frac{n(n+1)}{2}\left(\frac{r_1}{3}\right)^{n-1}, \\ S_n &= \theta_1^n+\theta_2^n+\theta_3^n=3\theta_1^n, \end{aligned}$$

respectively.

(d) *($\theta_1, \theta_2, \theta_3$: arbitrary)*

$$S_n = \theta_1^n + \theta_2^n + \theta_3^n.$$

Note that the above theorem can be given as follows:

$$U_n = \begin{cases} \frac{\theta_1^{n+1}}{(\theta_1-\theta_2)(\theta_1-\theta_3)} + \frac{\theta_2^{n+1}}{(\theta_2-\theta_1)(\theta_2-\theta_3)} + \frac{\theta_3^{n+1}}{(\theta_3-\theta_1)(\theta_3-\theta_2)}, \\ \frac{\theta_1^{n+2}}{r_1\theta_1^2+2s_1\theta_1+3t_1} + \frac{\theta_2^{n+2}}{r_1\theta_2^2+2s_1\theta_2+3t_1} + \frac{\theta_3^{n+2}}{r_1\theta_3^2+2s_1\theta_3+3t_1}, & , \quad \text{if } \theta_1 \neq \theta_2 \neq \theta_3 \text{ (Distinct Roots Case)} \\ \frac{\theta_1^{n+1}-(\theta_1+n(\theta_1-\theta_2))\theta_2^n}{(\theta_2-\theta_1)^2} & , \quad \text{if } \theta_1 \neq \theta_2 = \theta_3 \text{ (Two Distinct Roots Case)} \\ \frac{n(n+1)}{2}\theta_1^{n-1} = \frac{n(n+1)}{2}\left(\frac{r_1}{3}\right)^{n-1} & , \quad \text{if } \theta_1 = \theta_2 = \theta_3 = \frac{r_1}{3} \text{ (Single Root Case)} \end{cases}$$

and

$$S_n = \theta_1^n + \theta_2^n + \theta_3^n$$

where $\theta_1, \theta_2, \theta_3$ are arbitrary, i.e.,

$$S_n = \begin{cases} \theta_1^n + \theta_2^n + \theta_3^n & , \quad \text{if } \theta_1 \neq \theta_2 \neq \theta_3 \text{ (Distinct Roots Case)} \\ \theta_1^n + 2\theta_2^n & , \quad \text{if } \theta_1 \neq \theta_2 = \theta_3 \text{ (Two Distinct Roots Case)} \\ 3\theta_1^n = 3(\frac{r_1}{3})^n & , \quad \text{if } \theta_1 = \theta_2 = \theta_3 = \frac{r_1}{3} \text{ (Single Root Case)} \end{cases}$$

If some of the roots of characteristic equation is 1 then we get the following corollary as a special case of Theorem 26 .

Corollary 27. *For all integers n, Binet's formulas of (r,s,t)-co-Tribonacci and (r,s,t)-co-Tribonacci-Lucas polynomials are given as follows:*

(a) *(Three Distinct Roots Case: $\theta_1 \neq \theta_2 \neq \theta_3 = 1$)*

$$\begin{aligned} U_n &= \frac{\theta_1^{n+1}}{(\theta_1-\theta_2)(\theta_1-1)} + \frac{\theta_2^{n+1}}{(\theta_2-\theta_1)(\theta_2-1)} + \frac{1}{(1-\theta_1)(1-\theta_2)} \\ &= \frac{\theta_1^{n+2}}{r_1\theta_1^2+2s_1\theta_1+3t_1} + \frac{\theta_2^{n+2}}{r_1\theta_2^2+2s_1\theta_2+3t_1} + \frac{1}{r_1+2s_1+3t_1}, \\ S_n &= \theta_1^n + \theta_2^n + 1. \end{aligned}$$

(b) *(Two Distinct Roots Case: $\theta_1 \neq \theta_2 = \theta_3 = 1$)*

$$\begin{aligned} U_n &= \frac{\theta_1^{n+1} + ((1-\theta_1)n - \theta_1)}{(1-\theta_1)^2}, \\ S_n &= \theta_1^n + 2. \end{aligned}$$

(c) *(Single Root Case: $\theta_1 = \theta_2 = \theta_3 = 1 = \frac{r_1}{3}$)*

$$\begin{aligned} U_n &= \frac{n(n+1)}{2}, \\ S_n &= 3. \end{aligned}$$

If some of the roots of characteristic equation is -1, then we get the following corollary as a special case of Theorem 26.

Corollary 28. *For all integers n, Binet's formulas of (r,s,t)-co-Tribonacci and (r,s,t)-co-Tribonacci-Lucas polynomials are given as follows:*

(a) *(Three Distinct Roots Case: $\theta_1 \neq \theta_2 \neq \theta_3 = -1$)*

$$\begin{aligned} U_n &= \frac{\theta_1^{n+1}}{(\theta_1-\theta_2)(\theta_1+1)} + \frac{\theta_2^{n+1}}{(\theta_2-\theta_1)(\theta_2+1)} + \frac{(-1)^{n+1}}{(1+\theta_1)(1+\theta_2)} \\ &= \frac{\theta_1^{n+2}}{r_1\theta_1^2+2s_1\theta_1+3t_1} + \frac{\theta_2^{n+2}}{r_1\theta_2^2+2s_1\theta_2+3t_1} + \frac{(-1)^n}{r_1-2s_1+3t_1}, \\ S_n &= \theta_1^n + \theta_2^n + (-1)^n. \end{aligned}$$

(b) *(Two Distinct Roots Case: $\theta_1 \neq \theta_2 = \theta_3 = -1$)*

$$\begin{aligned} U_n &= \frac{\theta_1^{n+1} - (\theta_1 + (\theta_1+1)n)(-1)^n}{(1+\theta_1)^2}, \\ S_n &= \theta_1^n + 2(-1)^n. \end{aligned}$$

(c) *(Single Root Case: $\theta_1 = \theta_2 = \theta_3 = -1 = \frac{r_1}{3}$)*

$$\begin{aligned} U_n &= \frac{1}{2}n(n+1)(-1)^{n+1}, \\ S_n &= 3(-1)^n. \end{aligned}$$

Lemma 24 gives the following results as particular examples (generating functions of (r,s,t)-co-Tribonacci and (r,s,t)-co-Tribonacci-Lucas polynomials).

Corollary 29. *Generating functions of (r,s,t)-co-Tribonacci and (r,s,t)-co-Tribonacci-Lucas polynomials are*

$$\begin{aligned} \sum_{n=0}^{\infty} U_n u^n &= \frac{u}{1-r_1u-s_1u^2-t_1u^3} = \frac{u}{1+su+rtu^2-t^2u^3}, \\ \sum_{n=0}^{\infty} S_n u^n &= \frac{3-2r_1u-s_1u^2}{1-r_1u-s_1u^2-t_1u^3} = \frac{3+2su+rtu^2}{1+su+rtu^2-t^2u^3}, \end{aligned}$$

respectively.

3 Connections between G_n, H_n and U_n, S_n

S_n can be given as follows.

Lemma 30. *(α, β, γ; $\theta_1, \theta_2, \theta_3$: arbitrary) For all integers n, we have the following formula for S_n.*

$$S_n = \alpha^n \beta^n + \alpha^n \gamma^n + \beta^n \gamma^n.$$

Proof. Use the identities

$$\theta_1 = \beta\gamma,\ \theta_2 = \alpha\beta,\ \theta_3 = \alpha\gamma$$

and

$$S_n = \theta_1^n + \theta_2^n + \theta_3^n.$$

□

For the special cases α, β, γ; $\theta_1, \theta_2, \theta$, Lemma 30 can be written as follows.

Lemma 31. **(a)** *(Three Distinct Roots Case: $\alpha \neq \beta \neq \gamma$, $\theta_1 \neq \theta_2 \neq \theta_3$)*

$$S_n = \alpha^n \beta^n + \alpha^n \gamma^n + \beta^n \gamma^n.$$

(b) *(Two Distinct Roots Case: $\alpha \neq \beta = \gamma$, $\theta_1 \neq \theta_2 = \theta_3$)*

$$S_n = \alpha^n \beta^n + \alpha^n \gamma^n + \beta^n \gamma^n = 2\alpha^n \beta^n + \beta^{2n}.$$

(c) *(Single Root Case: $\alpha = \beta = \gamma = \dfrac{r}{3}$, $\theta_1 = \theta_2 = \theta_3 = \dfrac{r_1}{3}$)*

$$S_n = \alpha^n \beta^n + \alpha^n \gamma^n + \beta^n \gamma^n = 3\alpha^{2n}.$$

We can present the relations between U_n, S_n and G_n, H_n as follows.

Lemma 32. *For all integers n, we have the following formulas:*

(a) $S_n = \frac{1}{2}(H_n^2 - H_{2n})$.

(b) $U_n = t^n G_{-n-1}$ *and* $U_{-n} = t^{-n} G_{n-1}$.

(c) $S_n = t^n H_{-n}$ *and* $S_{-n} = t^{-n} H_n$.

Proof. Use Theorem 12, Theorem 26 and the identities

$$\begin{aligned} \alpha\beta\gamma &= t, \\ \theta_1 &= \beta\gamma,\ \theta_2 = \alpha\beta,\ \theta_3 = \alpha\gamma, \\ S_n &= \beta^n \gamma^n + \alpha^n \gamma^n + \alpha^n \beta^n. \end{aligned}$$

□

4 Simson's Formulas of Tribonacci Polynomials

The following theorem gives Simson's formula of the generalized Tribonacci polynomials $\{W_n\}$.

Theorem 33 (Simson's Formula of Generalized Tribonacci Polynomials)**.** *For all integers n, we have*

$$\begin{vmatrix} W_{n+2} & W_{n+1} & W_n \\ W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \end{vmatrix} = t^n \begin{vmatrix} W_2 & W_1 & W_0 \\ W_1 & W_0 & W_{-1} \\ W_0 & W_{-1} & W_{-2} \end{vmatrix}. \tag{4.1}$$

Proof. Eq. (4.1) can be proved by mathematical induction. For the proof of the case of generalized Tribonacci numbers, see Soykan [16, Theorem 2.2]. For an alternative proof, proof by matrix method, see Theorem 57 (b). □

The previous theorem gives the following results as particular examples.

Corollary 34. *For all integers n, Simson's formula of (r,s,t)-Tribonacci and (r,s,t)-Tribonacci-Lucas polynomials are given as*

$$\begin{vmatrix} G_{n+2} & G_{n+1} & G_n \\ G_{n+1} & G_n & G_{n-1} \\ G_n & G_{n-1} & G_{n-2} \end{vmatrix} = -t^{n-1},$$

$$\begin{vmatrix} H_{n+2} & H_{n+1} & H_n \\ H_{n+1} & H_n & H_{n-1} \\ H_n & H_{n-1} & H_{n-2} \end{vmatrix} = (-4r^3t + r^2s^2 - 18rst + 4s^3 - 27t^2)t^{n-2},$$

respectively.

Note also that (4.1) can be written as

$$\begin{vmatrix} W_{n+m+2} & W_{n+m+1} & W_{n+m} \\ W_{n+m+1} & W_{n+m} & W_{n+m-1} \\ W_{n+m} & W_{n+m-1} & W_{n+m-2} \end{vmatrix} = t^{n+m} \begin{vmatrix} W_2 & W_1 & W_0 \\ W_1 & W_0 & W_{-1} \\ W_0 & W_{-1} & W_{-2} \end{vmatrix}$$

for all integers n,m.

We define

$$\begin{aligned} \Lambda_W(n) &= W_{n+2}^3 + (t+rs)W_{n+1}^3 + t^2W_n^3 - 2rW_{n+1}W_{n+2}^2 - sW_nW_{n+2}^2 \\ &\quad + (r^2-s)W_{n+2}W_{n+1}^2 \\ &\quad + (rt+s^2)W_nW_{n+1}^2 + rtW_n^2W_{n+2} + 2stW_n^2W_{n+1} - (3t-rs)W_{n+2}W_{n+1}W_n. \end{aligned}$$

Then

$$\Lambda_W(0) = W_2^3+(t+rs)W_1^3+t^2W_0^3-2rW_1W_2^2-sW_0W_2^2+(r^2-s)W_2W_1^2 \quad (4.2)$$
$$+(rt+s^2)W_0W_1^2+rtW_0^2W_2+2stW_0^2W_1-(3t-rs)W_2W_1W_0.$$

Simson's formulas of $W_n, G_n\ H_n$ can be given in the following forms.

Lemma 35. *For all integers n, we have*

(a) $\Lambda_W(n)=t^n\Lambda_W(0)$, *i.e.,*

$$\begin{aligned} & W_{n+2}^3+(t+rs)W_{n+1}^3+t^2W_n^3-2rW_{n+1}W_{n+2}^2-sW_nW_{n+2}^2 \\ & +(r^2-s)W_{n+2}W_{n+1}^2 \qquad (4.3)\\ & +(rt+s^2)W_nW_{n+1}^2+rtW_n^2W_{n+2}+2stW_n^2W_{n+1}-(3t-rs)W_{n+2}W_{n+1}W_n \\ = \; & t^n(W_2^3+(t+rs)W_1^3+t^2W_0^3-2rW_1W_2^2-sW_0W_2^2+(r^2-s)W_2W_1^2 \\ & +(rt+s^2)W_0W_1^2+rtW_0^2W_2+2stW_0^2W_1-(3t-rs)W_2W_1W_0). \end{aligned}$$

(b) $\Lambda_G(n)=t^n\Lambda_G(0)=t^{n+1}$, *i.e.,*

$$\begin{aligned} & G_{n+2}^3+(t+rs)G_{n+1}^3+t^2G_n^3-2rG_{n+1}G_{n+2}^2-sG_nG_{n+2}^2 \\ & +(r^2-s)G_{n+2}G_{n+1}^2 \qquad (4.4)\\ & +(rt+s^2)G_nG_{n+1}^2+rtG_n^2G_{n+2}+2stG_n^2G_{n+1}-(3t-rs)G_{n+2}G_{n+1}G_n \\ = \; & t^{n+1}. \end{aligned}$$

(c) $\Lambda_H(n)=t^n\Lambda_H(0)=(4r^3t-r^2s^2+18rst-4s^3+27t^2)t^n$, *i.e.,*

$$\begin{aligned} & H_{n+2}^3+(t+rs)H_{n+1}^3+t^2H_n^3-2rH_{n+1}H_{n+2}^2-sH_nH_{n+2}^2 \\ & +(r^2-s)H_{n+2}H_{n+1}^2 \qquad (4.5)\\ & +(rt+s^2)H_nH_{n+1}^2+rtH_n^2H_{n+2}+2stH_n^2H_{n+1}-(3t-rs)H_{n+2}H_{n+1}H_n \\ = \; & (4r^3t-r^2s^2+18rst-4s^3+27t^2)t^n. \end{aligned}$$

Proof. **(a)** Note that (4.1) can be written in the following form:

$$\begin{aligned} & \begin{vmatrix} W_{n+2} & W_{n+1} & W_n \\ W_{n+1} & W_n & \frac{1}{t}(W_{n+2}-rW_{n+1}-sW_n) \\ W_n & \frac{1}{t}(W_{n+2}-rW_{n+1}-sW_n) & \frac{1}{t^2}(-sW_{n+2}+(rs+t)W_{n+1}+(s^2-rt)W_n) \end{vmatrix} \\ = \; & t^n \begin{vmatrix} W_2 & W_1 & W_0 \\ W_1 & W_0 & \frac{1}{t}\left(W_2-rW_1-sW_0\right) \\ W_0 & \frac{1}{t}\left(W_2-rW_1-sW_0\right) & \frac{1}{t^2}\left(-sW_2+tW_1+s^2W_0+rsW_1-rtW_0\right) \end{vmatrix} \\ = \; & -t^{n-2}\Lambda_W(0) \end{aligned}$$

since

$$\begin{aligned} W_{n-1} &= \frac{1}{t}(W_{n+2}-rW_{n+1}-sW_{n}), \\ W_{n-2} &= \frac{1}{t^{2}}(-sW_{n+2}+(rs+t)W_{n+1}+(s^{2}-rt)W_{n}), \\ W_{-1} &= \frac{1}{t}\left(W_{2}-rW_{1}-sW_{0}\right), \\ W_{-2} &= \frac{1}{t^{2}}\left(-sW_{2}+tW_{1}+s^{2}W_{0}+rsW_{1}-rtW_{0}\right). \end{aligned}$$

Note also that

$$\begin{aligned} & \begin{vmatrix} W_{n+2} & W_{n+1} & W_{n} \\ W_{n+1} & W_{n} & \frac{1}{t}(W_{n+2}-rW_{n+1}-sW_{n}) \\ W_{n} & \frac{1}{t}(W_{n+2}-rW_{n+1}-sW_{n}) & \frac{1}{t^{2}}(-sW_{n+2}+(rs+t)W_{n+1}+(s^{2}-rt)W_{n}) \end{vmatrix} \\ = & -\frac{1}{t^{2}}(W_{n+2}^{3}+(t+rs)W_{n+1}^{3}+t^{2}W_{n}^{3}-2rW_{n+1}W_{n+2}^{2}-sW_{n}W_{n+2}^{2}+(r^{2}-s)W_{n+2}W_{n+1}^{2} \\ & +(rt+s^{2})W_{n}W_{n+1}^{2}+rtW_{n}^{2}W_{n+2}+2stW_{n}^{2}W_{n+1}-(3t-rs)W_{n+2}W_{n+1}W_{n}). \end{aligned}$$

So we get the identity (4.3).

(b) Since

$$\begin{aligned} \Lambda_{G}(0) &= (G_{2}^{3}+(t+rs)G_{1}^{3}+t^{2}G_{0}^{3}+(r^{2}-s)G_{1}^{2}G_{2}-2rG_{1}G_{2}^{2}-sG_{0}G_{2}^{2}+rtG_{0}^{2}G_{2} \\ & +(s^{2}+rt)G_{0}G_{1}^{2}+2stG_{0}^{2}G_{1}+(rs-3t)G_{0}G_{1}G_{2}) \\ &= t, \end{aligned}$$

we get the identity (4.4).

(c) Since

$$\begin{aligned} \Lambda_{H}(0) &= (H_{2}^{3}+(t+rs)H_{1}^{3}+t^{2}H_{0}^{3}+(r^{2}-s)H_{1}^{2}H_{2}-2rH_{1}H_{2}^{2}-sH_{0}H_{2}^{2}+rtH_{0}^{2}H_{2} \\ & +(s^{2}+rt)H_{0}H_{1}^{2}+2stH_{0}^{2}H_{1}+(rs-3t)H_{0}H_{1}H_{2}) \\ &= 4r^{3}t-r^{2}s^{2}+18rst-4s^{3}+27t^{2}, \end{aligned}$$

we get the identity (4.5). □

We will use the identities (4.3), (4.4) and (4.5) in Section 10.

5 Some Identities of Generalized Tribonacci Polynomials

In this section, we obtain some identities of (r,s,t)-Tribonacci and (r,s,t)-Tribonacci-Lucas polynomials. First, we can give a few basic relations between $\{G_n\}$ and $\{H_n\}$.

Lemma 36. *The following equalities are true:*

(a) $t^3H_n = (-s^3+3t^2+3rst)G_{n+4} + (rs^3-5rt^2+s^2t-3r^2st)G_{n+3} + (-4st^2+2r^2t^2+s^4-4rs^2t)G_{n+2}$.

(b) $t^2H_n = (-2rt+s^2)G_{n+3} - (rs^2-2r^2t+st)G_{n+2} + (-s^3+3t^2+3rst)G_{n+1}$.

(c) $tH_n = -sG_{n+2} + (3t+rs)G_{n+1} + (-2rt+s^2)G_n$.

(d) $H_n = 3G_{n+1} - 2rG_n - sG_{n-1}$.

(e) $H_n = rG_n + 2sG_{n-1} + 3tG_{n-2}$.

(f) $(-4s^3t+4r^3t^2+27t^3+18rst^2-r^2s^2t)G_n = (r^2s-3rt+4s^2)H_{n+4} - (4rs^2+r^3s-r^2t+6st)H_{n+3} + (2r^3t-r^2s^2-4s^3+9t^2+10rst)H_{n+2}$.

(g) $(4r^3t-r^2s^2-4s^3+27t^2+18rst)G_n = -(6s+2r^2)H_{n+3} + (9t+7rs+2r^3)H_{n+2} + (r^2s-3rt+4s^2)H_{n+1}$.

(h) $(4r^3t-r^2s^2-4s^3+27t^2+18rst)G_n = (9t+rs)H_{n+2} - (r^2s+3rt+2s^2)H_{n+1} - (2r^2t+6st)H_n$.

(i) $(4r^3t-r^2s^2-4s^3+27t^2+18rst)G_n = -(-6rt+2s^2)H_{n+1} + (rs^2-2r^2t+3st)H_n + (9t^2+rst)H_{n-1}$.

(j) $(4r^3t-r^2s^2-4s^3+27t^2+18rst)G_n = (-rs^2+4r^2t+3st)H_n + (-2s^3+9t^2+7rst)H_{n-1} - (-6rt^2+2s^2t)H_{n-2}$.

Proof. Note that all the identities hold for all integers n. We prove (a). To show (a), writing

$$H_n = a\times G_{n+4} + b\times G_{n+3} + c\times G_{n+2}$$

and solving the system of equations

$$\begin{aligned} H_0 &= a\times G_4 + b\times G_3 + c\times G_2 \\ H_1 &= a\times G_5 + b\times G_4 + c\times G_3 \\ H_2 &= a\times G_6 + b\times G_5 + c\times G_4 \end{aligned}$$

we find that $a = \frac{1}{t^3}(-s^3 + 3t^2 + 3rst), b = \frac{1}{t^3}(rs^3 - 5rt^2 + s^2t - 3r^2st), c = \frac{1}{t^3}(-4st^2 + 2r^2t^2 + s^4 - 4rs^2t)$. The other equalities can be proved similarly. □

Note that all the identities in the above lemma can be proved by induction as well. Next, we give a few basic relations between $\{G_n\}$ and $\{W_n\}$.

Lemma 37. *The following equalities are true:*

(a) $(W_2^3 + (t + rs)W_1^3 + t^2W_0^3 + (r^2 - s)W_1^2W_2 - 2rW_1W_2^2 - sW_0W_2^2 + rtW_0^2W_2 + (s^2 + rt)W_0W_1^2 + 2stW_0^2W_1 + (rs - 3t)W_0W_1W_2)G_n = (rW_1^2 + tW_0^2 - W_1W_2 + sW_0W_1)W_{n+2} + (W_2^2 - rW_1W_2 - sW_0W_2 - tW_0W_1)W_{n+1} + (tW_1^2 - tW_0W_2)W_n.$

(b) $(W_2^3 + (t + rs)W_1^3 + t^2W_0^3 + (r^2 - s)W_1^2W_2 - 2rW_1W_2^2 - sW_0W_2^2 + rtW_0^2W_2 + (s^2 + rt)W_0W_1^2 + 2stW_0^2W_1 + (rs - 3t)W_0W_1W_2)G_n = (W_2^2 + r^2W_1^2 + rtW_0^2 - 2rW_1W_2 - sW_0W_2 + (rs - t)W_0W_1)W_{n+1} + ((t + rs)W_1^2 + stW_0^2 - sW_1W_2 - tW_0W_2 + s^2W_0W_1)W_n + (rtW_1^2 + t^2W_0^2 - tW_1W_2 + stW_0W_1)W_{n-1}.$

(c) $(W_2^3 + (t + rs)W_1^3 + t^2W_0^3 + (r^2 - s)W_1^2W_2 - 2rW_1W_2^2 - sW_0W_2^2 + rtW_0^2W_2 + (s^2 + rt)W_0W_1^2 + 2stW_0^2W_1 + (rs - 3t)W_0W_1W_2)G_n = (rW_2^2 + (r^3 + t + rs)W_1^2 + t(s + r^2)W_0^2 - (s + 2r^2)W_1W_2 - (t + rs)W_0W_2 + (r^2s + s^2 - rt)W_0W_1)W_n + (sW_2^2 + r(rs + t)W_1^2 + t(t + rs)W_0^2 - (t + 2rs)W_1W_2 - s^2W_0W_2 + rs^2W_0W_1)W_{n-1} + (tW_2^2 + r^2tW_1^2 + rt^2W_0^2 - 2rtW_1W_2 - stW_0W_2 + t(rs - t)W_0W_1)W_{n-2}.$

(d) $tW_n = (W_2 - rW_1 - sW_0)G_{n+2} + (-rW_2 + r^2W_1 + (t + rs)W_0)G_{n+1} + (-sW_2 + (t + rs)W_1 + (s^2 - rt)W_0)G_n.$

(e) $W_n = W_0G_{n+1} + (W_1 - rW_0)G_n + (W_2 - rW_1 - sW_0)G_{n-1}.$

(f) $W_n = W_1G_n + (W_2 - rW_1)G_{n-1} + tW_0G_{n-2}.$

Proof. We prove (f). The other identities can be proved similarly. Writing

$$W_n = a \times G_n + b \times G_{n-1} + c \times G_{n-2}$$

and solving the system of equations

$$\begin{aligned} W_0 &= a \times G_0 + b \times G_{-1} + c \times G_{-2} \\ W_1 &= a \times G_1 + b \times G_0 + c \times G_{-1} \\ W_2 &= a \times G_2 + b \times G_1 + c \times G_0 \end{aligned}$$

we find that $a = W_1, b = W_2 - rW_1, c = tW_0$. □

Now, we present a few basic relations between $\{H_n\}$ and $\{W_n\}$.

Lemma 38. *The following equalities are true:*

(a) $(W_2^3+(t+rs)W_1^3+t^2W_0^3+(r^2-s)W_1^2W_2-2rW_1W_2^2-sW_0W_2^2+rtW_0^2W_2+(s^2+rt)W_0W_1^2+2stW_0^2W_1+(rs-3t)W_0W_1W_2)H_n=(3W_2^2+(r^2-s)W_1^2+rtW_0^2-4rW_1W_2-2sW_0W_2+(rs-3t)W_0W_1)W_{n+2}+(-2rW_2^2+3tW_1^2-2sW_1W_2-3tW_0W_2+3rsW_1^2+2stW_0^2+2r^2W_1W_2+2s^2W_0W_1+rsW_0W_2+2rtW_0W_1)W_{n+1}+(-sW_2^2+(s^2+rt)W_1^2+3t^2W_0^2+(rs-3t)W_1W_2+2rtW_0W_2+4stW_0W_1)W_n.$

(b) $(W_2^3+(t+rs)W_1^3+t^2W_0^3+(r^2-s)W_1^2W_2-2rW_1W_2^2-sW_0W_2^2+rtW_0^2W_2+(s^2+rt)W_0W_1^2+2stW_0^2W_1+(rs-3t)W_0W_1W_2)H_n=(r^3W_1^2+rW_2^2+3tW_1^2+r^2tW_0^2-2sW_1W_2-3tW_0W_2+2rsW_1^2+2stW_0^2-2r^2W_1W_2+2s^2W_0W_1-rsW_0W_2-rtW_0W_1+r^2sW_0W_1)W_{n+1}+(3t^2W_0^2+2sW_2^2+r^2sW_1^2-3tW_1W_2+rtW_1^2-2s^2W_0W_2-3rsW_1W_2+2rtW_0W_2+stW_0W_1+rstW_0^2+rs^2W_0W_1)W_n+(3tW_2^2+t(r^2-s)W_1^2+rt^2W_0^2-4rtW_1W_2-2stW_0W_2+t(rs-3t)W_0W_1)W_{n-1}.$

(c) $(W_2^3+(t+rs)W_1^3+t^2W_0^3+(r^2-s)W_1^2W_2-2rW_1W_2^2-sW_0W_2^2+rtW_0^2W_2+(s^2+rt)W_0W_1^2+2stW_0^2W_1+(rs-3t)W_0W_1W_2)H_n=((r^2+2s)W_2^2+r(r^3+3rs+4t)W_1^2+t(r^3+3rs+3t)W_0^2-(3t+2r^3+5rs)W_1W_2-(r^2s+rt+2s^2)W_0W_2+(r^3s-r^2t+3rs^2+st)W_0W_1)W_n+((3t+rs)W_2^2+(r^3s+2rs^2+2st+r^2t)W_1^2+t(rt+r^2s+2s^2)W_0^2-2(2rt+s^2+r^2s)W_1W_2-s(rs+5t)W_0W_2+(2s^3+r^2s^2-3t^2)W_0W_1)W_{n-1}+(rtW_2^2+t(r^3+3t+2rs)W_1^2+t^2(2s+r^2)W_0^2-2t(s+r^2)W_1W_2-t(3t+rs)W_0W_2+t(2s^2+r^2s-rt)W_0W_1)W_{n-2}.$

(d) $(4r^3t-r^2s^2-4s^3+27t^2+18rst)W_n=(-2(r^2+3s)W_2+(2r^3+9t+7rs)W_1+(4s^2-3rt+r^2s)W_0)H_{n+2}+((2r^3+9t+7rs)W_2-2(r^4+4r^2s+6tr+s^2)W_1-(4rs^2+6ts-tr^2+r^3s)W_0)H_{n+1}+((-3rt+4s^2+r^2s)W_2-(4rs^2+r^3s-r^2t+6st)W_1+(-r^2s^2+2r^3t+9t^2-4s^3+10rst)W_0)H_n.$

(e) $(4r^3t-r^2s^2-4s^3+27t^2+18rst)W_n=((9t+rs)W_2-(r^2s+3tr+2s^2)W_1-2t(3s+r^2)W_0)H_{n+1}+(-(r^2s+2s^2+3rt)W_2+(r^3s+3rs^2+r^2t+3st)W_1+t(9t+2r^3+7rs)W_0)H_n+(-2t(r^2+3s)W_2+t(2r^3+9t+7rs)W_1+t(4s^2+r^2s-3rt)W_0)H_{n-1}.$

(f) $(4r^3t-r^2s^2-4s^3+27t^2+18rst)W_n=(2(3rt-s^2)W_2+(3st+rs^2-2r^2t)W_1+t(9t+rs)W_0)H_n+((rs^2-2r^2t+3st)W_2+(2r^3t-r^2s^2+4rst-2s^3+9t^2)W_1-t(2s^2+3rt+r^2s)W_0)H_{n-1}+(t(9t+rs)W_2-t(r^2s+3rt+2s^2)W_1-2t^2(3s+r^2)W_0)H_{n-2}.$

Proof. We prove (f). The other identities can be proved similarly. Writing

$$W_n=a\times H_n+b\times H_{n-1}+c\times H_{n-2}$$

and solving the system of equations

$$
\begin{aligned}
W_0 &= a \times H_0 + b \times H_{-1} + c \times H_{-2} \\
W_1 &= a \times H_1 + b \times H_0 + c \times H_{-1} \\
W_2 &= a \times H_2 + b \times H_1 + c \times H_0
\end{aligned}
$$

we find that

$$
\begin{aligned}
a &= \frac{2(3rt - s^2)W_2 + (3st + rs^2 - 2r^2t)W_1 + t(9t + rs)W_0}{4r^3t - r^2s^2 - 4s^3 + 27t^2 + 18rst}, \\
b &= \frac{(rs^2 - 2r^2t + 3st)W_2 + (2r^3t - r^2s^2 + 4rst - 2s^3 + 9t^2)W_1 - t(2s^2 + 3rt + r^2s)W_0}{4r^3t - r^2s^2 - 4s^3 + 27t^2 + 18rst}, \\
c &= \frac{t(9t + rs)W_2 - t(r^2s + 3rt + 2s^2)W_1 - 2t^2(3s + r^2)W_0}{4r^3t - r^2s^2 - 4s^3 + 27t^2 + 18rst}.
\end{aligned}
$$

□

6 Recurrence Properties of Generalized Tribonacci Polynomials

Now, we can propose a problem as follows: Whether and how can the generalized Tribonacci (sequence of) polynomials W_n at negative indices be expressed by the sequence itself at positive indices?

We present the following result which completely solves the above problem for the generalized Tribonacci polynomials W_n.

Theorem 39. *For $n \in \mathbb{Z}$, we have*

$$W_{-n} = t^{-n}(W_{2n} - H_n W_n + \frac{1}{2}(H_n^2 - H_{2n})W_0).$$

For the proof of Theorem 39, we need the following lemma.

Lemma 40.

(a) *(Three Distinct Roots Case: $\alpha \neq \beta \neq \gamma$) If A_1, A_2 and A_3 are as in Theorem 3 (a), then we have*

$$t^n W_{-n} = A_3 \alpha^n \beta^n + A_2 \alpha^n \gamma^n + A_1 \beta^n \gamma^n.$$

(b) *(Two Distinct Roots Case: $\alpha \neq \beta = \gamma$) If A_1, A_2 and A_3 are as in Theorem 3 (b), then we have*

$$\begin{aligned} t^n W_{-n} &= A_1\beta^{2n} + (A_2 - A_3 n)\alpha^n\beta^n \\ &= (2\alpha^n\beta^n + \beta^{2n})W_0 - n\alpha^n\beta^n A_3 - \beta^{2n}A_2 - \alpha^n\beta^n A_2 - 2\alpha^n\beta^n A_1. \end{aligned}$$

(c) *(Single Root Case: $\alpha = \beta = \gamma = \frac{r}{3}$) If A_1, A_2 and A_3 are as in Theorem 3 (c), then we have*

$$\begin{aligned} t^n W_{-n} &= (A_1 - A_2 n + A_3 n^2)\alpha^{2n} \\ &= \left(W_0 - \frac{3}{2r^2}(-3W_2 + 4rW_1 - r^2W_0)n + \frac{1}{2r^2}(9W_2 - 6rW_1 + r^2W_0)n^2\right)\left(\frac{r}{3}\right)^{2n}. \end{aligned}$$

Proof. **(a)** Since $\alpha\beta\gamma = t$ and $W_n = A_1\alpha^n + A_2\beta^n + A_3\gamma^n$, we get

$$\begin{aligned} W_{-n} &= A_1\alpha^{-n} + A_2\beta^{-n} + A_3\gamma^{-n} \\ &= A_1\beta^n\gamma^n t^{-n} + A_2\alpha^n\gamma^n t^{-n} + A_3\alpha^n\beta^n t^{-n} \end{aligned}$$

and so

$$t^n W_{-n} = A_3\alpha^n\beta^n + A_2\alpha^n\gamma^n + A_1\beta^n\gamma^n.$$

(b) Since $\alpha\beta\gamma = t$ (i.e., $\alpha\beta^2 = t$ in this case) and $W_n = A_1\alpha^n + (A_2 + A_3 n)\beta^n$, we obtain

$$\begin{aligned} W_{-n} &= A_1\alpha^{-n} + (A_2 - A_3 n)\beta^{-n} \\ &= A_1\left(\frac{t}{\beta^2}\right)^{-n} + (A_2 - A_3 n)\left(\frac{t}{\alpha\beta}\right)^{-n} \end{aligned}$$

and so

$$t^n W_{-n} = A_1\beta^{2n} + (A_2 - A_3 n)\alpha^n\beta^n.$$

Note also that by using A_1, A_2 and A_3, we get

$$A_1\beta^{2n} + (A_2 - A_3 n)\alpha^n\beta^n = (2\alpha^n\beta^n + \beta^{2n})W_0 - n\alpha^n\beta^n A_3 - \beta^{2n}A_2 - \alpha^n\beta^n A_2 - 2\alpha^n\beta^n A_1.$$

(c) Since $\alpha\beta\gamma = t$ (i.e., $\alpha^3 = t$ in this case) and $W_n = (A_1 + A_2 n + A_3 n^2)\alpha^n$, we get

$$W_{-n} = (A_1 - A_2 n + A_3 n^2)\alpha^{-n} = (A_1 - A_2 n + A_3 n^2) \times \frac{\alpha^{2n}}{t^n}$$

and so

$$t^n W_{-n} = (A_1 - A_2 n + A_3 n^2)\alpha^{2n}.$$

Note also that by using A_1, A_2 and A_3, we obtain

$$(A_1 - A_2 n + A_3 n^2)\alpha^{2n} = \left(W_0 - \frac{3}{2r^2}(-3W_2 + 4rW_1 - r^2W_0)n + \frac{1}{2r^2}(9W_2 - 6rW_1 + r^2W_0)n^2\right)\left(\frac{r}{3}\right)^{2n}.$$

□

Now, we shall complete the proof of Theorem 39.

The Proof of Theorem 39:

If $\alpha \neq \beta \neq \gamma$ (i.e., if we have the three distinct roots case) then by using Theorem 3 (a) (or Theorem 6 (a)), Theorem 12 (a), Lemma 40 (a), we get, for $n \in \mathbb{Z}$,

$$\begin{aligned} W_nH_n &= (A_1\alpha^n + A_2\beta^n + A_3\gamma^n)(\alpha^n + \beta^n + \gamma^n) \\ &= A_1\alpha^{2n} + A_2\beta^{2n} + A_3\gamma^{2n} + A_1\alpha^n\beta^n + A_1\alpha^n\gamma^n + A_2\alpha^n\beta^n + A_2\beta^n\gamma^n \\ &\quad + A_3\alpha^n\gamma^n + A_3\beta^n\gamma^n \\ &= W_{2n} + A_1\alpha^n\beta^n + A_1\alpha^n\gamma^n + A_2\alpha^n\beta^n + A_2\beta^n\gamma^n + A_3\alpha^n\gamma^n + A_3\beta^n\gamma^n \\ &= W_{2n} + (A_1 + A_2 + A_3)\alpha^n\beta^n + (A_1 + A_3 + A_2)\alpha^n\gamma^n \\ &\quad + (A_2 + A_3 + A_1)\beta^n\gamma^n - (A_3\alpha^n\beta^n + A_2\alpha^n\gamma^n + A_1\beta^n\gamma^n) \\ &= W_{2n} + (A_1 + A_2 + A_3)(\alpha^n\beta^n + \alpha^n\gamma^n + \beta^n\gamma^n) - (A_3\alpha^n\beta^n + A_2\alpha^n\gamma^n + A_1\beta^n\gamma^n) \\ &= W_{2n} + W_0S_n - t^nW_{-n}. \end{aligned}$$

If $\alpha \neq \beta = \gamma$ (i.e., if we have the two distinct roots case) then by using Theorem 3 (b) (or Theorem 6 (b)), Theorem 12 (b), Lemma 40 (b), we get, for $n \in \mathbb{Z}$,

$$\begin{aligned} W_nH_n &= (A_1\alpha^n + (A_2 + A_3n)\beta^n)(\alpha^n + 2\beta^n) \\ &= \alpha^{2n}A_1 + 2\beta^{2n}A_2 + 2n\beta^{2n}A_3 + 2\alpha^n\beta^nA_1 + \alpha^n\beta^nA_2 + n\alpha^n\beta^nA_3 \\ &= \alpha^{2n}A_1 + \beta^{2n}A_2 + 2n\beta^{2n}A_3 + \beta^{2n}A_2 + 2\alpha^n\beta^nA_1 + \alpha^n\beta^nA_2 + n\alpha^n\beta^nA_3 \\ &= (A_1\alpha^{2n} + (A_2 + A_3 \times 2n)\beta^{2n}) + \beta^{2n}A_2 + 2\alpha^n\beta^nA_1 + \alpha^n\beta^nA_2 + n\alpha^n\beta^nA_3 \\ &= W_{2n} + \beta^{2n}A_2 + 2\alpha^n\beta^nA_1 + \alpha^n\beta^nA_2 + n\alpha^n\beta^nA_3 \\ &= W_{2n} + \beta^{2n}A_2 + 2\alpha^n\beta^nA_1 + \alpha^n\beta^nA_2 + n\alpha^n\beta^nA_3 \\ &= W_{2n} + (2\alpha^n\beta^n + \beta^{2n})W_0 - (2\alpha^n\beta^n + \beta^{2n})W_0 + \beta^{2n}A_2 + 2\alpha^n\beta^nA_1 \\ &\quad + \alpha^n\beta^nA_2 + n\alpha^n\beta^nA_3 \\ &= W_{2n} + W_0S_n - (2\alpha^n\beta^n + \beta^{2n})W_0 + n\alpha^n\beta^nA_3 + (\beta^{2n} + \alpha^n\beta^n)A_2 + 2\alpha^n\beta^nA_1 \\ &= W_{2n} + W_0S_n - (A_1\beta^{2n} + (A_2 - A_3n)\alpha^n\beta^n) \\ &= W_{2n} + W_0S_n - t^nW_{-n}. \end{aligned}$$

If $\alpha = \beta = \gamma = \dfrac{r}{3}$ (i.e., if we have the single root case) then by using Theorem 3 (c) (or

Theorem 6 (c)), Theorem 12 (c), Lemma 40 (c), we get, for $n \in \mathbb{Z}$,

$$
\begin{aligned}
W_n H_n &= \frac{1}{18}(9n(n-1)W_2 - 6n(n-2)rW_1 + (n-1)(n-2)r^2W_0)\left(\frac{r}{3}\right)^{n-2} \\
&\quad \times \frac{1}{3}((r^2+3s)n^2 - (r^2+3s)n + r^2)\left(\frac{r}{3}\right)^{n-2} \\
&= \frac{1}{18}(9\times 2n(2n-1)W_2 - 6\times 2n(2n-2)rW_1 + (2n-1)(2n-2)r^2W_0)\left(\frac{r}{3}\right)^{2n-2} \\
&\quad + W_0 \times 3\left(\frac{r}{3}\right)^{2n} - (W_0 - \frac{3}{2r^2}(-3W_2 + 4rW_1 - r^2W_0)n \\
&\quad + \frac{1}{2r^2}(9W_2 - 6rW_1 + r^2W_0)n^2)\left(\frac{r}{3}\right)^{2n} \\
&= W_{2n} + W_0 S_n - t^n W_{-n}.
\end{aligned}
$$

Therefore, for all the case of the roots of characteristic equation (polynomial), we have the identity

$$W_n H_n = W_{2n} + W_0 S_n - t^n W_{-n}.$$

Then by Lemma 32 (a), it follows that

$$W_n H_n = W_{2n} + \frac{1}{2}(H_n^2 - H_{2n})W_0 - t^n W_{-n}$$

and so

$$W_{-n} = t^{-n}(W_{2n} - H_n W_n + \frac{1}{2}(H_n^2 - H_{2n})W_0).$$

This completes the proof of Theorem 39. □

Next, we present a remark which presents how H_n can be written in terms of W_n.

Remark 41. *To express W_{-n} by the sequence itself at positive indices we need that H_n can be written in terms of W_n. For this, writing*

$$H_n = a \times W_{n+2} + b \times W_{n+1} + c \times W_n$$

and solving the system of equations

$$
\begin{aligned}
H_0 &= a \times W_2 + b \times W_1 + c \times W_0 \\
H_1 &= a \times W_3 + b \times W_2 + c \times W_1 \\
H_2 &= a \times W_4 + b \times W_3 + c \times W_2
\end{aligned}
$$

or

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} W_2 & W_1 & W_0 \\ W_3 & W_2 & W_1 \\ W_4 & W_3 & W_2 \end{pmatrix}^{-1} \begin{pmatrix} H_0 \\ H_1 \\ H_2 \end{pmatrix}$$

we find a, b, c so that H_n can be written in terms of W_n and we can replace this H_n in Theorem 39.

Note that also, Lemma 38 (a) can be used to write H_n in terms of W_n.

Using Theorem 39 and Lemma 38 (a) or Remark 41, we have the following corollary.

Corollary 42. *For $n \in \mathbb{Z}$, we have*

(a) $G_{-n} = \frac{1}{t^{n+1}}((2rt - s^2)G_n^2 + tG_{2n} + sG_{n+2}G_n - (3t + rs)G_{n+1}G_n)$.

(b) $H_{-n} = \frac{1}{2t^n}(H_n^2 - H_{2n})$.

7 Linear Sums of Generalized Tribonacci Polynomials

The following theorem presents sum formula $\sum_{k=0}^{n} W_{mk+j}$ of generalized Tribonacci polynomials.

Theorem 43. *For all integers m and j, we have*

$$\sum_{k=0}^{n} W_{mk+j} = \frac{W_{mn+m+j} + W_{mn-m+j}t^m + (1 - H_{-m})t^m W_{mn+j} - W_{m+j} - W_{j-m}t^m + (H_m - 1)W_j}{H_m + (1 - H_{-m})t^m - 1}. \tag{7.1}$$

Proof. First, we assume that the roots of characteristic equation of $\{W_n\}$ are distict, i.e., $\alpha \neq \beta \neq \gamma$. Then, by using Binet's formula of $\{W_n\}$ we get

$$\begin{aligned}\sum_{k=0}^{n} W_{mk+j} &= W_{mn+j} + \sum_{k=0}^{n-1} W_{mk+j} = W_{mn+j} + \sum_{k=0}^{n-1}(A_1\alpha^{mk+j} + A_2\beta^{mk+j} + A_3\gamma^{mk+j}) \\ &= W_{mn+j} + A_1\alpha^j\left(\frac{\alpha^{mn} - 1}{\alpha^m - 1}\right) + A_2\beta^j\left(\frac{\beta^{mn} - 1}{\beta^m - 1}\right) + A_3\gamma^j\left(\frac{\gamma^{mn} - 1}{\gamma^m - 1}\right).\end{aligned}$$

Simplifying the last equalities in the last two expression imply (7.1) as required. The proof of the other two cases of the roots of characteristic equation of $\{W_n\}$ are similar. □

Note that (7.1) can be written in the following form:

$$\sum_{k=1}^{n} W_{mk+j} = \frac{W_{mn+m+j} + W_{mn-m+j}t^m + (1 - H_{-m})t^m W_{mn+j} - W_{m+j} - W_{j-m}t^m + t^m(H_{-m} - 1)W_j}{H_m + (1 - H_{-m})t^m - 1}.$$

As special cases of the above theorem, we have the following corollary.

Corollary 44. *For all integers m and j, we have*

$$\sum_{k=0}^{n} G_{mk+j} = \frac{G_{mn+m+j}+G_{mn-m+j}t^{m}+(1-H_{-m})t^{m}G_{mn+j}-G_{m+j}-G_{j-m}t^{m}+(H_{m}-1)G_{j}}{H_{m}+(1-H_{-m})t^{m}-1},$$

$$\sum_{k=0}^{n} H_{mk+j} = \frac{H_{mn+m+j}+H_{mn-m+j}t^{m}+(1-H_{-m})t^{m}H_{mn+j}-H_{m+j}-H_{j-m}t^{m}+(H_{m}-1)H_{j}}{H_{m}+(1-H_{-m})t^{m}-1}.$$

8 The Sum Formula $\sum_{k=0}^{n} z^k W_k$ of Generalized Tribonacci Polynomials via Generating Functions

From now on, through the paper, we suppose that z is a real or complex number. So we may assume that z is a scalar value (real or complex) number or function in $x \in \mathbb{R}$, for example $z = 7$, $z = 5 - 3i$, $z = \sin x$, $z = e^{6x+1}$, $z = 4 + 23x$, $z = e^{3ix} = \cos 3x + i \sin 3x$, $z = 16x^{13} - 9x^5 + 4x - 1, z = \dfrac{1}{2 + x^2}$ for $x \in \mathbb{R}$. We also suppose that $z \neq 0$ if necessary (needed).

In this section, we present the sum formula $\sum_{k=0}^{n} z^k W_k$ of generalized Tribonacci polynomials via generating functions. For this, we need another subseqence of $\{W_n\}$. $\{E_n\}_{n\geq 0}$ is defined by the third-order recurrence relation

$$E_{n+3} = rE_{n+2} + sE_{n+1} + tE_n, \qquad E_0 = 1, E_1 = r - 1, E_2 = -r + s + r^2.$$

The sequence $\{E_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$E_{-n} = -\frac{s}{t}E_{-(n-1)} - \frac{r}{t}E_{-(n-2)} + \frac{1}{t}E_{-(n-3)}.$$

for $n = 1, 2, 3, ...$ respectively.

Next, we present the first few values of the sequence of polynomials, $\{E_n\}$, with positive and negative subscripts:

Table 4: The first few values of E_n with positive and negative subscripts.

n	0	1	2	3	4
E_n	1	$r-1$	$-r+s+r^2$	$r^3-r^2+2sr-s+t$	$r^4-r^3+3r^2s-2rs+2tr+s^2-t$
E_{-n}		0	$-\frac{1}{t}$	$\frac{1}{t^2}(s+t)$	$\frac{1}{t^3}(rt-st-s^2)$

Lemma 9 gives the following result as a particular example.

Corollary 45. *Generating function of* $\{W_n\}$ *is*

$$\sum_{n=0}^{\infty} E_n z^n = \frac{1-z}{1-rz-sz^2-tz^3}.$$

We also need the ordinary generating function of the sequence $\{z^n W_n\}$ to find the sum formula of generalized Tribonacci polynomials. Next, we give the ordinary generating function $\sum\limits_{n=0}^{\infty} z^n W_n u^n$ of the sequence $\{z^n W_n\}$.

Lemma 46. *Suppose that* $f_{z^n W_n}(u) = \sum\limits_{n=0}^{\infty} z^n W_n u^n$ *is the ordinary generating function of the sequence* $\{z^n W_n\}_{n\geq 0}$*. Then,* $\sum\limits_{n=0}^{\infty} z^n W_n u^n$ *is given by*

$$\sum_{n=0}^{\infty} z^n W_n u^n = \frac{W_0 + z(W_1 - rW_0)u + z^2(W_2 - rW_1 - sW_0)u^2}{1 - rzu - sz^2u^2 - tz^3u^3}. \tag{8.1}$$

Proof. Note that

$$z^n W_n = z^n(rW_{n-1} + sW_{n-2} + tW_{n-3}).$$

Using the definition of generalized Tribonacci polynomials, and substracting $rzu\sum_{n=0}^{\infty} z^n W_n u^n$, $sz^2u^2\sum_{n=0}^{\infty} z^n W_n u^n$ and $tz^3u^3\sum_{n=0}^{\infty} z^n W_n u^n$ from $\sum_{n=0}^{\infty} z^n W_n u^n$ we obtain

$$\begin{aligned}
& (1 - rzu - sz^2u^2 - tz^3u^3)\sum_{n=0}^{\infty} z^n W_n u^n \\
= {} & \sum_{n=0}^{\infty} z^n W_n z^n - rzu\sum_{n=0}^{\infty} z^n W_n u^n - sz^2u^2\sum_{n=0}^{\infty} z^n W_n u^n - tz^3u^3\sum_{n=0}^{\infty} z^n W_n u^n \\
= {} & \sum_{n=0}^{\infty} z^n W_n u^n - r\sum_{n=0}^{\infty} z^{n+1} W_n u^{n+1} - s\sum_{n=0}^{\infty} z^{n+2} W_n u^{n+2} - tz^3\sum_{n=0}^{\infty} z^{n+3} W_n u^{n+3} \\
= {} & \sum_{n=0}^{\infty} z^n W_n u^n - r\sum_{n=1}^{\infty} z^n W_{n-1} u^n - s\sum_{n=2}^{\infty} z^n W_{n-2} u^n - t\sum_{n=3}^{\infty} z^n W_{n-3} u^n \\
= {} & (W_0 + zW_1u + z^2W_2u^2) - r(zW_0u + z^2W_1u^2) - sz^2W_0u^2 \\
& + \sum_{n=3}^{\infty} z^n(W_n - rW_{n-1} - sW_{n-2} - tW_{n-3})u^n \\
= {} & W_0 + zW_1u + z^2W_2u^2 - rzW_0u - rz^2W_1u^2 - sz^3W_0u^2 \\
= {} & W_0 + z(W_1 - rW_0)u + z^2(W_2 - rW_1 - sW_0)u^2.
\end{aligned}$$

Rearranging the above equation, we obtain (8.1). □

Lemma 46 gives the following results as particular examples.

Corollary 47. *Generating functions* $\sum_{n=0}^{\infty} z^n G_n u^n$, $\sum_{n=0}^{\infty} z^n H_n u^n$ *and* $\sum_{n=0}^{\infty} z^n E_n u^n$ *are*

$$\begin{aligned}\sum_{n=0}^{\infty} z^n G_n u^n &= \frac{zu}{1-rzu-sz^2u^2-tz^3u^3}, \\ \sum_{n=0}^{\infty} z^n H_n u^n &= \frac{3-2rzu-sz^2u^2}{1-rzu-sz^2u^2-tz^3u^3}, \\ \sum_{n=0}^{\infty} z^n E_n u^n &= \frac{1-zu}{1-rzu-sz^2u^2-tz^3u^3},\end{aligned}$$

respectively.

The following theorem presents sum formulas of generalized Tribonacci polynomials with positive subscripts.

Theorem 48. *Let* z *be a nonzero complex (or real) number.*

(a) *If* $1-rz-sz^2-tz^3 \neq 0$ *then*

$$\sum_{k=0}^{n} z^k W_k = \frac{\Phi(z)}{t(1-rz-sz^2-tz^3)} \tag{8.2}$$

and

$$\sum_{k=0}^{n} z^k W_k = \frac{\Psi(z)}{s(1-rz-sz^2-tz^3)} \tag{8.3}$$

where

$\Phi(z) = t(z^2W_2 - z(rz-1)W_1 - (sz^2+rz-1)W_0) + (-tz^3W_2 + tz^2(rz-1)W_1 + tz(sz^2+rz-1)W_0)z^nG_{n+2} + (tz^2(rz-1)W_2 - tz(rz-1)^2W_1 + tz(r-r^2z-tz^2-rsz^2)W_0)z^nG_{n+1} + (tz(sz^2+rz-1)W_2 + tz(r-r^2z-tz^2-rsz^2)W_1 + tz(rtz^2-s^2z^2-tz-rsz+s)W_0)z^nG_n$

and

$\Psi(z) = s(z^2W_2 - z(rz-1)W_1 - (sz^2+rz-1)W_0) - (z^2(s+3tz)W_2 - z(s+3tz)(rz-1)W_1 - tz(2sz^2+3rz-3)W_0)z^nG_{n+1} + (z(2rtz^2+rsz-s)W_2 - z(stz^2+2r^2tz^2+s^2z+r^2sz-2rtz-rs)W_1 + tz\left(-rsz^2-2r^2z-sz+2r\right)W_0)z^nG_n - (-tz^3W_2 + tz^2(rz-1)W_1 + tz(sz^2+rz-1)W_0)z^nH_n.$

(b) *If* $1-rz-sz^2-tz^3=u(z-a)(z-b)(z-c)=0$ *for some* $u,a,b,c\in\mathbb{C}$ *with* $u\neq 0$ *and* $a\neq b\neq c$, *i.e.,* $z=a$ *or* $z=b$ *or* $z=c$ *then*

$$\sum_{k=0}^{n} z^k W_k = \frac{\Phi_1(z)}{-t\left(3tz^2+2sz+r\right)}$$

where

$$\Phi_1(z) = t(2zW_2-(2rz-1)W_1-(r+2sz)W_0)+t(-z^2(n+3)W_2+z(3rz+nrz-n-2)W_1+(nsz^2+3sz^2+2rz+nrz-n-1)W_0)z^nG_{n+2}-t(z(n-3rz-nrz+2)W_2+(rz-1)(3rz+nrz-n-1)W_1+(2r^2z+3tz^2+3rsz^2+ntz^2+nrsz^2+nr^2z-r-nr)W_0)z^nG_{n+1} -t((-3sz^2-nsz^2-2rz-nrz+n+1)W_2+(ntz^2+3rsz^2+nrsz^2+3tz^2+2r^2z+nr^2z-r-nr)W_1+(3s^2z^2-nrtz^2-3rtz^2+ns^2z^2+2tz+ntz+2rsz+nrsz-s-ns)W_0)z^nG_n.$$

(c) *If* $1-rz-sz^2-tz^3=u(z-a)^2(z-b)=0$ *for some* $u,a,b\in\mathbb{C}$ *with* $u\neq 0$ *and* $a\neq b$, *i.e.,* $z=a$ *or* $z=b$ *then for* $z=a$ *we get*

$$\sum_{k=0}^{n} z^k W_k = \frac{\Phi_2(z)}{-t\left(2s+6tz\right)}$$

where

$$\Phi_2(z) = 2t(W_2-rW_1-sW_0)+t(-z^2(n+3)(n+2)W_2+z(n+2)(3rz+nrz-n-1)W_1+(5nsz^2+n^2sz^2+6sz^2+n^2rz+3nrz+2rz-n^2-n)W_0)z^{n-1}G_{n+2}-t(z(n+2)(-3rz-nrz+n+1)W_2+(6r^2z^2+n^2r^2z^2+5nr^2z^2-4rz-2n^2rz-6nrz+n^2+n)W_1+(5ntz^2+6rsz^2+n^2tz^2+5nrsz^2+n^2rsz^2+6tz^2+3nr^2z+n^2r^2z+2r^2z-nr-n^2r)W_0)z^{n-1}G_{n+1}-t((-6sz^2-5nsz^2-n^2sz^2-3nrz-n^2rz-2rz+n^2+n) W_2+(n^2tz^2+5nrsz^2+n^2rsz^2+6tz^2+5ntz^2+6rsz^2+2r^2z+3nr^2z+n^2r^2z-nr-n^2r) W_1+(6s^2z^2+n^2s^2z^2-6rtz^2-5nrtz^2-n^2rtz^2+5ns^2z^2+3ntz+2rsz+n^2rsz+3nrsz+n^2tz+2tz-n^2s-ns)W_0)z^{n-1}G_n$$

and for $z=b$ *we get*

$$\sum_{k=0}^{n} z^k W_k = \frac{\Phi_3(z)}{-t\left(3tz^2+2sz+r\right)}$$

where

$$\Phi_3(z) = t(2zW_2-(2rz-1)W_1-(r+2sz)W_0)+t(-z^2(n+3)W_2+z(3rz+nrz-n-2)W_1+(nsz^2+3sz^2+2rz+nrz-n-1)W_0)z^nG_{n+2}-t(z(n-3rz-nrz+2)W_2+(rz-1)(3rz+nrz-n-1)W_1+(2r^2z+3tz^2+3rsz^2+ntz^2+nrsz^2+nr^2z-r-nr)W_0)z^nG_{n+1} -t((-3sz^2-nsz^2-2rz-nrz+n+1)W_2+(ntz^2+3rsz^2+nrsz^2+3tz^2+2r^2z+nr^2z-r-nr)W_1+(3s^2z^2-nrtz^2-3rtz^2+ns^2z^2+2tz+ntz+2rsz+nrsz-s-ns)W_0)z^nG_n.$$

(d) *If* $1 - rz - sz^2 - tz^3 = u(z-a)^3 = 0$ *for some* $u, a \in \mathbb{C}$ *with* $u \neq 0$, *i.e.,* $z = a$, *then*

$$\sum_{k=0}^{n} z^k W_k = \frac{\Phi_4(z)}{-6t^2}$$

where

$$\Phi_4(z) = t(n+1)(-z^2(n+3)(n+2)W_2 + z(n+2)(3rz+nrz-n)W_1 + (6sz^2+5nsz^2+n^2sz^2+n^2rz+2nrz-n^2+n)W_0)z^{n-2}G_{n+2}$$

$$+t(n+1)(z(n+2)(3rz+nrz-n)W_2 - (6r^2z^2+n^2r^2z^2+5nr^2z^2-2n^2rz-4nrz+n^2-n)W_1 + (-6tz^2-n^2tz^2-5nrsz^2-n^2rsz^2-5ntz^2-6rsz^2-n^2r^2z-2nr^2z+n^2r-nr)W_0)z^{n-2}G_{n+1}$$

$$+t(n+1)((5nsz^2+n^2sz^2+6sz^2+n^2rz+2nrz-n^2+n)W_2 + (-n^2tz^2-5nrsz^2-n^2rsz^2-6tz^2-5ntz^2-6rsz^2-2nr^2z-n^2r^2z+n^2r-nr)W_1 + (5nrtz^2+n^2rtz^2-6s^2z^2-n^2s^2z^2+6rtz^2-5ns^2z^2-2ntz-n^2rsz-2nrsz-n^2tz+n^2s-ns)W_0)z^{n-2}G_n.$$

Proof. Let

$$M_n = \sum_{k=0}^{n} z^k W_k.$$

(a) Note that using generating functions, we get

$$\begin{aligned}
S(u) &= \sum_{n=0}^{\infty} M_n u^n = \frac{1}{1-u}\frac{W_0 + z(W_1 - rW_0)u + z^2(W_2 - rW_1 - sW_0)u^2}{1 - rzu - sz^2u^2 - tz^3u^3} \\
&= \frac{A}{1-u} + B\frac{zu}{1-rzu-sz^2u^2-tz^3u^3} + C\frac{3-2rzu-sz^2u^2}{1-rzu-sz^2u^2-tz^3u^3} \\
&\quad + D\frac{1-zu}{1-rzu-sz^2u^2-tz^3u^3} \\
&= A\sum_{n=0}^{\infty} u^n + B\sum_{n=0}^{\infty} z^n G_n u^n + C\sum_{n=0}^{\infty} z^n H_n u^n + D\sum_{n=0}^{\infty} z^n E_n u^n \\
&= \sum_{n=0}^{\infty} \left(A + Bz^nG_n + Cz^nH_n + Dz^nE_n\right) u^n
\end{aligned}$$

where

$$A = \frac{z^2W_2 - z(rz-1)W_1 - (sz^2+rz-1)W_0}{1-rz-sz^2-tz^3},$$

$$B = -\frac{\begin{array}{c} z(3tz^2-2rtz^2+sz-rsz+s)W_2 \\ +z(-3rtz^2+stz^2+2r^2tz^2+s^2z+3tz+r^2sz-rsz-2rtz+s-rs)W_1 \\ +tz(rsz^2-2sz^2+2r^2z-3rz+sz+3-2r)W_0 \end{array}}{s(1-rz-sz^2-tz^3)},$$

$$C = -\frac{-tz^3W_2 + tz^2(rz-1)W_1 + tz(sz^2+rz-1)W_0}{s(1-rz-sz^2-tz^3)},$$

$$D = -\frac{z^2(s+3tz)W_2 - z(s+3tz)(rz-1)W_1 - tz(2sz^2+3rz-3)W_0}{s(1-rz-sz^2-tz^3)},$$

i.e.,

$$\sum_{n=0}^{\infty} M_n u^n = \sum_{n=0}^{\infty} \left(A + Bz^nG_n + Cz^nH_n + Dz^nE_n\right) u^n.$$

Comparing on both sides, we obtain

$$\sum_{k=0}^{n} z^kW_k = M_n = \sum_{n=0}^{\infty} \left(A + Bz^nG_n + Cz^nH_n + Dz^nE_n\right).$$

Since

$$E_n = G_{n+1} - G_n$$

we get

$$\sum_{k=0}^{n} z^kW_k = A + Dz^nG_{n+1} + (B-D)z^nG_n + Cz^nH_n.$$

The last formula can be written as (8.3).

Note that using the identity

$$tH_n = -sG_{n+2} + (3t+rs)G_{n+1} + (-2rt+s^2)G_n$$

we obtain (8.2).

(b) We use (8.2). For $z = a$ or $z = b$ or $z = c$, the right hand side of the above sum formula (8.2) is an indeterminate form. Now, we can use L'Hospital rule. Then we get (b) by using

$$\begin{aligned}\sum_{k=0}^{n} a^k W_k &= \left.\frac{\frac{d}{dz}\Phi(z)}{\frac{d}{dz}(t(1-rz-sz^2-tz^3))}\right|_{z=a},\\ \sum_{k=0}^{n} b^k W_k &= \left.\frac{\frac{d}{dz}\Phi(z)}{\frac{d}{dz}(t(1-rz-sz^2-tz^3))}\right|_{z=b},\\ \sum_{k=0}^{n} c^k W_k &= \left.\frac{\frac{d}{dz}\Phi(z)}{\frac{d}{dz}(t(1-rz-sz^2-tz^3))}\right|_{z=c}.\end{aligned}$$

(c) We use (8.2). For $z = a$ and $z = b$, the right hand side of the above sum formula (8.2) is an indeterminate form. Now, we can use L'Hospital rule (two times). Then we get (c) by using

$$\sum_{k=0}^{n} a^k W_k = \left.\frac{\frac{d^2}{dz^2}\Phi(z)}{\frac{d^2}{dz^2}(t(1-rz-sz^2-tz^3))}\right|_{z=a},$$

and

$$\sum_{k=0}^{n} b^k W_k = \left.\frac{\frac{d}{dz}\Phi(z)}{\frac{d}{dz}(t(1-rz-sz^2-tz^3))}\right|_{z=b}.$$

(d) We use (8.2). For $z = a$, the right hand side of the above sum formula (8.2) is an indeterminate form. Now, we can use L'Hospital rule (three times). Then we get (d) by using

$$\sum_{k=0}^{n} a^k W_k = \left.\frac{\frac{d^3}{dz^3}\Phi(z)}{\frac{d^3}{dz^3}(t(1-rz-sz^2-tz^3))}\right|_{z=a}.$$

□

From the last Theorem, we have the following Corollary which gives linear sum formulas of (r, s, t)-Tribonacci polynomials (take $W_n = G_n$ with $G_0 = 0, G_1 = 1, G_2 = r$).

Corollary 49. *Let z be a nonzero complex (or real) number.*

(a) *If $1 - rz - sz^2 - tz^3 \neq 0$ then*

$$\sum_{k=0}^{n} z^k G_k = \frac{tz(-z^{n+1}G_{n+2} + (rz-1)z^n G_{n+1} - tz^{n+2}G_n + 1)}{t(1-rz-sz^2-tz^3)},$$

and

$$\sum_{k=0}^{n} z^k G_k = \frac{z(-z^n(s+3tz)G_{n+1} + z^{n+1}(2rt-s^2-stz)G_n + tz^{n+1}H_n + s)}{s(1-rz-sz^2-tz^3)}.$$

(b) *If* $1-rz-sz^2-tz^3=u(z-a)(z-b)(z-c)=0$ *for some* $u,a,b,c\in\mathbb{C}$ *with* $u\neq 0$ *and* $a\neq b\neq c$, *i.e.,* $z=a$ *or* $z=b$ *or* $z=c$ *then*
$\sum_{k=0}^{n} z^k G_k = \frac{1}{3tz^2+2sz+r}(z^{n+1}(n+2)G_{n+2}-z^n(2rz+nrz-n-1)G_{n+1}+tz^{n+2}(n+3)G_n-1).$

(c) *If* $1-rz-sz^2-tz^3=u(z-a)^2(z-b)=0$ *for some* $u,a,b\in\mathbb{C}$ *with* $u\neq 0$ *and* $a\neq b$, *i.e.,* $z=a$ *or* $z=b$ *then for* $z=a$ *we get*
$\sum_{k=0}^{n} z^k G_k = \frac{z^{n-1}}{2s+6tz}(z(n+2)(n+1)G_{n+2}-(n+1)(-n+2rz+nrz)G_{n+1}+tz^2(n+3)(n+2)G_n).$

and for $z=b$ *we get*
$\sum_{k=0}^{n} z^k G_k = \frac{1}{3tz^2+2sz+r}(z^{n+1}(n+2)G_{n+2}-z^n(-n+2rz+nrz-1)G_{n+1}+tz^{n+2}(n+3)G_n-1).$

(d) *If* $1-rz-sz^2-tz^3=u(z-a)^3=0$ *for some* $u,a\in\mathbb{C}$ *with* $u\neq 0$, *i.e.,* $z=a$, *then*
$\sum_{k=0}^{n} z^k G_k = \frac{(n+1)z^{n-2}}{6t}(nz(n+2)G_{n+2}-n(-n+2rz+nrz+1)G_{n+1}+tz^2(n+3)(n+2)G_n).$

Taking $W_n=H_n$ with $H_0=3,H_1=r,H_2=2s+r^2$ in the last Theorem, we have the following Corollary which gives linear sum formulas of (r,s,t)-Tribonacci-Lucas polynomials.

Corollary 50. *Let* z *be a nonzero complex (or real) number.*

(a) *If* $1-rz-sz^2-tz^3\neq 0$ *then*
$\sum_{k=0}^{n} z^k H_k = \frac{1}{1-rz-sz^2-tz^3}(z^{n+1}(sz^2+2rz-3)G_{n+2}+z^{n+1}(2r-2r^2z-3tz^2-2sz-rsz^2)G_{n+1}+z^{n+1}(s-s^2z^2-3tz+2rtz^2-rsz)G_n-sz^2-2rz+3)$

and
$\sum_{k=0}^{n} z^k H_k = \frac{1}{s(1-rz-sz^2-tz^3)}(z^{n+1}(6rtz-2zs^2-rs-9t)G_{n+1}-z^{n+1}(4r^2tz+3stz-6rt+2s^2-rs^2z)G_n-tz^{n+1}(sz^2+2rz-3)H_n+3s-s^2z^2-2rsz).$

(b) *If* $1-rz-sz^2-tz^3=u(z-a)(z-b)(z-c)=0$ *for some* $u,a,b,c\in\mathbb{C}$ *with* $u\neq 0$ *and* $a\neq b\neq c$, *i.e.,* $z=a$ *or* $z=b$ *or* $z=c$ *then*
$\sum_{k=0}^{n} z^k H_k = \frac{1}{3tz^2+2sz+r}(z^n(3n-3sz^2-4rz-nsz^2-2nrz+3)G_{n+2}+z^n(2nr^2z+3ntz^2-2nr+2nsz+nrsz^2+4r^2z+9tz^2+4sz+3rsz^2-2r)G_{n+1}+z^n(ns^2z^2-2nrtz^2+3ntz+nrsz-ns+3s^2z^2+6tz-6rtz^2+2rsz-s)G_n+2r+2sz).$

(c) *If* $1-rz-sz^2-tz^3=u(z-a)^2(z-b)=0$ *for some* $u,a,b\in\mathbb{C}$ *with* $u\neq 0$ *and* $a\neq b$, *i.e.,* $z=a$ *or* $z=b$ *then for* $z=a$ *we get*
$\sum_{k=0}^{n} z^k H_k = \frac{1}{z\,(2s+6tz)}(z^n(3n(n+1)-z(n+2)(2r+2nr+3sz+nsz))G_{n+2}+z^n(-2n^2r-2nr+z(n+2)(2(n+1)(s+r^2)+z(3t+rs)(n+3)))G_{n+1}+z^n(-n^2s-ns+z(n+2)((n+1)(3t+rs)+z(-2rt+s^2)(n+3)))G_n+2sz)$

and for $z=b$ *we get*
$\sum_{k=0}^{n} z^k H_k = \frac{1}{3tz^2+2sz+r}(z^n(3(n+1)-z(4r+2nr+sz(n+3)))G_{n+2}+z^n(-2r(n+1)+z(2(n+2)(s+r^2)+z(3t+rs)(n+3)))G_{n+1}+z^n(-s(n+1)+z((n+2)(3t+rs)+z(-2rt+s^2)(n+3)))G_n+2r+2sz).$

(d) *If* $1-rz-sz^2-tz^3=u(z-a)^3=0$ *for some* $u,a\in\mathbb{C}$ *with* $u\neq 0$, *i.e.,* $z=a$, *then*
$\sum_{k=0}^{n} z^k H_k = \frac{n+1}{-6t}(z^{n-2}(3n-3n^2+z(n+2)(2nr+3sz+nsz))G_{n+2}+z^{n-2}(2n^2r-2nr-z(n+2)(2n(s+r^2)+z(3t+rs)(n+3)))G_{n+1}+z^{n-2}(n^2s-ns-z(n+2)(n(3t+rs)+z(-2rt+s^2)(n+3)))G_n).$

9 Generalized Tribonacci Polynomials by Matrix Methods

In this section, we present matrix representations of the sequences W_n, G_n and H_n. We also introduce Simson matrix and investigate its properties.

9.1 Matrix Representations of the Sequences W_n, G_n and H_n

We define the square matrix A of order 3 as:

$$A=\begin{pmatrix} r & s & t\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}$$

such that $\det A=t$. Some properties of matrix A^n can be given as

$$\begin{aligned} A^n &= rA^{n-1}+sA^{n-2}+tA^{n-3},\\ A^{n+m} &= A^nA^m=A^mA^n,\\ \det(A^n) &= t^n,\end{aligned}$$

for all integers m and n.

From (1.1), we have

$$\begin{pmatrix} W_{n+2} \\ W_{n+1} \\ W_{n} \end{pmatrix} = \begin{pmatrix} r & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} W_{n+1} \\ W_{n} \\ W_{n-1} \end{pmatrix} \tag{9.1}$$

and using (9.1) and induction, we have the matrix formulation of W_n as

$$\begin{pmatrix} W_{n+2} \\ W_{n+1} \\ W_{n} \end{pmatrix} = \begin{pmatrix} r & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{n} \begin{pmatrix} W_{2} \\ W_{1} \\ W_{0} \end{pmatrix}. \tag{9.2}$$

If we take $W_n = G_n$ in (9.1) we have

$$\begin{pmatrix} G_{n+2} \\ G_{n+1} \\ G_{n} \end{pmatrix} = \begin{pmatrix} r & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} G_{n+1} \\ G_{n} \\ G_{n-1} \end{pmatrix}. \tag{9.3}$$

We also define

$$B_n = \begin{pmatrix} G_{n+1} & sG_n + tG_{n-1} & tG_n \\ G_n & sG_{n-1} + tG_{n-2} & tG_{n-1} \\ G_{n-1} & sG_{n-2} + tG_{n-3} & tG_{n-2} \end{pmatrix}$$

and

$$D_n = \begin{pmatrix} W_{n+1} & sW_n + tW_{n-1} & tW_n \\ W_n & sW_{n-1} + tW_{n-2} & tW_{n-1} \\ W_{n-1} & sW_{n-2} + tW_{n-3} & tW_{n-2} \end{pmatrix}.$$

Theorem 51. *For all integers m, n, we have the following properties:*

(a) $B_n = A^n$, *i.e.,*

$$\begin{pmatrix} r & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{n} = \begin{pmatrix} G_{n+1} & sG_n + tG_{n-1} & tG_n \\ G_n & sG_{n-1} + tG_{n-2} & tG_{n-1} \\ G_{n-1} & sG_{n-2} + tG_{n-3} & tG_{n-2} \end{pmatrix}.$$

(b) $D_1 A^n = A^n D_1$.

(c) $D_{n+m} = D_n B_m = B_m D_n$, *i.e.,*

$$\begin{pmatrix} W_{n+m+1} & sW_{n+m} + tW_{n+m-1} & tW_{n+m} \\ W_{n+m} & sW_{n+m-1} + tW_{n+m-2} & tW_{n+m-1} \\ W_{n+m-1} & sW_{n+m-2} + tW_{n+m-3} & tW_{n+m-2} \end{pmatrix}$$

$$= \begin{pmatrix} W_{n+1} & sW_n + tW_{n-1} & tW_n \\ W_n & sW_{n-1} + tW_{n-2} & tW_{n-1} \\ W_{n-1} & sW_{n-2} + tW_{n-3} & tW_{n-2} \end{pmatrix} \begin{pmatrix} G_{m+1} & sG_m + tG_{m-1} & tG_m \\ G_m & sG_{m-1} + tG_{m-2} & tG_{m-1} \\ G_{m-1} & sG_{m-2} + tG_{m-3} & tG_{m-2} \end{pmatrix}$$

$$= \begin{pmatrix} G_{m+1} & sG_m + tG_{m-1} & tG_m \\ G_m & sG_{m-1} + tG_{m-2} & tG_{m-1} \\ G_{m-1} & sG_{m-2} + tG_{m-3} & tG_{m-2} \end{pmatrix} \begin{pmatrix} W_{n+1} & sW_n + tW_{n-1} & tW_n \\ W_n & sW_{n-1} + tW_{n-2} & tW_{n-1} \\ W_{n-1} & sW_{n-2} + tW_{n-3} & tW_{n-2} \end{pmatrix}.$$

(d)

$$A^n = G_{n-1}A^2 + (sG_{n-2} + tG_{n-3})A + tG_{n-2}I,$$

i.e.,

$A^n = \frac{1}{t}((G_{n+2} - rG_{n+1} - sG_n)A^2 + (-rG_{n+2} + r^2G_{n+1} + (rs+t)G_n)A + (-sG_{n+2} + (rs+t)G_{n+1} + (s^2 - rt)G_n)I)$,

that is,

$A^n = \frac{1}{t}(G_{n+2}(A^2 - rA - sI) + G_{n+1}(-rA^2 + r^2A + (rs+t)I) + G_n(-sA^2 + (rs+t)A + (s^2 - rt)I))$,

where

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Proof. **(a)** We use induction on n. First we assume that $n \geq 0$. For $n = 0$, $n = 1$ and $n = 2$, respectively, we get

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} G_1 & sG_0 + tG_{-1} & tG_0 \\ G_0 & sG_{-1} + tG_{-2} & tG_{-1} \\ G_{-1} & sG_{-2} + tG_{-3} & tG_{-2} \end{pmatrix},$$

which is true because $G_{-3} = -\frac{s}{t^2}, G_{-2} = \frac{1}{t}, G_{-1} = 0, G_0 = 0, G_1 = 1$. Suppose that the relation holds for all k with $0 \leq k \leq n$. Then, we get

$$\begin{aligned}\begin{pmatrix} r & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{n+1} &= \begin{pmatrix} r & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{n} \begin{pmatrix} r & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} G_{n+1} & sG_n + tG_{n-1} & tG_n \\ G_n & sG_{n-1} + tG_{n-2} & tG_{n-1} \\ G_{n-1} & sG_{n-2} + tG_{n-3} & tG_{n-2} \end{pmatrix} \begin{pmatrix} r & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\end{aligned}$$

and so

$$\begin{aligned}\begin{pmatrix} r & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{n+1} &= \begin{pmatrix} rG_{n+1} + sG_n + tG_{n-1} & tG_n + sG_{n+1} & tG_{n+1} \\ rG_n + sG_{n-1} + tG_{n-2} & sG_n + tG_{n-1} & tG_n \\ rG_{n-1} + sG_{n-2} + tG_{n-3} & sG_{n-1} + tG_{n-2} & tG_{n-1} \end{pmatrix} \\ &= \begin{pmatrix} G_{n+2} & tG_n + sG_{n+1} & tG_{n+1} \\ G_{n+1} & sG_n + tG_{n-1} & tG_n \\ G_n & sG_{n-1} + tG_{n-2} & tG_{n-1} \end{pmatrix}.\end{aligned}$$

For $n \leq 0$, we use induction on $v = |n| = -n$. For $v = 0$, the relation already been verified. Assume now that it holds for all v with $0 \leq v \leq |n|$. Then, we obtain

$$\begin{aligned}\begin{pmatrix} r & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{-(n+1)} &= \begin{pmatrix} r & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{-n} \begin{pmatrix} r & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} G_{-n+1} & sG_{-n} + tG_{-n-1} & tG_{-n} \\ G_{-n} & sG_{-n-1} + tG_{-n-2} & tG_{-n-1} \\ G_{-n-1} & sG_{-n-2} + tG_{-n-3} & tG_{-n-2} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{t} & -\frac{r}{t} & -\frac{s}{t} \end{pmatrix},\end{aligned}$$

that is

$$\begin{pmatrix} r & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{-(n+1)} = \begin{pmatrix} G_{-n} & G_{-n+1} - rG_{-n} & tG_{-n-1} \\ G_{-n-1} & G_{-n} - rG_{-n-1} & tG_{-n-2} \\ G_{-n-2} & G_{-n-1} - rG_{-n-2} & tG_{-n-3} \end{pmatrix}.$$

Then, using recurrence relation of G_n, it follows that

$$\begin{aligned}\begin{pmatrix} r & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{-(n+1)} &= \begin{pmatrix} G_{-n} & sG_{-n-1} + tG_{-n-2} & tG_{-n-1} \\ G_{-n-1} & sG_{-n-2} + tG_{-n-3} & tG_{-n-2} \\ G_{-n-2} & sG_{-n-3} + tG_{-n-4} & tG_{-n-3} \end{pmatrix} \\ &= \begin{pmatrix} G_{-(n+1)+1} & sG_{-(n+1)} + tG_{-(n+1)-1} & tG_{-(n+1)} \\ G_{-(n+1)} & sG_{-(n+1)-1} + tG_{-(n+1)-2} & tG_{-(n+1)-1} \\ G_{-(n+1)-1} & sG_{-(n+1)-2} + tG_{-(n+1)-3} & tG_{-(n+1)-2} \end{pmatrix}\end{aligned}$$

which completes the proof by using induction. Note that proof of the case $n \geq 0$ can also be given as follows. By expanding the vectors on the both sides of (9.3) to 3-colums and multiplying the obtained on the right-hand side by A, we get

$$B_n = AB_{n-1}.$$

By induction argument, from the last equation, we obtain

$$B_n = A^{n-1}B_1.$$

But $B_1 = A$. It follows that $B_n = A^n$.

(b) Using (a) and definition of D_1, (b) follows.

(c) We have

$$\begin{aligned} AD_{n-1} &= \begin{pmatrix} r & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} W_n & sW_{n-1} + tW_{n-2} & tW_{n-1} \\ W_{n-1} & sW_{n-2} + tW_{n-3} & tW_{n-2} \\ W_{n-2} & sW_{n-3} + tW_{n-4} & tW_{n-3} \end{pmatrix} \\ &= \begin{pmatrix} W_{n+1} & sW_n + tW_{n-1} & tW_n \\ W_n & sW_{n-1} + tW_{n-2} & tW_{n-1} \\ W_{n-1} & sW_{n-2} + tW_{n-3} & tW_{n-2} \end{pmatrix} = D_n. \end{aligned}$$

i.e. $D_n = AD_{n-1}$. From the last equation, using induction we obtain $D_n = A^{n-1}D_1$. Now

$$D_{n+m} = A^{n+m-1}D_1 = A^{n-1}A^m D_1 = A^{n-1}D_1A^m = D_nB_m$$

and similarly

$$D_{n+m} = B_mD_n.$$

(d) Proof can be given by mathematical induction. But, we present a direct proof as follows. By using (a), we get

$$\begin{aligned}
& G_{n-1}A^2 + (sG_{n-2} + tG_{n-3})A + tG_{n-2}I \\
= \; & G_{n-1}\begin{pmatrix} r & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^2 + (sG_{n-2} + tG_{n-3})\begin{pmatrix} r & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + tG_{n-2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
= \; & \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & sG_{n-1} + tG_{n-2} & tG_{n-1} \\ G_{n-1} & sG_{n-2} + tG_{n-3} & tG_{n-2} \end{pmatrix} = \begin{pmatrix} G_{n+1} & sG_n + tG_{n-1} & tG_n \\ G_n & sG_{n-1} + tG_{n-2} & tG_{n-1} \\ G_{n-1} & sG_{n-2} + tG_{n-3} & tG_{n-2} \end{pmatrix} \\
= \; & \begin{pmatrix} r & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = A^n
\end{aligned}$$

where

$$\begin{aligned}
x_{11} &= r^2G_{n-1} + sG_{n-1} + tG_{n-2} + rsG_{n-2} + rtG_{n-3} \\
&= r^2G_{n-1} + sG_{n-1} + tG_{n-2} + rsG_{n-2} + r(G_n - rG_{n-1} - sG_{n-2}) \\
&= rG_n + sG_{n-1} + tG_{n-2} \\
&= G_{n+1}, \\
x_{12} &= s^2G_{n-2} + rsG_{n-1} + stG_{n-3} + tG_{n-1} = s(rG_{n-1} + sG_{n-2} + tG_{n-3}) + tG_{n-1} \\
&= sG_n + tG_{n-1}, \\
x_{13} &= t\,(rG_{n-1} + sG_{n-2} + tG_{n-3}) = tG_n, \\
x_{21} &= rG_{n-1} + sG_{n-2} + tG_{n-3} = G_n.
\end{aligned}$$

Then it follows that

$A^n = \frac{1}{t}((G_{n+2} - rG_{n+1} - sG_n)A^2 + (-rG_{n+2} + r^2G_{n+1} + (rs+t)G_n)A + (-sG_{n+2} + (rs+t)G_{n+1} + (s^2 - rt)G_n)I)$

since

$$\begin{aligned}
G_{n-1} &= \frac{1}{t}(G_{n+2} - rG_{n+1} - sG_n), \\
G_{n-2} &= \frac{1}{t^2}(-sG_{n+2} + (rs+t)G_{n+1} + (s^2 - rt)G_n), \\
G_{n-3} &= \frac{1}{t^3}((s^2 - rt)G_{n+2} + (r^2t - rs^2 - st)G_{n+1} + (-s^3 + t^2 + 2rst)G_n). \\
sG_{n-2} + tG_{n-3} &= \frac{1}{t}(-rG_{n+2} + r^2G_{n+1} + (rs+t)G_n).
\end{aligned}$$

So, after rearranging, we get

$A^n = \frac{1}{t}(G_{n+2}(A^2 - rA - sI) + G_{n+1}(-rA^2 + r^2A + (rs+t)I) + G_n(-sA^2 + (rs+t)A + (s^2 - rt)I))$.

□

Next, we present matrix formulas for the generalized Tribonacci polynomials and (r,s,t)-Tribonacci-Lucas polynomials.

In the next Corollary, we use $\Lambda_W(0)$ given in (4.2), i.e.,

$\Lambda_W(0) = W_2^3 + (t+rs)W_1^3 + t^2W_0^3 - 2rW_1W_2^2 - sW_0W_2^2 + (r^2-s)W_2W_1^2 + (rt+s^2)W_0W_1^2 + rtW_0^2W_2 + 2stW_0^2W_1 - (3t-rs)W_2W_1W_0$.

Corollary 52. *For all integers n, we have the following formulas for generalized Tribonacci polynomials and (r,s,t)-Tribonacci-Lucas polynomials.*

(a) *Generalized Tribonacci polynomials.*

$$\begin{pmatrix} r & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \frac{1}{\Lambda_W(0)} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

where

$a_{11} = (rW_1^2 + tW_0^2 - W_1W_2 + sW_0W_1)W_{n+3} + (W_2^2 - rW_1W_2 - sW_0W_2 - tW_0W_1)W_{n+2} + (tW_1^2 - tW_0W_2)W_{n+1}$,

$a_{21} = (rW_1^2 + tW_0^2 - W_1W_2 + sW_0W_1)W_{n+2} + (W_2^2 - rW_1W_2 - sW_0W_2 - tW_0W_1)W_{n+1} + (tW_1^2 - tW_0W_2)W_n$,

$a_{31} = (rW_1^2 + tW_0^2 - W_1W_2 + sW_0W_1)W_{n+1} + (W_2^2 - rW_1W_2 - sW_0W_2 - tW_0W_1)W_n + (tW_1^2 - tW_0W_2)W_{n-1}$,

$a_{12} = s((rW_1^2 + tW_0^2 - W_1W_2 + sW_0W_1)W_{n+2} + (W_2^2 - rW_1W_2 - sW_0W_2 - tW_0W_1)W_{n+1} + (tW_1^2 - tW_0W_2)W_n) + t((rW_1^2 + tW_0^2 - W_1W_2 + sW_0W_1)W_{n+1} + (W_2^2 - rW_1W_2 - sW_0W_2 - tW_0W_1)W_n + (tW_1^2 - tW_0W_2)W_{n-1})$,

$a_{22} = s((rW_1^2 + tW_0^2 - W_1W_2 + sW_0W_1)W_{n+1} + (W_2^2 - rW_1W_2 - sW_0W_2 - tW_0W_1)W_n + (tW_1^2 - tW_0W_2)W_{n-1}) + t((rW_1^2 + tW_0^2 - W_1W_2 + sW_0W_1)W_n + (W_2^2 - rW_1W_2 - sW_0W_2 - tW_0W_1)W_{n-1} + (tW_1^2 - tW_0W_2)W_{n-2})$,

$a_{32} = s((rW_1^2 + tW_0^2 - W_1W_2 + sW_0W_1)W_n + (W_2^2 - rW_1W_2 - sW_0W_2 - tW_0W_1)W_{n-1} + (tW_1^2 - tW_0W_2)W_{n-2}) + t((rW_1^2 + tW_0^2 - W_1W_2 + sW_0W_1)W_{n-1} + (W_2^2 - rW_1W_2 - sW_0W_2 - tW_0W_1)W_{n-2} + (tW_1^2 - tW_0W_2)W_{n-3})$,

$a_{13} = t((rW_1^2 + tW_0^2 - W_1W_2 + sW_0W_1)W_{n+2} + (W_2^2 - rW_1W_2 - sW_0W_2 - tW_0W_1)W_{n+1} + (tW_1^2 - tW_0W_2)W_n),$

$a_{23} = t((rW_1^2 + tW_0^2 - W_1W_2 + sW_0W_1)W_{n+1} + (W_2^2 - rW_1W_2 - sW_0W_2 - tW_0W_1)W_n + (tW_1^2 - tW_0W_2)W_{n-1}),$

$a_{33} = t((rW_1^2 + tW_0^2 - W_1W_2 + sW_0W_1)W_n + (W_2^2 - rW_1W_2 - sW_0W_2 - tW_0W_1)W_{n-1} + (tW_1^2 - tW_0W_2)W_{n-2}),$

(b) (r,s,t)*-Tribonacci-Lucas polynomials.*

$$\begin{pmatrix} r & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \frac{1}{4r^3t - r^2s^2 - 4s^3 + 27t^2 + 18rst} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

where

$b_{11} = (9t + rs)H_{n+3} - (r^2s + 2s^2 + 3rt)H_{n+2} - 2t(r^2 + 3s)H_{n+1},$

$b_{21} = (9t + rs)H_{n+2} - (r^2s + 2s^2 + 3rt)H_{n+1} - 2t(r^2 + 3s)H_n,$

$b_{31} = (9t + rs)H_{n+1} - (r^2s + 2s^2 + 3rt)H_n - 2t(r^2 + 3s)H_{n-1},$

$b_{12} = s((9t+rs)H_{n+2} - (r^2s + 2s^2 + 3rt)H_{n+1} - 2t(r^2 + 3s)H_n) + t((9t+rs)H_{n+1} - (r^2s + 2s^2 + 3rt)H_n - 2t(r^2 + 3s)H_{n-1}),$

$b_{22} = s((9t + rs)H_{n+1} - (r^2s + 2s^2 + 3rt)H_n - 2t(r^2 + 3s)H_{n-1}) + t((9t + rs)H_n - (r^2s + 2s^2 + 3rt)H_{n-1} - 2t(r^2 + 3s)H_{n-2}),$

$b_{32} = s((9t+rs)H_n - (r^2s + 2s^2 + 3rt)H_{n-1} - 2t(r^2 + 3s)H_{n-2}) + t((9t+rs)H_{n-1} - (r^2s + 2s^2 + 3rt)H_{n-2} - 2t(r^2 + 3s)H_{n-3}),$

$b_{13} = t((9t + rs)H_{n+2} - (r^2s + 2s^2 + 3rt)H_{n+1} - 2t(r^2 + 3s)H_n),$

$b_{23} = t((9t + rs)H_{n+1} - (r^2s + 2s^2 + 3rt)H_n - 2t(r^2 + 3s)H_{n-1}),$

$b_{33} = t((9t + rs)H_n - (r^2s + 2s^2 + 3rt)H_{n-1} - 2t(r^2 + 3s)H_{n-2}).$

Proof. **(a)** Use Lemma 37 (a) and Theorem 51 (a).

(b) Use Lemma 36 (h) and Theorem 51 (a) or set $W_n = H_n$ with $H_0 = 3, H_1 = r, H_2 = 2s + r^2$ in (a).

□

Note that, (by using (1.1)), a_{12}, a_{22}, a_{32} and b_{12}, b_{22}, b_{32} can be written in the following form:

$a_{12} = (sW_2^2+r(t+rs)W_1^2+t(t+rs)W_0^2-(t+2rs)W_1W_2-s^2W_0W_2+rs^2W_0W_1)W_{n+1}+(tW_2^2+s(t+rs)W_1^2+s^2tW_0^2-(rt+s^2)W_1W_2-2stW_0W_2+(s^3-t^2)W_0W_1)W_n+t((t+rs)W_1^2+stW_0^2-sW_1W_2-tW_0W_2+s^2W_0W_1)W_{n-1}$,

$a_{22} = (sW_2^2+r(t+rs)W_1^2+t(t+rs)W_0^2-(t+2rs)W_1W_2-s^2W_0W_2+rs^2W_0W_1)W_n+(tW_2^2+s(t+rs)W_1^2+s^2tW_0^2-(rt+s^2)W_1W_2-2stW_0W_2+(s^3-t^2)W_0W_1)W_{n-1}+t((t+rs)W_1^2+stW_0^2-sW_1W_2-tW_0W_2+s^2W_0W_1)W_{n-2}$,

$a_{32} = (sW_2^2+r(t+rs)W_1^2+t(t+rs)W_0^2-(t+2rs)W_1W_2-s^2W_0W_2+rs^2W_0W_1)W_{n-1}+(tW_2^2+s(t+rs)W_1^2+s^2tW_0^2-(rt+s^2)W_1W_2-2stW_0W_2+(s^3-t^2)W_0W_1)W_{n-2}+t((t+rs)W_1^2+stW_0^2-sW_1W_2-tW_0W_2+s^2W_0W_1)W_{n-3}$,

and

$b_{12} = (-2s^3+9t^2+7rst)H_{n+1}+(t+rs)(-3rt+s^2)H_n+t(rs^2-2r^2t+3st)H_{n-1}$,

$b_{22} = (-2s^3+9t^2+7rst)H_n+(t+rs)(-3rt+s^2)H_{n-1}+t(rs^2-2r^2t+3st)H_{n-2}$,

$b_{32} = (-2s^3+9t^2+7rst)H_{n-1}+(t+rs)(-3rt+s^2)H_{n-2}+t(rs^2-2r^2t+3st)H_{n-3}$.

Now, we present an identity for W_{n+m}.

Theorem 53. *(Honsberger's Identity) For all integers m and n, we have*

$$\begin{aligned} W_{n+m} &= W_nG_{m+1}+W_{n-1}(sG_m+tG_{m-1})+tW_{n-2}G_m \qquad (9.4) \\ &= W_nG_{m+1}+(sW_{n-1}+tW_{n-2})G_m+tW_{n-1}G_{m-1}. \end{aligned}$$

Proof. From the equation $D_{n+m} = D_nB_m = B_mD_n$ we see that an element of D_{n+m} is the product of row D_n and a column B_m. From the last equation we say that an element of D_{n+m} is the product of a row D_n and column B_m. We just compare the linear combination of the 2nd row and 1st column entries of the matrices D_{n+m} and D_nB_m. This completes the proof. □

Proof can also be given by using induction as follows. For $m \geq 1$ and $m \leq 0$, we proceed by induction on m. First we assume that $m \geq 1$. For $m = 1$, (9.4) is true because we have, by definition of W_n and the values $G_0 = 0, G_1 = 1, G_2 = r$,

$$W_nG_2+W_{n-1}(sG_1+tG_0)+tW_{n-2}G_1 = rW_n+sW_{n-1}+tW_{n-2} = W_{n+1}.$$

For $m = 2$, (9.4) is true because, we get again by definition of W_n and the values $G_1 = 1, G_2 = r, G_3 = s+r^2$,

$$\begin{aligned} W_{n+2} &= rW_{n+1}+sW_n+tW_{n-1} \\ &= r(rW_n+sW_{n-1}+tW_{n-2})+sW_n+tW_{n-1} \\ &= (r^2+s)W_n+(rs+t)W_{n-1}+rtW_{n-2} \\ &= W_nG_3+W_{n-1}(sG_2+tG_1)+tW_{n-2}G_2. \end{aligned}$$

For $m=3$, (9.4) is true because, we get again by definition of W_n and the values $G_2 = r, G_3 = s + r^2, G_4 = r^3 + 2sr + t$,

$$\begin{aligned}
W_{n+3} &= rW_{n+2} + sW_{n+1} + tW_n \\
&= r((r^2+s)W_n + (rs+t)W_{n-1} + rtW_{n-2}) + s(rW_n + sW_{n-1} + tW_{n-2}) + tW_n \\
&= W_nG_4 + W_{n-1}(sG_3 + tG_2) + tW_{n-2}G_3.
\end{aligned}$$

Suppose now that (9.4) holds for all m with $1 \leq m \leq k+2$. Then, by assumption, for $m=k$, $m=k+1$ and $m=k+2$, we have, respectively,

$$\begin{aligned}
W_{n+k} &= W_nG_{k+1} + W_{n-1}(sG_k + tG_{k-1}) + tW_{n-2}G_k, \\
W_{n+k+1} &= W_nG_{k+1+1} + W_{n-1}(sG_{k+1} + tG_{k+1-1}) + tW_{n-2}G_{k+1}, \\
W_{n+k+2} &= W_nG_{k+2+1} + W_{n-1}(sG_{k+2} + tG_{k+2-1}) + tW_{n-2}G_{k+2}.
\end{aligned}$$

By adding up these three equations (after multiplying each side of the these three equations by r, s, t, respectively), we get

$$\begin{aligned}
& rW_{n+k+2} + sW_{n+k+1} + tW_{n+k} \\
= \; & r(W_nG_{k+2+1} + W_{n-1}(sG_{k+2} + tG_{k+2-1}) + tW_{n-2}G_{k+2}) \\
& + s(W_nG_{k+1+1} + W_{n-1}(sG_{k+1} + tG_{k+1-1}) + tW_{n-2}G_{k+1}) \\
& + t(W_nG_{k+1} + W_{n-1}(sG_k + tG_{k-1}) + tW_{n-2}G_k) \\
= \; & W_n(rG_{k+3} + sG_{k+2} + tG_{k+1}) \\
& + W_{n-1}(s(rG_{k+2} + sG_{k+1} + tG_k) + t(rG_{k+1} + sG_k + tG_{k-1})) \\
& + tW_{n-2}(rG_{k+2} + sG_{k+1} + tG_k) \\
= \; & W_nG_{k+4} + W_{n-1}(sG_{k+3} + tG_{k+2}) + tW_{n-2}G_{k+3},
\end{aligned}$$

i.e.,

$$W_{n+k+3} = W_nG_{k+3+1} + W_{n-1}(sG_{k+3} + tG_{k+3-1}) + tW_{n-2}G_{k+3},$$

which yields the (9.4) for $m = k+3$.

Now, if $m \leq 0$ then we proceed by induction on $|m| = -m = v$. For $v = 0$, that is $m = 0$, (9.4) is true because

$$\begin{aligned}
W_nG_{0+1} + W_{n-1}(sG_0 + tG_{0-1}) + tW_{n-2}G_0 &= W_nG_{0+1} + W_{n-1}(sG_0 + tG_{0-1}) + tW_{n-2}G_0 \\
&= W_n = W_{n+0}
\end{aligned}$$

where $G_{-1} = 0, G_0 = 0$ and $G_1 = 1$. For $v = 1$, that is $m = -1$, (9.4) is true because

$$W_{n-1} = W_nG_0 + W_{n-1}(sG_{-1} + tG_{-2}) + tW_{n-2}G_{-1}$$

where $G_{-2} = \frac{1}{t}$, $G_{-1} = 0$ and $G_0 = 0$. For $v = 2$, that is $m = -2$, (9.4) is true because

$$W_{n-2} = W_n G_{-1} + W_{n-1}(sG_{-2} + tG_{-3}) + tW_{n-2}G_{-2}$$

where $G_{-3} = -\frac{s}{t^2}$, $G_{-2} = \frac{1}{t}$ and $G_{-1} = 0$.

Suppose now that (9.4) holds for all $v = |m| = -m$ with $1 \leq v \leq k+2$. Then, by assumption, for $v = k, v = k+1$ and $v = k+2$ we have, respectively,

$$\begin{aligned} W_{n-k} &= W_n G_{-k+1} + W_{n-1}(sG_{-k} + tG_{-k-1}) + tW_{n-2}G_{-k}, & (9.5)\\ W_{n-(k+1)} &= W_n G_{-(k+1)+1} + W_{n-1}(sG_{-(k+1)} + tG_{-(k+1)-1}) + tW_{n-2}G_{-(k+1)}, & (9.6)\\ W_{n-(k+2)} &= W_n G_{-(k+2)+1} + W_{n-1}(sG_{-(k+2)} + tG_{-(k+2)-1}) + tW_{n-2}G_{-(k+2)}. & (9.7) \end{aligned}$$

We have to show that

$$\begin{aligned} W_{n-(k+3)} &= W_n G_{-(k+3)+1} + W_{n-1}(sG_{-(k+3)} + tG_{-(k+3)-1}) + tW_{n-2}G_{-(k+3)} \\ &= W_n G_{-k-2} + W_{n-1}(sG_{-k-3} + tG_{-k-4}) + tW_{n-2}G_{-k-3}. \end{aligned}$$

By adding up these three equations given in (9.7), (9.6) and (9.5), (after multiplying each side of the these three equations by $\frac{1}{t}, -\frac{r}{t}, -\frac{s}{t}$, respectively), we obtain

$$\begin{aligned} & -\frac{s}{t}W_{n-(k+2)} - \frac{r}{t}W_{n-(k+1)} + \frac{1}{t}W_{n-k} \\ = \; & -\frac{s}{t}(W_n G_{-(k+2)+1} + W_{n-1}(sG_{-(k+2)} + tG_{-(k+2)-1}) + tW_{n-2}G_{-(k+2)}) \\ & -\frac{r}{t}(W_n G_{-(k+1)+1} + W_{n-1}(sG_{-(k+1)} + tG_{-(k+1)-1}) + tW_{n-2}G_{-(k+1)}) \\ & +\frac{1}{t}(W_n G_{-k+1} + W_{n-1}(sG_{-k} + tG_{-k-1}) + tW_{n-2}G_{-k}) \\ = \; & \frac{1}{t}(W_n(-sG_{-k-1} - rG_{-k} + G_{-k+1}) + W_{n-1}(t(-rG_{-k-2} - sG_{-k-3} + G_{-k-1}) \\ & +s(-sG_{-k-2} - rG_{-k-1} + G_{-k})) + tW_{n-2}(-rG_{-k-1} - sG_{-k-2} + G_{-k})) \\ = \; & (W_n(-\frac{s}{t}G_{-k-1} - \frac{r}{t}G_{-k} + \frac{1}{t}G_{-k+1}) + W_{n-1}(s(-\frac{s}{t}G_{-k-2} - \frac{r}{t}G_{-k-1} + \frac{1}{t}G_{-k}) \\ & +t(-\frac{s}{t}G_{-k-3} - \frac{r}{t}G_{-k-2} + \frac{1}{t}G_{-k-1})) + tW_{n-2}(-\frac{s}{t}G_{-k-2} - \frac{r}{t}G_{-k-1} + \frac{1}{t}G_{-k})) \\ = \; & W_n G_{-k-2} + W_{n-1}(sG_{-k-3} + tG_{-k-4}) + tW_{n-2}G_{-k-3} \\ = \; & W_n G_{-(k+3)+1} + W_{n-1}(sG_{-(k+3)} + tG_{-(k+3)-1}) + tW_{n-2}G_{-(k+3)}, \end{aligned}$$

i.e.,

$$W_{n-(k+3)} = W_n G_{-(k+3)+1} + W_{n-1}(sG_{-(k+3)} + tG_{-(k+3)-1}) + tW_{n-2}G_{-(k+3)},$$

thus we get 9.4) for $v = |m| = k+3$. □

Corollary 54. *For all integers m, n, we have the following properties:*

$$\begin{aligned} G_{n+m} &= G_nG_{m+1} + G_{n-1}(sG_m + tG_{m-1}) + tG_{n-2}G_m, \\ H_{n+m} &= H_nG_{m+1} + H_{n-1}(sG_m + tG_{m-1}) + tH_{n-2}G_m. \end{aligned}$$

Proof. Set $W_n = H_n$ with $H_0 = 3, H_1 = r, H_2 = 2s + r^2$ and $W_n = G_n$ with $G_0 = 0, G_1 = 1, G_2 = r$ in Theorem 53, respectively. □

Corollary 55. *For all integers m, n, j, we have the following properties:*

$$W_{mn+j} = G_{mn-1}W_{j+2} + (sG_{mn-2} + tG_{mn-3})\,W_{j+1} + tG_{mn-2}W_j, \tag{9.8}$$

$$G_{mn+j} = G_{mn-1}G_{j+2} + (sG_{mn-2} + tG_{mn-3})\,G_{j+1} + tG_{mn-2}G_j, \tag{9.9}$$

$$H_{mn+j} = G_{mn-1}H_{j+2} + (sG_{mn-2} + tG_{mn-3})\,H_{j+1} + tG_{mn-2}H_j. \tag{9.10}$$

Proof. If we make the following changes

$$\begin{aligned} n &\Leftrightarrow a \\ m &\Leftrightarrow b \end{aligned}$$

in (9.4), i.e.,

$$W_{n+m} = W_nG_{m+1} + W_{n-1}(sG_m + tG_{m-1}) + tW_{n-2}G_m,$$

we get

$$W_{a+b} = W_aG_{b+1} + W_{a-1}(sG_b + tG_{b-1}) + tW_{a-2}G_b. \tag{9.11}$$

Now, if we make the following changes

$$\begin{aligned} a &\Leftrightarrow j+2 \\ b &\Leftrightarrow mn-2 \end{aligned}$$

in (9.11) we obtain

$$W_{mn+j} = G_{mn-1}W_{j+2} + (sG_{mn-2} + tG_{mn-3})\,W_{j+1} + tG_{mn-2}W_j.$$

To complete the proof, set $W_n = G_n$ and $W_n = H_n$ in the last identity, respectively. □

9.2 Simson Matrix and its Properties

For $n \in \mathbb{Z}$, we define

$$f_W(n) = \begin{pmatrix} W_{n+2} & W_{n+1} & W_n \\ W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \end{pmatrix}.$$

We call this matrix as Simson matrix of the sequence W_n. Similarly, as special cases of W_n, Simson matrices of the sequences G_n and H_n are

$$f_G(n) = \begin{pmatrix} G_{n+2} & G_{n+1} & G_n \\ G_{n+1} & G_n & G_{n-1} \\ G_n & G_{n-1} & G_{n-2} \end{pmatrix},$$

and

$$f_H(n) = \begin{pmatrix} H_{n+2} & H_{n+1} & H_n \\ H_{n+1} & H_n & H_{n-1} \\ H_n & H_{n-1} & H_{n-2} \end{pmatrix},$$

respectively.

Lemma 56. *For all integers n, m and j, the followings hold.*

(a) $f_W(n) = rf_W(n-1) + sf_W(n-2) + tf_W(n-3)$.

(b) $f_W(n) = Af_W(n-1)$ *and* $f_W(n) = A^n f_W(0)$, *i.e.,*

$$\begin{pmatrix} W_{n+2} & W_{n+1} & W_n \\ W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \end{pmatrix} = \begin{pmatrix} r & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \\ W_{n-1} & W_{n-2} & W_{n-3} \end{pmatrix}$$

and

$$\begin{pmatrix} W_{n+2} & W_{n+1} & W_n \\ W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \end{pmatrix} = \begin{pmatrix} r & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_2 & W_1 & W_0 \\ W_1 & W_0 & W_{-1} \\ W_0 & W_{-1} & W_{-2} \end{pmatrix}.$$

(c) $f_W(n+m) = A^n f_W(m)$ *and* $f_W(n+m) = A^m f_W(n)$ *i.e.,*

$$\begin{pmatrix} W_{n+m+2} & W_{n+m+1} & W_{n+m} \\ W_{n+m+1} & W_{n+m} & W_{n+m-1} \\ W_{n+m} & W_{n+m-1} & W_{n+m-2} \end{pmatrix} = \begin{pmatrix} r & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_{m+2} & W_{m+1} & W_m \\ W_{m+1} & W_m & W_{m-1} \\ W_m & W_{m-1} & W_{m-2} \end{pmatrix},$$

and

$$\begin{pmatrix} W_{m+n+2} & W_{m+n+1} & W_{m+n} \\ W_{m+n+1} & W_{m+n} & W_{m+n-1} \\ W_{m+n} & W_{m+n-1} & W_{m+n-2} \end{pmatrix} = \begin{pmatrix} r & m & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^m \begin{pmatrix} W_{n+2} & W_{n+1} & W_n \\ W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \end{pmatrix},$$

and $f_W(n) = A^m f_W(n-m)$, *i.e.*,

$$\begin{pmatrix} W_{n+2} & W_{n+1} & W_n \\ W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \end{pmatrix} = \begin{pmatrix} r & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^m \begin{pmatrix} W_{n-m+2} & W_{n-m+1} & W_{n-m} \\ W_{n-m+1} & W_{n-m} & W_{n-m-1} \\ W_{n-m} & W_{n-m-1} & W_{n-m-2} \end{pmatrix}.$$

(d)

$$f_W(mn+j) = A^{mn} f_W(j)$$

and

$$f_W(mn+j) = (G_{n-1}A^2 + (sG_{n-2} + tG_{n-3})A + tG_{n-2}I)^m f_W(j).$$

(e)

$$\begin{pmatrix} W_{n+2} & W_{n+1} & W_n \\ W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \end{pmatrix} = \begin{pmatrix} G_{n+1} & sG_n + tG_{n-1} & tG_n \\ G_n & sG_{n-1} + tG_{n-2} & tG_{n-1} \\ G_{n-1} & sG_{n-2} + tG_{n-3} & tG_{n-2} \end{pmatrix} \begin{pmatrix} W_2 & W_1 & W_0 \\ W_1 & W_0 & W_{-1} \\ W_0 & W_{-1} & W_{-2} \end{pmatrix}$$

and

$$\begin{pmatrix} W_{n+m+2} & W_{n+m+1} & W_{n+m} \\ W_{n+m+1} & W_{n+m} & W_{n+m-1} \\ W_{n+m} & W_{n+m-1} & W_{n+m-2} \end{pmatrix} = \begin{pmatrix} G_{n+1} & sG_n + tG_{n-1} & tG_n \\ G_n & sG_{n-1} + tG_{n-2} & tG_{n-1} \\ G_{n-1} & sG_{n-2} + tG_{n-3} & tG_{n-2} \end{pmatrix} \begin{pmatrix} W_{m+2} & W_{m+1} & W_m \\ W_{m+1} & W_m & W_{m-1} \\ W_m & W_{m-1} & W_{m-2} \end{pmatrix}$$

and

$$\begin{pmatrix} W_{n+2} & W_{n+1} & W_n \\ W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \end{pmatrix} = \begin{pmatrix} G_{m+1} & sG_m + tG_{m-1} & tG_m \\ G_m & sG_{m-1} + tG_{m-2} & tG_{m-1} \\ G_{m-1} & sG_{m-2} + tG_{m-3} & tG_{m-2} \end{pmatrix} \begin{pmatrix} W_{n-m+2} & W_{n-m+1} & W_{n-m} \\ W_{n-m+1} & W_{n-m} & W_{n-m-1} \\ W_{n-m} & W_{n-m-1} & W_{n-m-2} \end{pmatrix}.$$

(f)

$$\begin{pmatrix} W_{n+2} & W_{n+1} & W_n \\ W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \end{pmatrix} = \frac{1}{\Lambda_W(0)} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} W_2 & W_1 & W_0 \\ W_1 & W_0 & W_{-1} \\ W_0 & W_{-1} & W_{-2} \end{pmatrix}$$

and

$$\begin{pmatrix} W_{n+m+2} & W_{n+m+1} & W_{n+m} \\ W_{n+m+1} & W_{n+m} & W_{n+m-1} \\ W_{n+m} & W_{n+m-1} & W_{n+m-2} \end{pmatrix} = \frac{1}{\Lambda_W(0)} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} W_{m+2} & W_{m+1} & W_m \\ W_{m+1} & W_m & W_{m-1} \\ W_m & W_{m-1} & W_{m-2} \end{pmatrix}$$

and

$$\begin{pmatrix} W_{n+2} & W_{n+1} & W_n \\ W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \end{pmatrix} = \frac{1}{\Lambda_W(0)} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} W_{n-m+2} & W_{n-m+1} & W_{n-m} \\ W_{n-m+1} & W_{n-m} & W_{n-m-1} \\ W_{n-m} & W_{n-m-1} & W_{n-m-2} \end{pmatrix}$$

where $a_{11}, a_{21}, a_{31}, a_{12}, a_{22}, a_{32}, a_{13}, a_{23}, a_{33}$ *and* $\Lambda_W(0)$ *are as in Corollary 52 (a) (in the last identity above, we replace* n *with* m *in* $a_{11}, a_{21}, a_{31}, a_{12}, a_{22}, a_{32}, a_{13}, a_{23}, a_{33}$*).*

(g)

$$\begin{pmatrix} W_{n+2} & W_{n+1} & W_n \\ W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \end{pmatrix} = \frac{\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}\begin{pmatrix} W_2 & W_1 & W_0 \\ W_1 & W_0 & W_{-1} \\ W_0 & W_{-1} & W_{-2} \end{pmatrix}}{4r^3t - r^2s^2 - 4s^3 + 27t^2 + 18rst}$$

and

$$\begin{pmatrix} W_{n+m+2} & W_{n+m+1} & W_{n+m} \\ W_{n+m+1} & W_{n+m} & W_{n+m-1} \\ W_{n+m} & W_{n+m-1} & W_{n+m-2} \end{pmatrix} = \frac{\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}\begin{pmatrix} W_{m+2} & W_{m+1} & W_m \\ W_{m+1} & W_m & W_{m-1} \\ W_m & W_{m-1} & W_{m-2} \end{pmatrix}}{4r^3t - r^2s^2 - 4s^3 + 27t^2 + 18rst}$$

and

$$\begin{pmatrix} W_{n+2} & W_{n+1} & W_n \\ W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \end{pmatrix} = \frac{\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}\begin{pmatrix} W_{n-m+2} & W_{n-m+1} & W_{n-m} \\ W_{n-m+1} & W_{n-m} & W_{n-m-1} \\ W_{n-m} & W_{n-m-1} & W_{n-m-2} \end{pmatrix}}{4r^3t - r^2s^2 - 4s^3 + 27t^2 + 18rst}$$

where $b_{11}, b_{21}, b_{31}, b_{12}, b_{22}, b_{32}, b_{13}, b_{23}, b_{33}$ *are as in Corollary 52 (b) (in the last identity above, we replace* n *with* m *in* $b_{11}, b_{21}, b_{31}, b_{12}, b_{22}, b_{32}, b_{13}, b_{23}, b_{33}$*).*

Proof. **(a)** Use (1.1), i.e. $W_n = rW_{n-1} + sW_{n-2} + tW_{n-3}$.

(b) By using the definition of W_n, i.e., $W_n = rW_{n-1} + sW_{n-2} + tW_{n-3}$, we get

$$\begin{aligned} Af_W(n-1) &= \begin{pmatrix} r & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \\ W_{n-1} & W_{n-2} & W_{n-3} \end{pmatrix} \\ &= \begin{pmatrix} rW_{n+1} + sW_n + tW_{n-1} & rW_n + sW_{n-1} + tW_{n-2} & rW_{n-1} + sW_{n-2} + tW_{n-3} \\ W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \end{pmatrix} \\ &= \begin{pmatrix} W_{n+2} & W_{n+1} & W_n \\ W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \end{pmatrix} \\ &= f_W(n). \end{aligned}$$

Now it follows that $f_W(n) = A^n f_W(0)$.

(c) By using (b) we get

$$f_W(n+m) = A^{n+m} f_W(0) = A^n A^m f_W(0) = A^n f_W(m).$$

By interchanging m and n in $f_W(n+m) = A^n f_W(m)$, we get $f_W(n+m) = A^m f_W(n)$.

Then it follows that

$$\begin{aligned} f_W(n+m) &= A^n f_W(m) \Leftrightarrow A^{-n} f_W(n+m) = f_W(m) \\ &\Leftrightarrow A^m f_W(-m+n) = f_W(n) \Leftrightarrow f_W(n) = A^m f_W(n-m). \end{aligned}$$

(d) By using Theorem 51 (d), i.e.,

$$G_{n-1}A^2 + (sG_{n-2} + tG_{n-3})A + tG_{n-2}I = A^n$$

and the identity in (9.8), i.e.,

$$W_{mn+j} = G_{mn-1}W_{j+2} + (sG_{mn-2} + tG_{mn-3})\,W_{j+1} + tG_{mn-2}W_j,$$

we get

$$\begin{aligned} & (G_{n-1}A^2 + (sG_{n-2} + tG_{n-3})A + tG_{n-2}I)^m f_W(j) \\ = \; & A^{mn} f_W(j) \\ = \; & \begin{pmatrix} G_{mn+1} & sG_{mn} + tG_{mn-1} & tG_{mn} \\ G_{mn} & sG_{mn-1} + tG_{mn-2} & tG_{mn-1} \\ G_{mn-1} & sG_{mn-2} + tG_{mn-3} & tG_{mn-2} \end{pmatrix} \begin{pmatrix} W_{j+2} & W_{j+1} & W_j \\ W_{j+1} & W_j & W_{j-1} \\ W_j & W_{j-1} & W_{j-2} \end{pmatrix} \\ = \; & \begin{pmatrix} W_{mn+j+2} & W_{mn+j+1} & W_{mn+j} \\ W_{mn+j+1} & W_{mn+j} & W_{mn+j-1} \\ W_{mn+j} & W_{mn+j-1} & W_{mn+j-2} \end{pmatrix} \\ = \; & f_W(mn+j). \end{aligned}$$

Note that $A^{mn} f_W(j) = f_W(mn+j)$ also follows from the identity $f_W(n+m) = A^n f_W(m)$ which is given in (c), by replacing n and m by mn and j respectively in $f_W(n+m) = A^n f_W(m)$.

(e) Use (b), (c) and Theorem 51 (a) which states that

$$A^n = \begin{pmatrix} r & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} G_{n+1} & sG_n + tG_{n-1} & tG_n \\ G_n & sG_{n-1} + tG_{n-2} & tG_{n-1} \\ G_{n-1} & sG_{n-2} + tG_{n-3} & tG_{n-2} \end{pmatrix}.$$

(f) Use (b), (c) and Corollary 52 (a).

(g) Use (b), (c) and Corollary 52 (b).

□

Note that since

$$\begin{aligned} W_{-1} &= \frac{1}{t}(W_2 - rW_1 - sW_0), \\ W_{-2} &= \frac{1}{t^2}(-sW_2 + (rs+t)W_1 + (s^2 - rt)W_0), \end{aligned}$$

we get

$$\begin{pmatrix} W_2 & W_1 & W_0 \\ W_1 & W_0 & W_{-1} \\ W_0 & W_{-1} & W_{-2} \end{pmatrix} = \begin{pmatrix} W_2 & W_1 & W_0 \\ W_1 & W_0 & \frac{1}{t}(W_2 - rW_1 - sW_0) \\ W_0 & \frac{1}{t}(W_2 - rW_1 - sW_0) & \frac{1}{t^2}(-sW_2 + (rs+t)W_1 + (s^2 - rt)W_0) \end{pmatrix}.$$

Taking the determinant of both sides of the identities given in Lemma 56, we obtain the following Theorem.

Theorem 57. *For all integers n and m, the following identities hold.*

(a) *Catalan's Identity:*

$$\det(f_W(n+m)) = t^n \det(f_W(m)),$$

and

$$\det(f_W(n)) = t^m \det(f_W(n-m)),$$

i.e.,

$$\begin{vmatrix} W_{n+m+2} & W_{n+m+1} & W_{n+m} \\ W_{n+m+1} & W_{n+m} & W_{n+m-1} \\ W_{n+m} & W_{n+m-1} & W_{n+m-2} \end{vmatrix} = t^n \begin{vmatrix} W_{m+2} & W_{m+1} & W_m \\ W_{m+1} & W_m & W_{m-1} \\ W_m & W_{m-1} & W_{m-2} \end{vmatrix},$$

and

$$\begin{vmatrix} W_{n+2} & W_{n+1} & W_n \\ W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \end{vmatrix} = t^m \begin{vmatrix} W_{n-m+2} & W_{n-m+1} & W_{n-m} \\ W_{n-m+1} & W_{n-m} & W_{n-m-1} \\ W_{n-m} & W_{n-m-1} & W_{n-m-2} \end{vmatrix}.$$

(b) *(see Theorem 33) Simson's (or Cassini's) Identity:*

$$\det(f_W(n)) = t^n \det(f_W(0)),$$

i.e.,

$$\begin{vmatrix} W_{n+2} & W_{n+1} & W_n \\ W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \end{vmatrix} = t^n \begin{vmatrix} W_2 & W_1 & W_0 \\ W_1 & W_0 & W_{-1} \\ W_0 & W_{-1} & W_{-2} \end{vmatrix}.$$

Proof. **(a)** Taking the determinant of both sides of the identities

$$f_W(n+m) = A^n f_W(m)$$

and

$$f_W(n) = A^m f_W(n-m)$$

which are given in Lemma 56 (c), we get the required results.

(b) Take $m = 0$ in $\det(f_W(n+m)) = t^n \det(f_W(m))$ in (a) or take the determinant of both sides of the identity $f_W(n) = A^n f_W(0)$ which is given in Lemma 56 (b). □

Remark 58. *To prove the second matrix identity in Lemma 56 (d), we used a consequence (Corollary 55) of Honsberger's Identity (Theorem 53). However, firstly, the second matrix identity in Lemma 56 (d) can be proved by induction and then Honsberger's Identity, i.e.,*

$$\begin{aligned} W_{m+n} &= W_{m+2}G_{n-1} + W_{m+1}(sG_{n-2} + tG_{n-3}) + tW_m G_{n-2} \\ &= W_{m+2}G_{n-1} + (sW_{m+1} + tW_m)G_{n-2} + tW_{m+1}G_{n-3}, \end{aligned}$$

can be obtained just comparing the linear combination of the 3rd row and 1st column entries of the matrices.

From the last Theorem, we have the following Corollary which gives determinantal formulas of (r, s, t)-Tribonacci polynomials (take $W_n = G_n$ with $G_0 = 0, G_1 = 1, G_2 = r$).

Corollary 59. *For all integers n and m, the following identities hold.*

(a) *Catalan's Identity:*

$$\det(f_G(n+m)) = t^n \det(f_G(m)),$$

and

$$\det(f_G(n)) = t^m \det(f_G(n-m)),$$

i.e.,

$$\begin{vmatrix} G_{n+m+2} & G_{n+m+1} & G_{n+m} \\ G_{n+m+1} & G_{n+m} & G_{n+m-1} \\ G_{n+m} & G_{n+m-1} & G_{n+m-2} \end{vmatrix} = t^n \begin{vmatrix} G_{m+2} & G_{m+1} & G_m \\ G_{m+1} & G_m & G_{m-1} \\ G_m & G_{m-1} & G_{m-2} \end{vmatrix},$$

and

$$\begin{vmatrix} G_{n+2} & G_{n+1} & G_n \\ G_{n+1} & G_n & G_{n-1} \\ G_n & G_{n-1} & G_{n-2} \end{vmatrix} = t^m \begin{vmatrix} G_{n-m+2} & G_{n-m+1} & G_{n-m} \\ G_{n-m+1} & G_{n-m} & G_{n-m-1} \\ G_{n-m} & G_{n-m-1} & G_{n-m-2} \end{vmatrix}.$$

(b) *Simson's (or Cassini's) Identity:*

$$\det(f_G(n)) = t^n \det(f_G(0)),$$

i.e.,

$$\begin{vmatrix} G_{n+2} & G_{n+1} & G_n \\ G_{n+1} & G_n & G_{n-1} \\ G_n & G_{n-1} & G_{n-2} \end{vmatrix} = -t^{n-1}.$$

Taking $W_n = H_n$ with $H_0 = 3, H_1 = r, H_2 = 2s + r^2$ in the last Theorem, we have the following Corollary which gives determinantal formulas of (r, s, t)-Tribonacci-Lucas polynomials.

Corollary 60. *For all integers n and m, the following identities hold.*

(a) *Catalan's Identity:*

$$\det(f_H(n+m)) = t^n \det(f_H(m)),$$

and

$$\det(f_H(n)) = t^m \det(f_H(n-m)),$$

i.e.,

$$\begin{vmatrix} H_{n+m+2} & H_{n+m+1} & H_{n+m} \\ H_{n+m+1} & H_{n+m} & H_{n+m-1} \\ H_{n+m} & H_{n+m-1} & H_{n+m-2} \end{vmatrix} = t^n \begin{vmatrix} H_{m+2} & H_{m+1} & H_m \\ H_{m+1} & H_m & H_{m-1} \\ H_m & H_{m-1} & H_{m-2} \end{vmatrix},$$

and

$$\begin{vmatrix} H_{n+2} & H_{n+1} & H_n \\ H_{n+1} & H_n & H_{n-1} \\ H_n & H_{n-1} & H_{n-2} \end{vmatrix} = t^m \begin{vmatrix} H_{n-m+2} & H_{n-m+1} & H_{n-m} \\ H_{n-m+1} & H_{n-m} & H_{n-m-1} \\ H_{n-m} & H_{n-m-1} & H_{n-m-2} \end{vmatrix}.$$

(b) *Simson's (or Cassini's) Identity:*

$$\det(f_H(n)) = t^n \det(f_H(0)),$$

i.e.,

$$\begin{vmatrix} H_{n+2} & H_{n+1} & H_n \\ H_{n+1} & H_n & H_{n-1} \\ H_n & H_{n-1} & H_{n-2} \end{vmatrix} = (r^2s^2 + 4s^3 - 4r^3t - 18rst - 27t^2)t^{n-2}.$$

10 The Sum Formula $\sum_{k=0}^{n} z^k W_{mk+j}$ of Generalized Tribonacci Polynomials via Matrix Methods

In this section, we give the sum formula $\sum_{k=0}^{n} z^k W_{mk+j}$ of generalized Tribonacci polynomials via matrix methods. First, we need to present matrix notations to use linear algebra (matrix) method. Suppose that M is a $n \times n$ matrix. Then

$$MAdj(M) = \det(M)I$$

where I is the identity matrix and $Adj(M)$ is the adjugate of M. The *adjugate* or *classical adjoint* of a square matrix M is the transpose of the matrix of cofactors of M. The i,j *cofactor* M_{ij} of M is the scalar $(-1)^{i+j}\det M(i|j)$, where $M(i|j)$ denotes the matrix that you obtain from M by removing the ith row and jth column. Since,

$$\det(I-M)I = (I-M)Adj(I-M)$$

and

$$\left(\sum_{k=0}^{n} M^k\right)(I-M) = (I-M^{n+1})$$

for any square matrix M, we get

$$\left(\sum_{k=0}^{n} M^k\right)\det(I-M) = (I-M^{n+1})Adj(I-M). \tag{10.1}$$

Note also that

$$\begin{aligned} f_W(j) &= \begin{pmatrix} W_{j+2} & W_{j+1} & W_j \\ W_{j+1} & W_j & W_{j-1} \\ W_j & W_{j-1} & W_{j-2} \end{pmatrix} \\ &= \begin{pmatrix} W_{j+2} & W_{j+1} & W_j \\ W_{j+1} & W_j & \frac{1}{t}(W_{j+2}-rW_{j+1}-sW_j) \\ W_j & \frac{1}{t}(W_{j+2}-rW_{j+1}-sW_j) & \frac{1}{t^2}(-sW_{j+2}+(rs+t)W_{j+1}+(s^2-rt)W_j) \end{pmatrix} \end{aligned}$$

where

$$\begin{aligned} W_{j-1} &= \frac{1}{t}(W_{j+2}-rW_{j+1}-sW_j) \\ W_{j-2} &= \frac{1}{t^2}(-sW_{j+2}+(rs+t)W_{j+1}+(s^2-rt)W_j). \end{aligned}$$

If

$$\begin{aligned} M &= zA^m = z\begin{pmatrix} r & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^m \\ &= z\begin{pmatrix} G_{m+1} & sG_m + tG_{m-1} & tG_m \\ G_m & sG_{m-1} + tG_{m-2} & tG_{m-1} \\ G_{m-1} & sG_{m-2} + tG_{m-3} & tG_{m-2} \end{pmatrix}, \end{aligned}$$

where

$$\begin{aligned} G_{m-1} &= \frac{1}{t}(G_{m+2} - rG_{m+1} - sG_m), \\ G_{m-2} &= \frac{1}{t^2}(-sG_{m+2} + (rs+t)G_{m+1} + (s^2 - rt)G_m), \\ G_{m-3} &= \frac{1}{t^3}((s^2 - rt)G_{m+2} + (r^2t - rs^2 - st)G_{m+1} + (-s^3 + t^2 + 2rst)G_m), \end{aligned}$$

then we have

$$\det(I - zA^m)\left(\sum_{k=0}^{n} z^k A^{mk}\right) = (I - z^{n+1}A^{mn+m})Adj(I - zA^m),$$

and then, since $f_W(mk+j) = A^{mk} f_W(j)$ by Lemma 56 (d), we get

$$\begin{aligned} \det(I - zA^m)\left(\sum_{k=0}^{n} z^k f_W(mk+j)\right) &= \det(I - zA^m)\left(\sum_{k=0}^{n} z^k A^{mk}\right) f_W(j) \\ &= (I - z^{n+1}A^{mn+m})Adj(I - zA^m) f_W(j). \end{aligned}$$

10.1 The Sum Formula $\sum_{k=0}^{n} z^k W_{mk+j}$ of Generalized Tribonacci Polynomials in Terms of Generalized Tribonacci Polynomials

By using Corollary 52 (a), we can give the sum formula $\sum_{k=0}^{n} z^k W_{mk+j}$ of generalized Tribonacci polynomials via matrix methods (in terms of elements of the sequence of generalized Tribonacci polynomials).

Theorem 61. *For all integers m and j, we have the following sum formulas.*

(a) *If* $z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4 \neq 0$ *then*

$$\sum_{k=0}^{n} z^k W_{mk+j} = \frac{z^{n+3}\Theta_1 + z^{n+2}\Theta_2 + z^{n+1}\Theta_3 + z^2\Theta_4 + z\Theta_5 + \Theta_6}{z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4} = \frac{\Theta_W(z)}{\Gamma_W(z)} \quad (10.2)$$

where

$\Theta_W(z) = z^{n+3}\Theta_1 + z^{n+2}\Theta_2 + z^{n+1}\Theta_3 + z^2\Theta_4 + z\Theta_5 + \Theta_6,$

$z^{n+3}\Theta_1 = z^{n+3}(-W_jW_{m+2}^2W_{m+mn+2} + (-W_{j+1} + rW_j)W_{m+2}^2W_{m+mn+1} + (-W_{j+2} + rW_{j+1} + sW_j)W_{m+2}^2W_{m+mn} + (-W_{j+2} + sW_j)W_{m+1}^2W_{m+mn+2} - (t + rs)W_jW_{m+1}^2W_{m+mn+1} + (sW_{j+2} - (t + rs)W_{j+1} - s^2W_j)W_{m+1}^2W_{m+mn} - tW_m^2W_{j+1}W_{m+mn+2} + t(-W_{j+2} + rW_{j+1})W_m^2W_{m+mn+1} - t^2W_jW_m^2W_{m+mn} + (W_{j+1} + rW_j)W_{m+2}W_{m+1}W_{m+mn+2} + (W_{j+2} - rW_{j+1})W_{m+2}W_mW_{m+mn+2} + (-sW_{j+1} + tW_j)W_{m+1}W_mW_{m+mn+2} + (W_{j+2} - r^2W_j)W_{m+2}W_{m+1}W_{m+mn+1} + (-rW_{j+2} + (r^2 + s)W_{j+1} + tW_j)W_{m+2}W_mW_{m+mn+1} + (-sW_{j+2} + (t + rs)W_{j+1} - rtW_j)W_{m+1}W_mW_{m+mn+1} + rW_{j+2}W_{m+1}W_{m+2}W_{m+mn} + tW_jW_{m+1}W_{m+2}W_{m+mn} - rsW_jW_{m+1}W_{m+2}W_{m+mn} - r^2W_{j+1}W_{m+1}W_{m+2} W_{m+mn} + t(W_{j+1} - rW_j)W_{m+2}W_mW_{m+mn} + t(W_{j+2} - rW_{j+1} - 2sW_j) W_{m+1}W_mW_{m+mn}),$

$z^{n+2}\Theta_2 = z^{n+2}((-W_0W_{j+2}+(rW_0-W_1)W_{j+1}+(2W_2-rW_1)W_j)W_{m+2}W_{m+mn+2}+ (2W_1W_{j+2} + (sW_0 - W_2)W_{j+1} - (rW_2 + 2sW_1 + tW_0)W_j)W_{m+1}W_{m+mn+2} + (-W_2W_{j+2}+(rW_2+sW_1+2tW_0)W_{j+1}-tW_1W_j)W_mW_{m+mn+2}+((rW_0-W_1)W_{j+2}+ (2W_2-r^2W_0-sW_0)W_{j+1}+(-2rW_2+r^2W_1-tW_0)W_j)W_{m+2}W_{m+mn+1}+((-W_2+ sW_0)W_{j+2}-(rsW_0+tW_0)W_{j+1}+(r^2W_2+2(t+rs)W_1+rtW_0)W_j)W_{m+1}W_{m+mn+1}+ ((rW_2+sW_1+2tW_0)W_{j+2}-(r^2W_2+rsW_1+2rtW_0+sW_2+tW_1)W_{j+1}+t(rW_1- W_2)W_j)W_mW_{m+mn+1}+((2W_2-rW_1)W_{j+2}+(r^2W_1-2rW_2-tW_0)W_{j+1}+(rsW_1- 2sW_2 - tW_1 + rtW_0)W_j)W_{m+2}W_{m+mn} + (-(rW_2 + 2sW_1 + tW_0)W_{j+2} + (r^2W_2 + 2(t + rs)W_1 + rtW_0)W_{j+1} + ((rs - t)W_2 + 2s^2W_1 + 2stW_0)W_j)W_{m+1}W_{m+mn} + t(-W_1W_{j+2} + (rW_1 - W_2)W_{j+1} + (rW_2 + 2sW_1 + 2tW_0)W_j)W_mW_{m+mn}),$

$z^{n+1}\Theta_3 = z^{n+1}(((W_0W_2 - W_1^2)W_{j+2} + (-tW_0^2 + W_1W_2 - rW_0W_2 - sW_0W_1)W_{j+1} + (-W_2^2 + sW_1^2 + rW_1W_2 + tW_0W_1)W_j)W_{m+mn+2} + ((-tW_0^2 + W_1W_2 - rW_0W_2 - sW_0W_1)W_{j+2} + (-W_2^2 + rtW_0^2 + (r^2 + s)W_0W_2 + (t+rs)W_0W_1)W_{j+1} + (rW_2^2 - (rs+ t)W_1^2 - r^2W_1W_2 + tW_0W_2 - rtW_0W_1)W_j)W_{m+mn+1} + ((-W_2^2 + sW_1^2 + rW_1W_2 + tW_0W_1)W_{j+2} + (rW_2^2 - (rs+t)W_1^2 - r^2W_1W_2 + tW_0W_2 - rtW_0W_1)W_{j+1} + (sW_2^2 - s^2W_1^2 - t^2W_0^2 + (t - rs)W_1W_2 - rtW_0W_2 - 2stW_0W_1)W_j)W_{m+mn}),$

$z^2\Theta_4 = z^2((W_0W_{j+2} + (W_1 - rW_0)W_{j+1} + (W_2 - rW_1 - sW_0)W_j)W_{m+2}^2 + ((W_2 - sW_0)W_{j+2} + (tW_0 + rsW_0)W_{j+1} + (s^2W_0 + (rs+t)W_1 - sW_2)W_j)W_{m+1}^2 + t(W_1W_{j+2} + (W_2 - rW_1)W_{j+1} + tW_0W_j)W_m^2 + (-(W_1 + rW_0)W_{j+2} + (r^2W_0 - W_2)W_{j+1} + (-rW_2 + r^2W_1 + (rs - t)W_0)W_j)W_{m+1}W_{m+2} + ((rW_1 - W_2)W_{j+2} + (rW_2 - (s + r^2)W_1 - tW_0)W_{j+1} + t(rW_0 - W_1)W_j)W_{m+2}W_m + ((sW_1 - tW_0)W_{j+2} + (sW_2 - (rs+t)W_1 + rtW_0)W_{j+1} + t(-W_2 + rW_1 + 2sW_0)W_j)W_{m+1}W_m),$

$z\Theta_5 = z(((W_1^2 - W_0W_2)W_{j+2} + (tW_0^2 - W_1W_2 + rW_0W_2 + sW_0W_1)W_{j+1} + (-2W_2^2 - r^2W_1^2 - trW_0^2 + 3rW_1W_2 + 2sW_0W_2 + (2t - sr)W_0W_1)W_j)W_{m+2} + ((tW_0^2 - W_1W_2 + rW_0W_2 + sW_0W_1)W_{j+2} + (W_2^2 - (r^2 + s)W_0W_2 - (rs + t)W_0W_1)W_{j+1} + (rW_2^2 - 2(t + rs)W_1^2 - 2stW_0^2 + (2s - r^2)W_1W_2 + (2t - rs)W_0W_2 - (2s^2 + rt)W_0W_1)W_j) W_{m+1} + ((W_2^2 - sW_1^2 - rW_1W_2 - tW_0W_1)W_{j+2} + (-rW_2^2 + (rs+t)W_1^2 + r^2W_1W_2 - tW_0W_2 + rtW_0W_1)W_{j+1} + t(-rW_1^2 - 2tW_0^2 + 2W_1W_2 - rW_0W_2 - 2sW_0W_1)W_j) W_m - rtW_0^2W_{j+1}W_{m+1}),$

$\Theta_6 = (W_2^3 + (t + rs)W_1^3 + t^2W_0^3 - 2rW_1W_2^2 - sW_0W_2^2 + (r^2 - s)W_1^2W_2 + (s^2 + rt) W_0W_1^2 + rtW_0^2W_2 + 2stW_0^2W_1 + (rs - 3t)W_0W_1W_2)W_j,$

and

$\Gamma_W(z) = z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4,$

$z^3\Gamma_1 = z^3(-t^m(W_2^3 + (t + rs)W_1^3 + t^2W_0^3 + (r^2 - s)W_1^2W_2 - 2rW_1W_2^2 - sW_0W_2^2 + rtW_0^2W_2 + (s^2 + rt)W_0W_1^2 + 2stW_0^2W_1 + (rs - 3t)W_0W_1W_2)),$

$z^2\Gamma_2 = z^2((3W_2 - 2rW_1 - sW_0)W_{m+2}^2 + ((r^2 - s)W_2 + (3rs + 3t)W_1 + (s^2 + rt)W_0) W_{m+1}^2 + t(rW_2 + 2sW_1 + 3tW_0)W_m^2 + (-4rW_2 + 2(r^2 - s)W_1 + (rs - 3t)W_0) W_{m+2}W_{m+1} + (-2sW_2 + (rs - 3t)W_1 + 2rtW_0)W_{m+2}W_m + ((rs - 3t)W_2 + 2(s^2 + rt)W_1 + 4stW_0)W_{m+1}W_m),$

$z\Gamma_3 = z((-3W_2^2 + (s - r^2)W_1^2 - trW_0^2 + 4rW_1W_2 + 2sW_0W_2 + (3t - sr)W_0W_1)W_{m+2} + (2rW_2^2 - (3rs + 3t)W_1^2 - 2stW_0^2 + (2s - 2r^2)W_1W_2 + (3t - rs)W_0W_2 - 2(s^2 + rt)W_0W_1) W_{m+1} + (sW_2^2 - (s^2 + rt)W_1^2 - 3t^2W_0^2 + (3t - rs)W_1W_2 - 2rtW_0W_2 - 4stW_0W_1) W_m),$

$\Gamma_4 = W_2^3 + (t + rs)W_1^3 + t^2W_0^3 - 2rW_1W_2^2 + (r^2 - s)W_1^2W_2 - sW_0W_2^2 + (s^2 + rt)W_0W_1^2 + rtW_0^2W_2 + 2stW_0^2W_1 + (rs - 3t)W_0W_1W_2.$

(b) *If* $z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4 = u(z - a)(z - b)(z - c) = 0$ *for some* $u, a, b, c \in \mathbb{C}$ *with* $u \neq 0$ *and* $a \neq b \neq c$, *i.e.,* $z = a$ *or* $z = b$ *or* $z = c$ *then*

$$\sum_{k=0}^{n} z^k W_{mk+j} = \frac{(n+3)z^{n+2}\Theta_1 + (n+2)z^{n+1}\Theta_2 + (n+1)z^n\Theta_3 + 2z\Theta_4 + \Theta_5}{3z^2\Gamma_1 + 2z\Gamma_2 + \Gamma_3}.$$

(c) *If* $z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4 = u(z-a)^2(z-b) = 0$ *for some* $u, a, b \in \mathbb{C}$ *with* $u \neq 0$ *and* $a \neq b$, *i.e.,* $z = a$ *or* $z = b$ *then for* $z = a$ *we get*

$$\sum_{k=0}^{n} z^k W_{mk+j} = \frac{(n+3)(n+2)z^{n+1}\Theta_1 + (n+2)(n+1)z^n\Theta_2 + (n+1)nz^{n-1}\Theta_3 + 2\Theta_4}{6z\Gamma_1 + 2\Gamma_2},$$

and for $z = b$ *we get*

$$\sum_{k=0}^{n} z^k W_{mk+j} = \frac{(n+3)z^{n+2}\Theta_1 + (n+2)z^{n+1}\Theta_2 + (n+1)z^n\Theta_3 + 2z\Theta_4 + \Theta_5}{3z^2\Gamma_1 + 2z\Gamma_2 + \Gamma_3}.$$

(d) *If* $z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4 = u(z-a)^3 = 0$ *for some* $u, a \in \mathbb{C}$ *with* $u \neq 0$, *i.e.,* $z = a$, *then*

$$\sum_{k=0}^{n} z^k W_{mk+j} = \frac{(n+3)(n+2)(n+1)z^n\Theta_1 + (n+2)(n+1)nz^{n-1}\Theta_2 + (n+1)n(n-1)z^{n-2}\Theta_3}{6\Gamma_1}.$$

Proof. **(a)** We set

$$\begin{aligned} M &= zA^m = z\begin{pmatrix} G_{m+1} & sG_m + tG_{m-1} & tG_m \\ G_m & sG_{m-1} + tG_{m-2} & tG_{m-1} \\ G_{m-1} & sG_{m-2} + tG_{m-3} & tG_{m-2} \end{pmatrix} \\ &= z \times \frac{1}{\Lambda_W(0)}\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \end{aligned}$$

in (10.1), where $a_{11}, a_{21}, a_{31}, a_{12}, a_{22}, a_{32}, a_{13}, a_{23}, a_{33}$ and $\Lambda_W(0)$ are as in Corollary 52 (a) (by replacing n with m in $a_{11}, a_{21}, a_{31}, a_{12}, a_{22}, a_{32}, a_{13}, a_{23}, a_{33}$). Also, use the identities

$$\begin{aligned} W_{m+3} &= rW_{m+2} + sW_{m+1} + tW_m, \\ W_{m-1} &= \frac{1}{t}(W_{m+2} - rW_{m+1} - sW_m), \\ W_{m-2} &= \frac{1}{t^2}(-sW_{m+2} + (rs+t)W_{m+1} + (s^2 - rt)W_m), \\ W_{m-3} &= \frac{1}{t^3}((s^2 - rt)W_{m+2} + (r^2t - rs^2 - st)W_{m+1} + (-s^3 + t^2 + 2rst)W_m), \end{aligned}$$

in the formula of A^m and as well as in $a_{11}, a_{21}, a_{31}, a_{12}, a_{22}, a_{32}, a_{13}, a_{23}, a_{33}$. After some calculations, we see that

$$\det(I - zA^m) = \frac{1}{\Lambda_W(0)}\Gamma_W(z) = \frac{1}{\Lambda_W(0)}(z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4)$$

and the 3rd row and 1st column entry of matrix $(I-z^{n+1}A^{mn+m})Adj(I-zA^m)f_W(j)$ is equal to

$$\frac{1}{\Lambda_W(0)}(z^{n+3}\Theta_1+z^{n+2}\Theta_2+z^{n+1}\Theta_3+z^2\Theta_4+z\Theta_5+\Theta_6)$$

where $\Gamma_W(z)=z^3\Gamma_1+z^2\Gamma_2+z\Gamma_3+\Gamma_4, z^3\Gamma_1, z^2\Gamma_2, z\Gamma_3, \Gamma_4$, and $\Theta_W(z)=z^{n+3}\Theta_1+z^{n+2}\Theta_2+z^{n+1}\Theta_3+z^2\Theta_4+z\Theta_5+\Theta_6, z^{n+3}\Theta_1, z^{n+2}\Theta_2, z^{n+1}\Theta_3, z^2\Theta_4, z\Theta_5, \Theta_6$, are as in the statement of (a) of Theorem.

Note that, since $\det(I-zA^m)(\sum_{k=0}^{n} z^k f_W(mk+j)=(I-z^{n+1}A^{mn+m})Adj(I-zA^m)f_W(j)$, i.e., matrices over the either side is equal, the 3rd row and 1st column entry of matrix of matrix $\det(I-zA^m)(\sum_{k=0}^{n} z^k f_W(mk+j))$ is equal to the 3rd row and 1st column entry of matrix $(I-z^{n+1}A^{mn+m})Adj(I-zA^m)f_W(j)$. So, to complete the proof, we will just compare the linear combination of the 3rd row and 1st column entries of the matrices. Then, we get

$$\frac{1}{\Lambda_W(0)}(z^3\Gamma_1+z^2\Gamma_2+z\Gamma_3+\Gamma_4)\sum_{k=0}^{n} z^k W_{mk+j}=\frac{1}{\Lambda_W(0)}(z^{n+3}\Theta_1+z^{n+2}\Theta_2+z^{n+1}\Theta_3+z^2\Theta_4+z\Theta_5+\Theta_6)$$

and so

$$\begin{aligned}\sum_{k=0}^{n} z^k W_{mk+j} &= \frac{z^{n+3}\Theta_1+z^{n+2}\Theta_2+z^{n+1}\Theta_3+z^2\Theta_4+z\Theta_5+\Theta_6}{z^3\Gamma_1+z^2\Gamma_2+z\Gamma_3+\Gamma_4}\\ &= \frac{\Theta_W(z)}{\Gamma_W(z)}.\end{aligned}$$

(b) We use (10.2). For $z=a$ or $z=b$ or $z=c$, the right hand side of the above sum formula (10.2) is an indeterminate form. Now, we can use L'Hospital rule. Then we get (b) by using

$$\begin{aligned}\sum_{k=0}^{n} a^k W_k &= \left.\frac{\frac{d}{dz}(z^{n+3}\Theta_1+z^{n+2}\Theta_2+z^{n+1}\Theta_3+z^2\Theta_4+z\Theta_5+\Theta_6)}{\frac{d}{dz}(z^3\Gamma_1+z^2\Gamma_2+z\Gamma_3+\Gamma_4)}\right|_{z=a}\\ &= \left.\frac{(n+3)z^{n+2}\Theta_1+(n+2)z^{n+1}\Theta_2+(n+1)z^n\Theta_3+2z\Theta_4+\Theta_5}{3z^2\Gamma_1+2z\Gamma_2+\Gamma_3}\right|_{z=a},\\ \sum_{k=0}^{n} b^k W_k &= \left.\frac{(n+3)z^{n+2}\Theta_1+(n+2)z^{n+1}\Theta_2+(n+1)z^n\Theta_3+2z\Theta_4+\Theta_5}{3z^2\Gamma_1+2z\Gamma_2+\Gamma_3}\right|_{z=b},\\ \sum_{k=0}^{n} c^k W_k &= \left.\frac{(n+3)z^{n+2}\Theta_1+(n+2)z^{n+1}\Theta_2+(n+1)z^n\Theta_3+2z\Theta_4+\Theta_5}{3z^2\Gamma_1+2z\Gamma_2+\Gamma_3}\right|_{z=c}.\end{aligned}$$

(c) We use (10.2). For $z=a$ and $z=b$, the right hand side of the above sum formula 10.2) is an indeterminate form. Now, we can use L'Hospital rule (two times). Then we get (c) by using

$$\sum_{k=0}^{n} a^k W_k = \left. \frac{\frac{d^2}{dz^2}(z^{n+3}\Theta_1 + z^{n+2}\Theta_2 + z^{n+1}\Theta_3 + z^2\Theta_4 + z\Theta_5 + \Theta_6)}{\frac{d^2}{dz^2}(z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4)} \right|_{z=a},$$

and

$$\sum_{k=0}^{n} b^k W_k = \left. \frac{\frac{d}{dz}(z^{n+3}\Theta_1 + z^{n+2}\Theta_2 + z^{n+1}\Theta_3 + z^2\Theta_4 + z\Theta_5 + \Theta_6)}{\frac{d}{dz}(z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4)} \right|_{z=b}.$$

(d) We use (10.2). For $z = a$, the right hand side of the above sum formula (10.2) is an indeterminate form. Now, we can use L'Hospital rule (three times). Then we get (d) by using

$$\sum_{k=0}^{n} a^k W_k = \left. \frac{\frac{d^3}{dz^3}(z^{n+3}\Theta_1 + z^{n+2}\Theta_2 + z^{n+1}\Theta_3 + z^2\Theta_4 + z\Theta_5 + \Theta_6)}{\frac{d^3}{dz^3}(z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4)} \right|_{z=a}.$$

□

Note that from Theorem 61 (a) we see that

$$\begin{aligned} \Gamma_4 &= \Lambda_W(0), \\ \Theta_6 &= \Gamma_4 W_j = \Lambda_W(0) W_j, \\ \Gamma_1 &= -t^m \Gamma_4 = -t^m \Lambda_W(0), \end{aligned}$$

where $\Lambda_W(0)$ is given in (4.2).

Now, we consider special cases of Theorem 61.

Theorem 62. *We have the following sum formulas.*

(a) $(m = 1, j = 0)$.

(i) *If* $z^3(-t) + z^2(-1)s + z(-1)r + 1 \neq 0$ *then*

$$\sum_{k=0}^{n} z^k W_k = \frac{\Omega_1}{z^3(-t) + z^2(-1)s + z(-1)r + 1}$$

where

$\Omega_1 = z^{n+3}(-1)tW_n + z^{n+2}(rW_{n+1} - W_{n+2}) + z^{n+1}(-1)W_{n+1} + z^2(W_2 - rW_1 - sW_0) + z(W_1 - rW_0) + W_0.$

(ii) *If* $z^3(-t)+z^2(-1)s+z(-1)r+1 = u(z-a)(z-b)(z-c) = 0$ *for some* $u, a, b, c \in \mathbb{C}$ *with* $u \neq 0$ *and* $a \neq b \neq c$, *i.e.,* $z = a$ *or* $z = b$ *or* $z = c$ *then*

$$\sum_{k=0}^{n} z^k W_k = \frac{\Omega_2}{3z^2(-t) + 2z(-1)s + (-1)r}$$

where

$\Omega_2 = (n+3)z^{n+2}(-1)tW_n + (n+2)z^{n+1}(rW_{n+1} - W_{n+2}) + (n+1)z^n(-1)W_{n+1} + 2z(W_2 - rW_1 - sW_0) + (W_1 - rW_0)$.

(iii) *If* $z^3(-t) + z^2(-1)s + z(-1)r + 1 = u(z-a)^2(z-b) = 0$ *for some* $u, a, b \in \mathbb{C}$ *with* $u \neq 0$ *and* $a \neq b$, *i.e.,* $z = a$ *or* $z = b$ *then for* $z = a$ *we get*

$$\sum_{k=0}^{n} z^k W_k = \frac{\Omega_3}{6z(-t) + 2(-1)s}$$

where

$\Omega_3 = (n+3)(n+2)z^{n+1}(-1)tW_n + (n+2)(n+1)z^n(rW_{n+1} - W_{n+2}) + (n+1)nz^{n-1}(-1)W_{n+1} + 2(W_2 - rW_1 - sW_0)$

and for $z = b$ *we get*

$$\sum_{k=0}^{n} z^k W_k = \frac{\Omega_4}{3z^2(-t) + 2z(-1)s + (-1)r}$$

where

$\Omega_4 = (n+3)z^{n+2}(-1)tW_n + (n+2)z^{n+1}(rW_{n+1} - W_{n+2}) + (n+1)z^n(-1)W_{n+1} + 2z(W_2 - rW_1 - sW_0) + (W_1 - rW_0)$.

(iv) *If* $z^3(-t) + z^2(-1)s + z(-1)r + 1 = u(z-a)^3 = 0$ *for some* $u, a \in \mathbb{C}$ *with* $u \neq 0$, *i.e.,* $z = a$, *then*

$$\sum_{k=0}^{n} z^k W_k = \frac{\Omega_5}{6(-t)}$$

where

$\Omega_5 = (n+3)(n+2)(n+1)z^n(-1)tW_n + (n+2)(n+1)nz^{n-1}(rW_{n+1} - W_{n+2}) + (n+1)n(n-1)z^{n-2}(-1)W_{n+1}$.

(b) $(m = 2, j = 0)$.

(i) *If* $z^3(-t^2) + z^2(-2rt + s^2) + z(-1)(2s + r^2) + 1 \neq 0$ *then*

$$\sum_{k=0}^{n} z^k W_{2k} = \frac{\Omega_1}{z^3(-t^2) + z^2(-2rt + s^2) + z(-1)(2s + r^2) + 1}$$

where

$\Omega_1 = z^{n+3}(-1)t^2W_{2n} + z^{n+2}(sW_{2n+2} - (rs+t)W_{2n+1} - rtW_{2n}) + z^{n+1}(-1)W_{2n+2} + z^2(-sW_2 + (t+rs)W_1 + (s^2-rt)W_0) + z(W_2 - (r^2+2s)W_0) + W_0$.

(ii) *If* $z^3(-t^2)+z^2(-2rt+s^2)+z(-1)(2s+r^2)+1 = u(z-a)(z-b)(z-c) = 0$ *for some* $u,a,b,c \in \mathbb{C}$ *with* $u \neq 0$ *and* $a \neq b \neq c$, *i.e.,* $z = a$ *or* $z = b$ *or* $z = c$ *then*

$$\sum_{k=0}^{n} z^k W_{2k} = \frac{\Omega_2}{3z^2(-t^2)+2z(-2rt+s^2)+(-1)(2s+r^2)}$$

where

$\Omega_2 = (n+3)z^{n+2}(-1)t^2W_{2n} + (n+2)z^{n+1}(sW_{2n+2} - (rs+t)W_{2n+1} - rtW_{2n}) + (n+1)z^n(-1)W_{2n+2} + 2z(-sW_2 + (t+rs)W_1 + (s^2-rt)W_0) + (W_2 - (r^2+2s)W_0)$.

(iii) *If* $z^3(-t^2)+z^2(-2rt+s^2)+z(-1)(2s+r^2)+1 = u(z-a)^2(z-b) = 0$ *for some* $u,a,b \in \mathbb{C}$ *with* $u \neq 0$ *and* $a \neq b$, *i.e.,* $z = a$ *or* $z = b$ *then for* $z = a$ *we get*

$$\sum_{k=0}^{n} z^k W_{2k} = \frac{\Omega_3}{6z(-t^2)+2(-2rt+s^2)}$$

where

$\Omega_3 = (n+3)(n+2)z^{n+1}(-1)t^2W_{2n} + (n+2)(n+1)z^n(sW_{2n+2} - (rs+t)W_{2n+1} - rtW_{2n}) + (n+1)nz^{n-1}(-1)W_{2n+2} + 2(-sW_2 + (t+rs)W_1 + (s^2-rt)W_0)$

and for $z = b$ *we get*

$$\sum_{k=0}^{n} z^k W_{2k} = \frac{\Omega_4}{3z^2(-t^2)+2z(-2rt+s^2)+(-1)(2s+r^2)}$$

where

$\Omega_4 = (n+3)z^{n+2}(-1)t^2W_{2n} + (n+2)z^{n+1}(sW_{2n+2} - (rs+t)W_{2n+1} - rtW_{2n}) + (n+1)z^n(-1)W_{2n+2} + 2z(-sW_2 + (t+rs)W_1 + (s^2-rt)W_0) + (W_2 - (r^2+2s)W_0)$.

(iv) *If* $z^3(-t^2)+z^2(-2rt+s^2)+z(-1)(2s+r^2)+1 = u(z-a)^3 = 0$ *for some* $u,a \in \mathbb{C}$ *with* $u \neq 0$, *i.e.,* $z = a$, *then*

$$\sum_{k=0}^{n} z^k W_{2k} = \frac{\Omega_5}{6(-t^2)}$$

where

$\Omega_5 = (n+3)(n+2)(n+1)z^n(-1)t^2W_{2n} + (n+2)(n+1)nz^{n-1}(sW_{2n+2} - (rs+t)W_{2n+1} - rtW_{2n}) + (n+1)n(n-1)z^{n-2}(-1)W_{2n+2}$.

(c) $(m=2, j=1)$.

(i) *If* $z^3(-t^2)+z^2(-2rt+s^2)+z(-1)(2s+r^2)+1\neq 0$ *then*

$$\sum_{k=0}^{n} z^k W_{2k+1} = \frac{\Omega_1}{z^3(-t^2)+z^2(-2rt+s^2)+z(-1)(2s+r^2)+1}$$

where

$\Omega_1 = z^{n+3}(-t^2W_{2n+1}) + z^{n+2}(-tW_{2n+2}+(s^2-rt)W_{2n+1}+stW_{2n}) + z^{n+1}(-1)(rW_{2n+2}+sW_{2n+1}+tW_{2n}) + z^2t(W_2-rW_1-sW_0) + z(rW_2-(r^2+s)W_1+tW_0)+W_1$.

(ii) *If* $z^3(-t^2)+z^2(-2rt+s^2)+z(-1)(2s+r^2)+1 = u(z-a)(z-b)(z-c)=0$ *for some* $u,a,b,c\in\mathbb{C}$ *with* $u\neq 0$ *and* $a\neq b\neq c$, *i.e.,* $z=a$ *or* $z=b$ *or* $z=c$ *then*

$$\sum_{k=0}^{n} z^k W_{2k+1} = \frac{\Omega_2}{3z^2(-t^2)+2z(-2rt+s^2)+(-1)(2s+r^2)}$$

where

$\Omega_2 = (n+3)z^{n+2}(-t^2W_{2n+1}) + (n+2)z^{n+1}(-tW_{2n+2}+(s^2-rt)W_{2n+1}+stW_{2n}) + (n+1)z^n(-1)(rW_{2n+2}+sW_{2n+1}+tW_{2n}) + 2zt(W_2-rW_1-sW_0) + (rW_2-(r^2+s)W_1+tW_0)$.

(iii) *If* $z^3(-t^2)+z^2(-2rt+s^2)+z(-1)(2s+r^2)+1 = u(z-a)^2(z-b)=0$ *for some* $u,a,b\in\mathbb{C}$ *with* $u\neq 0$ *and* $a\neq b$, *i.e.,* $z=a$ *or* $z=b$ *then for* $z=a$ *we get*

$$\sum_{k=0}^{n} z^k W_{2k+1} = \frac{\Omega_3}{6z(-t^2)+2(-2rt+s^2)}$$

where

$\Omega_3 = (n+3)(n+2)z^{n+1}(-t^2W_{2n+1}) + (n+2)(n+1)z^n(-tW_{2n+2}+(s^2-rt)W_{2n+1}+stW_{2n}) + (n+1)nz^{n-1}(-1)(rW_{2n+2}+sW_{2n+1}+tW_{2n}) + 2t(W_2-rW_1-sW_0)$

and for $z=b$ *we get*

$$\sum_{k=0}^{n} z^k W_{2k+1} = \frac{\Omega_4}{3z^2(-t^2)+2z(-2rt+s^2)+(-1)(2s+r^2)}$$

where

$\Omega_4 = (n+3)z^{n+2}(-t^2W_{2n+1}) + (n+2)z^{n+1}(-tW_{2n+2}+(s^2-rt)W_{2n+1}+stW_{2n}) + (n+1)z^n(-1)(rW_{2n+2}+sW_{2n+1}+tW_{2n}) + 2zt(W_2-rW_1-sW_0) + (rW_2-(r^2+s)W_1+tW_0)$.

(iv) *If* $z^3(-t^2)+z^2(-2rt+s^2)+z(-1)(2s+r^2)+1=u(z-a)^3=0$ *for some* $u,a\in\mathbb{C}$ *with* $u\neq 0$, *i.e.,* $z=a$, *then*

$$\sum_{k=0}^{n} z^k W_{2k+1}=\frac{\Omega_5}{6(-t^2)}$$

where

$\Omega_5=(n+3)(n+2)(n+1)z^n(-t^2W_{2n+1})+(n+2)(n+1)nz^{n-1}(-tW_{2n+2}+(s^2-rt)W_{2n+1}+stW_{2n})+(n+1)n(n-1)z^{n-2}(-1)(rW_{2n+2}+sW_{2n+1}+tW_{2n})$.

(d) $(m=-1,j=0)$.

(i) *If* $z^3(-1)+z^2r+zs+t\neq 0$ *then*

$$\sum_{k=0}^{n} z^k W_{-k}=\frac{\Omega_1}{z^3(-1)+z^2r+zs+t}$$

where

$\Omega_1=z^{n+3}(-1)W_{-n}+z^{n+2}(rW_{-n}-W_{-n+1})+z^{n+1}(-1)(W_{-n+2}-rW_{-n+1}-sW_{-n})+z^2W_1+z(W_2-rW_1)+tW_0$.

(ii) *If* $z^3(-1)+z^2r+zs+t=u(z-a)(z-b)(z-c)=0$ *for some* $u,a,b,c\in\mathbb{C}$ *with* $u\neq 0$ *and* $a\neq b\neq c$, *i.e.,* $z=a$ *or* $z=b$ *or* $z=c$ *then*

$$\sum_{k=0}^{n} z^k W_{-k}=\frac{\Omega_2}{3z^2(-1)+2zr+s}$$

where

$\Omega_2=(n+3)z^{n+2}(-1)W_{-n}+(n+2)z^{n+1}(rW_{-n}-W_{-n+1})+(n+1)z^n(-1)(W_{-n+2}-rW_{-n+1}-sW_{-n})+2zW_1+(W_2-rW_1)$.

(iii) *If* $z^3(-1)+z^2r+zs+t=u(z-a)^2(z-b)=0$ *for some* $u,a,b\in\mathbb{C}$ *with* $u\neq 0$ *and* $a\neq b$, *i.e.,* $z=a$ *or* $z=b$ *then for* $z=a$ *we get*

$$\sum_{k=0}^{n} z^k W_{-k}=\frac{\Omega_3}{6z(-1)+2r}$$

where

$\Omega_3=(n+3)(n+2)z^{n+1}(-1)W_{-n}+(n+2)(n+1)z^n(rW_{-n}-W_{-n+1})+(n+1)nz^{n-1}(-1)(W_{-n+2}-rW_{-n+1}-sW_{-n})+2W_1$

and for $z = b$ *we get*

$$\sum_{k=0}^{n} z^k W_{-k} = \frac{\Omega_4}{3z^2(-1) + 2zr + s}$$

where

$\Omega_4 = (n+3)z^{n+2}(-1)W_{-n} + (n+2)z^{n+1}(rW_{-n} - W_{-n+1}) + (n+1)z^n(-1)(W_{-n+2} - rW_{-n+1} - sW_{-n}) + 2zW_1 + (W_2 - rW_1)$.

(iv) *If* $z^3(-1) + z^2r + zs + t = u(z-a)^3 = 0$ *for some* $u, a \in \mathbb{C}$ *with* $u \neq 0$, *i.e.,* $z = a$, *then*

$$\sum_{k=0}^{n} z^k W_{-k} = \frac{\Omega_5}{6(-1)}$$

where

$\Omega_5 = (n+3)(n+2)(n+1)z^n(-1)W_{-n} + (n+2)(n+1)nz^{n-1}(rW_{-n} - W_{-n+1}) + (n+1)n(n-1)z^{n-2}(-1)(W_{-n+2} - rW_{-n+1} - sW_{-n})$.

(e) $(m = -2, j = 0)$.

(i) *If* $z^3(-1) + z^2(r^2+2s) + z(2rt - s^2) + t^2 \neq 0$ *then*

$$\sum_{k=0}^{n} z^k W_{-2k} = \frac{\Omega_1}{z^3(-1) + z^2(r^2+2s) + z(2rt - s^2) + t^2}$$

where

$\Omega_1 = z^{n+3}(-1)W_{-2n} + z^{n+2}(-1)(-sW_{-2n} + (t+rs)W_{-2n-1} + rtW_{-2n-2}) + z^{n+1}(-1)t^2W_{-2n-2} + z^2W_2 + z(-sW_2 + (t+rs)W_1 + rtW_0) + t^2W_0$.

(ii) *If* $z^3(-1) + z^2(r^2+2s) + z(2rt - s^2) + t^2 = u(z-a)(z-b)(z-c) = 0$ *for some*

$u, a, b, c \in \mathbb{C}$ *with* $u \neq 0$ *and* $a \neq b \neq c$, *i.e.,* $z = a$ *or* $z = b$ *or* $z = c$ *then*

$$\sum_{k=0}^{n} z^k W_{-2k} = \frac{\Omega_2}{3z^2(-1) + 2z(r^2+2s) + (2rt - s^2)}$$

where

$\Omega_2 = (n+3)z^{n+2}(-1)W_{-2n} + (n+2)z^{n+1}(-1)(-sW_{-2n} + (t+rs)W_{-2n-1} + rtW_{-2n-2}) + (n+1)z^n(-1)t^2W_{-2n-2} + 2zW_2 + (-sW_2 + (t+rs)W_1 + rtW_0)$.

(iii) *If* $z^3(-1)+z^2(r^2+2s)+z(2rt-s^2)+t^2=u(z-a)^2(z-b)=0$ *for some* $u,a,b\in\mathbb{C}$ *with* $u\neq 0$ *and* $a\neq b$, *i.e.,* $z=a$ *or* $z=b$ *then for* $z=a$ *we get*

$$\sum_{k=0}^{n} z^k W_{-2k} = \frac{\Omega_3}{6z(-1)+2(r^2+2s)}$$

where

$\Omega_3=(n+3)(n+2)z^{n+1}(-1)W_{-2n}+(n+2)(n+1)z^n(-1)(-sW_{-2n}+(t+rs)W_{-2n-1}+rtW_{-2n-2})+(n+1)nz^{n-1}(-1)t^2W_{-2n-2}+2W_2$

and for $z=b$ *we get*

$$\sum_{k=0}^{n} z^k W_{-2k} = \frac{\Omega_4}{3z^2(-1)+2z(r^2+2s)+(2rt-s^2)}$$

where

$\Omega_4=(n+3)z^{n+2}(-1)W_{-2n}+(n+2)z^{n+1}(-1)(-sW_{-2n}+(t+rs)W_{-2n-1}+rtW_{-2n-2})+(n+1)z^n(-1)t^2W_{-2n-2}+2zW_2+(-sW_2+(t+rs)W_1+rtW_0)$.

(iv) *If* $z^3(-1)+z^2(r^2+2s)+z(2rt-s^2)+t^2=u(z-a)^3=0$ *for some* $u,a\in\mathbb{C}$ *with* $u\neq 0$, *i.e.,* $z=a$, *then*

$$\sum_{k=0}^{n} z^k W_{-2k} = \frac{\Omega_5}{6(-1)}$$

where

$\Omega_5=(n+3)(n+2)(n+1)z^n(-1)W_{-2n}+(n+2)(n+1)nz^{n-1}(-1)(-sW_{-2n}+(t+rs)W_{-2n-1}+rtW_{-2n-2})+(n+1)n(n-1)z^{n-2}(-1)t^2W_{-2n-2}$.

(f) $(m=-2, j=1)$.

(i) *If* $z^3(-1)+z^2(r^2+2s)+z(2rt-s^2)+t^2\neq 0$ *then*

$$\sum_{k=0}^{n} z^k W_{-2k+1} = \frac{\Omega_1}{z^3(-1)+z^2(r^2+2s)+z(2rt-s^2)+t^2}$$

where

$\Omega_1=z^{n+3}(-1)(sW_{-2n-1}+tW_{-2n-2}+rW_{-2n})+z^{n+2}(-tW_{-2n}+(s^2-rt)W_{-2n-1}+stW_{-2n-2})+z^{n+1}(-1)t^2W_{-2n-1}+z^2(rW_2+sW_1+tW_0)+z(tW_2+(rt-s^2)W_1-stW_0)+t^2W_1$.

(ii) *If* $z^3(-1)+z^2(r^2+2s)+z(2rt-s^2)+t^2=u(z-a)(z-b)(z-c)=0$ *for some* $u,a,b,c\in\mathbb{C}$ *with* $u\neq 0$ *and* $a\neq b\neq c$, *i.e.,* $z=a$ *or* $z=b$ *or* $z=c$ *then*

$$\sum_{k=0}^{n} z^k W_{-2k+1} = \frac{\Omega_2}{3z^2(-1)+2z(r^2+2s)+(2rt-s^2)}$$

where

$\Omega_2=(n+3)z^{n+2}(-1)(sW_{-2n-1}+tW_{-2n-2}+rW_{-2n})+(n+2)z^{n+1}(-tW_{-2n}+(s^2-rt)W_{-2n-1}+stW_{-2n-2})+(n+1)z^n(-1)t^2W_{-2n-1}+2z(rW_2+sW_1+tW_0)+(tW_2+(rt-s^2)W_1-stW_0)$.

(iii) *If* $z^3(-1)+z^2(r^2+2s)+z(2rt-s^2)+t^2=u(z-a)^2(z-b)=0$ *for some* $u,a,b\in\mathbb{C}$ *with* $u\neq 0$ *and* $a\neq b$, *i.e.,* $z=a$ *or* $z=b$ *then for* $z=a$ *we get*

$$\sum_{k=0}^{n} z^k W_{-2k+1} = \frac{\Omega_3}{6z(-1)+2(r^2+2s)}$$

where

$\Omega_3=(n+3)(n+2)z^{n+1}(-1)(sW_{-2n-1}+tW_{-2n-2}+rW_{-2n})+(n+2)(n+1)z^n(-tW_{-2n}+(s^2-rt)W_{-2n-1}+stW_{-2n-2})+(n+1)nz^{n-1}(-1)t^2W_{-2n-1}+2(rW_2+sW_1+tW_0)$

and for $z=b$ *we get*

$$\sum_{k=0}^{n} z^k W_{-2k+1} = \frac{\Omega_4}{3z^2(-1)+2z(r^2+2s)+(2rt-s^2)}$$

where

$\Omega_4=(n+3)z^{n+2}(-1)(sW_{-2n-1}+tW_{-2n-2}+rW_{-2n})+(n+2)z^{n+1}(-tW_{-2n}+(s^2-rt)W_{-2n-1}+stW_{-2n-2})+(n+1)z^n(-1)t^2W_{-2n-1}+2z(rW_2+sW_1+tW_0)+(tW_2+(rt-s^2)W_1-stW_0)$.

(iv) *If* $z^3(-1)+z^2(r^2+2s)+z(2rt-s^2)+t^2=u(z-a)^3=0$ *for some* $u,a\in\mathbb{C}$ *with* $u\neq 0$, *i.e.,* $z=a$, *then*

$$\sum_{k=0}^{n} z^k W_{-2k+1} = \frac{\Omega_5}{6(-1)}$$

where

$\Omega_5=(n+3)(n+2)(n+1)z^n(-1)(sW_{-2n-1}+tW_{-2n-2}+rW_{-2n})+(n+2)(n+1)nz^{n-1}(-tW_{-2n}+(s^2-rt)W_{-2n-1}+stW_{-2n-2})+(n+1)n(n-1)z^{n-2}(-1)t^2W_{-2n-1}$.

10.2 The Sum Formula $\sum_{k=0}^{n} z^k W_{mk+j}$ of Generalized Tribonacci Polynomials in Terms of Generalized Tribonacci Polynomials and (r,s,t)-Tribonacci Polynomials

By using Theorem 51 (a), we can give the sum formula $\sum_{k=0}^{n} z^k W_{mk+j}$ of generalized Tribonacci polynomials via matrix methods (in terms of elements of the sequence of generalized Tribonacci polynomials and (r,s,t)-Tribonacci polynomials).

Theorem 63. *For all integers m and j, we have the following sum formulas.*

(a) *If $z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4 \neq 0$ then*

$$\begin{aligned}\sum_{k=0}^{n} z^k W_{mk+j} &= \frac{z^{n+3}\Theta_1 + z^{n+2}\Theta_2 + z^{n+1}\Theta_3 + z^2\Theta_4 + z\Theta_5 + \Theta_6}{z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4} \qquad (10.3)\\ &= \frac{\Theta_G(z)}{\Gamma_G(z)}\end{aligned}$$

where

$\Theta_G(z) = z^{n+3}\Theta_1 + z^{n+2}\Theta_2 + z^{n+1}\Theta_3 + z^2\Theta_4 + z\Theta_5 + \Theta_6,$

$z^{n+3}\Theta_1 = z^{n+3}((-W_jG_{m+2}^2 + (-W_{j+2} + sW_j)G_{m+1}^2 - tW_{j+1}G_m^2 + (W_{j+1} + rW_j)G_{m+2}G_{m+1} + (W_{j+2} - rW_{j+1})G_{m+2}G_m + (-sW_{j+1} + tW_j)G_{m+1}G_m)G_{m+mn+2} + ((-W_{j+1} + rW_j)G_{m+2}^2 - (t+rs)W_jG_{m+1}^2 + t(-W_{j+2} + rW_{j+1})G_m^2 + (W_{j+2} - r^2W_j)G_{m+2}G_{m+1} + (-rW_{j+2} + (r^2+s)W_{j+1} + tW_j)G_{m+2}G_m + (-sW_{j+2} + (t+rs)W_{j+1} - rtW_j)G_{m+1}G_m)G_{m+mn+1} + ((-W_{j+2} + rW_{j+1} + sW_j)G_{m+2}^2 + (sW_{j+2} - (t+rs)W_{j+1} - s^2W_j)G_{m+1}^2 - W_jt^2G_m^2 + (W_{j+2}r - r^2W_{j+1} + (t-rs)W_j)G_{m+2}G_{m+1} + t(W_{j+1} - rW_j)G_{m+2}G_m + t(W_{j+2} - rW_{j+1} - 2sW_j)G_{m+1}G_m)G_{m+mn}),$

$z^{n+2}\Theta_2 = z^{n+2}((((rW_j - W_{j+1})G_{m+2} + (2W_{j+2} - rW_{j+1} - (r^2+2s)W_j)G_{m+1} + (-rW_{j+2} + (r^2+s)W_{j+1} - tW_j)G_m)G_{m+mn+2} + ((2rW_{j+1} - W_{j+2} - r^2W_j)G_{m+2} + (-rW_{j+2} + (r^3 + 2t + 2rs)W_j)G_{m+1} + ((r^2+s)W_{j+2} - (r^3 + 2rs + t)W_{j+1})G_m)G_{m+mn+1} + ((rW_{j+2} - r^2W_{j+1} - (rs+t)W_j)G_{m+2} + (-(r^2+2s)W_{j+2} + (r^3 + 2rs + 2t)W_{j+1} + (r^2s + 2s^2 - rt)W_j)G_{m+1} + 2stW_jG_m - tG_mW_{j+2} + r^2tW_jG_m)G_{m+mn}),$

$z^{n+1}\Theta_3 = z^{n+1}((-W_{j+2} + rW_{j+1} + sW_j)G_{m+mn+2} + (rW_{j+2} - r^2W_{j+1} - (rs+t)W_j)G_{m+mn+1} + (sW_{j+2} - (rs+t)W_{j+1} - (s^2 - rt)W_j)G_{m+mn}),$

$z^2\Theta_4 = z^2(G_{m+2}^2W_{j+1} + (rW_{j+2} + tW_j)G_{m+1}^2 + tG_m^2W_{j+2} - (rW_{j+1} + W_{j+2})G_{m+1}G_{m+2} - (sW_{j+1} + tW_j)G_mG_{m+2} + (sW_{j+2} - tW_{j+1})G_mG_{m+1}),$

$z\Theta_5 = z((W_{j+2} - rW_{j+1})G_{m+2} + (-rW_{j+2} + r^2W_{j+1} - 2tW_j)G_{m+1} + (-sW_{j+2} + (tW_{j+1} + rsW_{j+1}) + rtW_j)G_m),$

$\Theta_6 = tW_j$,

and

$\Gamma_G(z) = \Gamma_1(z) + \Gamma_2(z) + \Gamma_3(z) + \Gamma_4(z)$,

$z^3\Gamma_1 = z^3(-t^{m+1})$,

$z^2\Gamma_2 = z^2(rG_{m+2}^2 + (r^3 + 2rs + 3t)G_{m+1}^2 + (r^2t + 2st)G_m^2 - 2(r^2 + s)G_{m+1}G_{m+2} - (rs + 3t)G_mG_{m+2} + (r^2s + 2s^2 - rt)G_mG_{m+1})$,

$z\Gamma_3 = z(sG_{m+2} - (rs + 3t)G_{m+1} + (2rt - s^2)G_m)$,

$\Gamma_4 = t$.

(b) *If* $z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4 = u(z-a)(z-b)(z-c) = 0$ *for some* $u, a, b, c \in \mathbb{C}$ *with* $u \neq 0$ *and* $a \neq b \neq c$*, i.e.,* $z = a$ *or* $z = b$ *or* $z = c$ *then*

$$\sum_{k=0}^{n} z^k W_{mk+j} = \frac{(n+3)z^{n+2}\Theta_1 + (n+2)z^{n+1}\Theta_2 + (n+1)z^n\Theta_3 + 2z\Theta_4 + \Theta_5}{3z^2\Gamma_1 + 2z\Gamma_2 + \Gamma_3}.$$

(c) *If* $z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4 = u(z-a)^2(z-b) = 0$ *for some* $u, a, b \in \mathbb{C}$ *with* $u \neq 0$ *and* $a \neq b$*, i.e.,* $z = a$ *or* $z = b$ *then for* $z = a$ *we get*

$$\sum_{k=0}^{n} z^k W_{mk+j} = \frac{(n+3)(n+2)z^{n+1}\Theta_1 + (n+2)(n+1)z^n\Theta_2 + (n+1)nz^{n-1}\Theta_3 + 2\Theta_4}{6z\Gamma_1 + 2\Gamma_2},$$

and for $z = b$ *we get*

$$\sum_{k=0}^{n} z^k W_{mk+j} = \frac{(n+3)z^{n+2}\Theta_1 + (n+2)z^{n+1}\Theta_2 + (n+1)z^n\Theta_3 + 2z\Theta_4 + \Theta_5}{3z^2\Gamma_1 + 2z\Gamma_2 + \Gamma_3}.$$

(d) *If* $z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4 = u(z-a)^3 = 0$ *for some* $u, a \in \mathbb{C}$ *with* $u \neq 0$*, i.e.,* $z = a$*, then*

$$\sum_{k=0}^{n} z^k W_{mk+j} = \frac{(n+3)(n+2)(n+1)z^n\Theta_1 + (n+2)(n+1)nz^{n-1}\Theta_2 + (n+1)n(n-1)z^{n-2}\Theta_3}{6\Gamma_1}.$$

Proof. We only prove (a). The proof of (b), (c) and (d) are as in Theorem 61 (b), (c) and (d), respectively.

Proof of (a). We use the same method as in Theorem 61 by setting

$$\begin{aligned} M &= zA^m = z\begin{pmatrix} G_{m+1} & sG_m + tG_{m-1} & tG_m \\ G_m & sG_{m-1}+tG_{m-2} & tG_{m-1} \\ G_{m-1} & sG_{m-2}+tG_{m-3} & tG_{m-2} \end{pmatrix} \\ &= z\begin{pmatrix} G_{m+1} & G_{m+2}-rG_{m+1} & tG_m \\ G_m & G_{m+1}-rG_m & G_{m+2}-sG_m-rG_{m+1} \\ e_{31} & e_{32} & e_{33} \end{pmatrix} \end{aligned}$$

in (10.1), by using

$$\begin{aligned} G_{m-1} &= \frac{1}{t}(G_{m+2}-rG_{m+1}-sG_m), \\ G_{m-2} &= \frac{1}{t^2}(-sG_{m+2}+(rs+t)G_{m+1}+(s^2-rt)G_m), \\ G_{m-3} &= \frac{1}{t^3}((s^2-rt)G_{m+2}+(r^2t-rs^2-st)G_{m+1}+(-s^3+t^2+2rst)G_m), \end{aligned}$$

where

$$\begin{aligned} e_{31} &= -\frac{1}{t}\left(sG_m - G_{m+2} + rG_{m+1}\right), \\ e_{32} &= \frac{1}{t}\left(tG_m + r^2G_{m+1} - rG_{m+2} + rsG_m\right), \\ e_{33} &= \frac{1}{t}\left(s^2G_m - sG_{m+2} + tG_{m+1} - rtG_m + rsG_{m+1}\right). \end{aligned}$$

Then we get

$$I - zA^m = \begin{pmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{pmatrix}$$

where

$f_{11} = -(zG_{m+1}-1)$,
$f_{21} = -zG_m$,
$f_{31} = zt^{-1}(-G_{m+2}+rG_{m+1}+sG_m)$,
$f_{12} = z(-G_{m+2}+rG_{m+1})$,
$f_{22} = -zG_{m+1}+rzG_m+1$,
$f_{32} = zt^{-1}(rG_{m+2}-r^2G_{m+1}-(t+rs)G_m)$,
$f_{13} = -tzG_m$,
$f_{23} = z(-G_{m+2}+rG_{m+1}+sG_m)$,
$f_{33} = t^{-1}(szG_{m+2}-z(t+rs)G_{m+1}+z(rt-s^2)G_m+t)$.

After some calculations, we see that

$$\begin{aligned}\det(I-zA^{m}) &= \frac{1}{t}(z^{3}(-G_{m+2}^{3}-(t+rs)G_{m+1}^{3}-t^{2}G_{m}^{3}+2rG_{m+1}G_{m+2}^{2}+sG_{m}G_{m+2}^{2}\\ &\quad -(r^{2}-s)G_{m+2}G_{m+1}^{2}-(rt+s^{2})G_{m}G_{m+1}^{2}-rtG_{m}^{2}G_{m+2}-2stG_{m}^{2}G_{m+1}\\ &\quad +(3t-rs)G_{m+2}G_{m+1}G_{m})\\ &\quad +z^{2}(rG_{m+2}^{2}+(r^{3}+2rs+3t)G_{m+1}^{2}+(r^{2}t+2st)G_{m}^{2}-2(r^{2}+s)G_{m+1}G_{m+2}\\ &\quad -(rs+3t)G_{m}G_{m+2}+(r^{2}s+2s^{2}-rt)G_{m}G_{m+1})\\ &\quad +z(sG_{m+2}-(rs+3t)G_{m+1}+(2rt-s^{2})G_{m})+t).\end{aligned}$$

Since

$$\begin{aligned}&-G_{n+2}^{3}-(t+rs)G_{n+1}^{3}-t^{2}G_{n}^{3}+2rG_{n+1}G_{n+2}^{2}+sG_{n}G_{n+2}^{2}-(r^{2}-s)G_{n+2}G_{n+1}^{2}\\ &-(rt+s^{2})G_{n}G_{n+1}^{2}-rtG_{n}^{2}G_{n+2}-2stG_{n}^{2}G_{n+1}+(3t-rs)G_{n+2}G_{n+1}G_{n}\\ = &-t^{m+1},\end{aligned}$$

see (4.4), it follows that

$$\begin{aligned}\det(I-zA^{m}) &= \frac{1}{t}(z^{3}(-t^{m+1})\\ &\quad +z^{2}(rG_{m+2}^{2}+(r^{3}+2rs+3t)G_{m+1}^{2}+(r^{2}t+2st)G_{m}^{2}\\ &\quad -2(r^{2}+s)G_{m+1}G_{m+2}-(rs+3t)G_{m}G_{m+2}+(r^{2}s+2s^{2}-rt)G_{m}G_{m+1})\\ &\quad +z(sG_{m+2}-(rs+3t)G_{m+1}+(2rt-s^{2})G_{m})+t)\\ &= \frac{1}{t}\Gamma_{G}(z)=\frac{1}{t}(z^{3}\Gamma_{1}+z^{2}\Gamma_{2}+z\Gamma_{3}+\Gamma_{4})\end{aligned}$$

where $\Gamma_W(z)=z^3\Gamma_1+z^2\Gamma_2+z\Gamma_3+\Gamma_4, z^3\Gamma_1, z^2\Gamma_2, z\Gamma_3, \Gamma_4$ are as in the statement of (a) of Theorem.

The 3rd row and 1st column entry of matrix $(I-z^{n+1}A^{mn+m})Adj(I-zA^m)f_W(j)$ is equal to

$$\frac{1}{t}(z^{n+3}\Theta_1+z^{n+2}\Theta_2+z^{n+1}\Theta_3+z^2\Theta_4+z\Theta_5+\Theta_6)$$

where $\Theta_W(z) = z^{n+3}\Theta_1 + z^{n+2}\Theta_2 + z^{n+1}\Theta_3 + z^2\Theta_4 + z\Theta_5 + \Theta_6, z^{n+3}\Theta_1, z^{n+2}\Theta_2, z^{n+1}\Theta_3, z^2\Theta_4, z\Theta_5, \Theta_6$, are as in the statement of (a) of Theorem.

Note that, since $\det(I-zA^m)(\sum_{k=0}^{n} z^k f_W(mk+j) = (I-z^{n+1}A^{mn+m})Adj(I-zA^m)f_W(j)$, i.e., matrices over the either side is equal, the 3rd row and 1st column entry of matrix of matrix $\det(I-zA^m)(\sum_{k=0}^{n} z^k f_W(mk+j))$ is equal to the 3rd row and 1st column entry of matrix $(I-z^{n+1}A^{mn+m})Adj(I-zA^m)f_W(j)$. So, to complete the proof, we will just compare the linear combination of the 3rd row and 1st column entries of the matrices. Then, we get

$$\frac{1}{t}(z^3\Gamma_1+z^2\Gamma_2+z\Gamma_3+\Gamma_4)\sum_{k=0}^{n}z^kW_{mk+j}=\frac{1}{t}(z^{n+3}\Theta_1+z^{n+2}\Theta_2+z^{n+1}\Theta_3+z^2\Theta_4+z\Theta_5+\Theta_6)$$

and so

$$\begin{aligned}\sum_{k=0}^{n}z^kW_{mk+j} &= \frac{z^{n+3}\Theta_1+z^{n+2}\Theta_2+z^{n+1}\Theta_3+z^2\Theta_4+z\Theta_5+\Theta_6}{z^3\Gamma_1+z^2\Gamma_2+z\Gamma_3+\Gamma_4}\\ &= \frac{\Theta_W(z)}{\Gamma_W(z)}.\end{aligned}$$

□

Now, we consider special cases of Theorem 63.

Theorem 64. *We have the following sum formulas.*

(a) $(m=1, j=0)$.

(i) *If* $z^3(-1)t+z^2(-s)+z(-r)+1\neq 0$ *then*

$$\sum_{k=0}^{n}z^kW_k=\frac{\Omega_1}{z^3(-1)t+z^2(-s)+z(-r)+1}$$

where

$\Omega_1 = z^{n+3}((-W_2+rW_1+sW_0)G_{n+2}+(rW_2-r^2W_1-(rs+t)W_0)G_{n+1}+(sW_2-(rs+t)W_1+(rt-s^2)W_0)G_n)+z^{n+2}((-W_1+rW_0)G_{n+2}-(W_2-2rW_1+r^2W_0)G_{n+1}+(rW_2-r^2W_1-(rs+t)W_0)G_n)+z^{n+1}(-W_0G_{n+2}+(-W_1+rW_0)G_{n+1}+(-W_2+rW_1+sW_0)G_n)+z^2(W_2-rW_1-sW_0)+z(W_1-rW_0)+W_0.$

(ii) *If* $z^3(-1)t+z^2(-s)+z(-r)+1=u(z-a)(z-b)(z-c)=0$ *for some* $u,a,b,c\in\mathbb{C}$ *with* $u\neq 0$ *and* $a\neq b\neq c$, *i.e.,* $z=a$ *or* $z=b$ *or* $z=c$ *then*

$$\sum_{k=0}^{n}z^kW_k=\frac{\Omega_2}{3z^2(-1)t+2z(-s)+(-r)}$$

where

$\Omega_2=(n+3)z^{n+2}((-W_2+rW_1+sW_0)G_{n+2}+(rW_2-r^2W_1-(rs+t)W_0)G_{n+1}+(sW_2-(rs+t)W_1+(rt-s^2)W_0)G_n)+(n+2)z^{n+1}((-W_1+rW_0)G_{n+2}-(W_2-2rW_1+r^2W_0)G_{n+1}+(rW_2-r^2W_1-(rs+t)W_0)G_n)+(n+1)z^n(-W_0G_{n+2}+(-W_1+rW_0)G_{n+1}+(-W_2+rW_1+sW_0)G_n)+2z(W_2-rW_1-sW_0)+(W_1-rW_0).$

(iii) *If* $z^3(-1)t + z^2(-s) + z(-r) + 1 = u(z-a)^2(z-b) = 0$ *for some* $u, a, b \in \mathbb{C}$ *with* $u \neq 0$ *and* $a \neq b$, *i.e.,* $z = a$ *or* $z = b$ *then for* $z = a$ *we get*

$$\sum_{k=0}^{n} z^k W_k = \frac{\Omega_3}{6z(-1)t + 2(-s)}$$

where

$\Omega_3 = (n+3)(n+2)z^{n+1}((-W_2 + rW_1 + sW_0)G_{n+2} + (rW_2 - r^2W_1 - (rs+t)W_0)G_{n+1} + (sW_2 - (rs+t)W_1 + (rt - s^2)W_0)G_n) + (n+2)(n+1)z^n((-W_1 + rW_0)G_{n+2} - (W_2 - 2rW_1 + r^2W_0)G_{n+1} + (rW_2 - r^2W_1 - (rs+t)W_0)G_n) + (n+1)nz^{n-1}(-W_0G_{n+2} + (-W_1 + rW_0)G_{n+1} + (-W_2 + rW_1 + sW_0)G_n) + 2(W_2 - rW_1 - sW_0)$

and for $z = b$ *we get*

$$\sum_{k=0}^{n} z^k W_k = \frac{\Omega_4}{3z^2(-1)t + 2z(-s) + (-r)}$$

where

$\Omega_4 = (n+3)z^{n+2}((-W_2 + rW_1 + sW_0)G_{n+2} + (rW_2 - r^2W_1 - (rs+t)W_0)G_{n+1} + (sW_2 - (rs+t)W_1 + (rt - s^2)W_0)G_n) + (n+2)z^{n+1}((-W_1 + rW_0)G_{n+2} - (W_2 - 2rW_1 + r^2W_0)G_{n+1} + (rW_2 - r^2W_1 - (rs+t)W_0)G_n) + (n+1)z^n(-W_0G_{n+2} + (-W_1 + rW_0)G_{n+1} + (-W_2 + rW_1 + sW_0)G_n) + 2z(W_2 - rW_1 - sW_0) + (W_1 - rW_0)$

(iv) *If* $z^3(-1)t + z^2(-s) + z(-r) + 1 = u(z-a)^3 = 0$ *for some* $u, a \in \mathbb{C}$ *with* $u \neq 0$, *i.e.,* $z = a$, *then*

$$\sum_{k=0}^{n} z^k W_k = \frac{\Omega_5}{6(-1)t}$$

where

$\Omega_5 = (n+3)(n+2)(n+1)z^n((-W_2 + rW_1 + sW_0)G_{n+2} + (rW_2 - r^2W_1 - (rs+t)W_0)G_{n+1} + (sW_2 - (rs+t)W_1 + (rt - s^2)W_0)G_n) + (n+2)(n+1)nz^{n-1}((-W_1 + rW_0)G_{n+2} - (W_2 - 2rW_1 + r^2W_0)G_{n+1} + (rW_2 - r^2W_1 - (rs+t)W_0)G_n) + (n+1)n(n-1)z^{n-2}(-W_0G_{n+2} + (-W_1 + rW_0)G_{n+1} + (-W_2 + rW_1 + sW_0)G_n)$

(b) $(m = 2, j = 0)$.

(i) *If* $z^3(-1)t^2 + z^2(s^2 - 2rt) + z(-(r^2 + 2s)) + 1 \neq 0$ *then*

$$\sum_{k=0}^{n} z^k W_{2k} = \frac{\Omega_1}{z^3(-1)t^2 + z^2(s^2 - 2rt) + z(-(r^2 + 2s)) + 1}$$

where

$\Omega_1 = z^{n+3}t((-W_2 + rW_1 + sW_0)G_{2n+2} + (rW_2 - r^2W_1 - (t + rs)W_0)G_{2n+1} + (sW_2 - (t+rs)W_1 + (rt-s^2)W_0)G_{2n}) + z^{n+2}((-rW_2 + (r^2+s)W_1 - tW_0)G_{2n+2} + ((r^2 + s)W_2 - (r^3 + 2rs + t)W_1)G_{2n+1} + t(-W_2 + 2sW_0 + r^2W_0)G_{2n}) + z^{n+1}(-W_1G_{2n+2} + (rW_1 - W_2)G_{2n+1} - tW_0G_{2n}) + z^2(-sW_2 + (rs + t)W_1 + (s^2 - rt)W_0) + z(W_2 - (r^2 + 2s)W_0) + W_0.$

(ii) *If* $z^3(-1)t^2 + z^2(s^2 - 2rt) + z(-(r^2 + 2s)) + 1 = u(z - a)(z - b)(z - c) = 0$ *for some* $u, a, b, c \in \mathbb{C}$ *with* $u \neq 0$ *and* $a \neq b \neq c$*, i.e.,* $z = a$ *or* $z = b$ *or* $z = c$ *then*

$$\sum_{k=0}^{n} z^k W_{2k} = \frac{\Omega_2}{3z^2(-1)t^2 + 2z(s^2 - 2rt) + (-(r^2 + 2s))}$$

where

$\Omega_2 = (n + 3)z^{n+2}t((-W_2 + rW_1 + sW_0)G_{2n+2} + (rW_2 - r^2W_1 - (t + rs)W_0)G_{2n+1} + (sW_2 - (t + rs)W_1 + (rt - s^2)W_0)G_{2n}) + (n + 2)z^{n+1}((-rW_2 + (r^2 + s)W_1 - tW_0)G_{2n+2} + ((r^2 + s)W_2 - (r^3 + 2rs + t)W_1)G_{2n+1} + t(-W_2 + 2sW_0 + r^2W_0)G_{2n}) + (n + 1)z^n(-W_1G_{2n+2} + (rW_1 - W_2)G_{2n+1} - tW_0G_{2n}) + 2z(-sW_2 + (rs + t)W_1 + (s^2 - rt)W_0) + (W_2 - (r^2 + 2s)W_0).$

(iii) *If* $z^3(-1)t^2 + z^2(s^2 - 2rt) + z(-(r^2 + 2s)) + 1 = u(z - a)^2(z - b) = 0$ *for some* $u, a, b \in \mathbb{C}$ *with* $u \neq 0$ *and* $a \neq b$*, i.e.,* $z = a$ *or* $z = b$ *then for* $z = a$ *we get*

$$\sum_{k=0}^{n} z^k W_{2k} = \frac{\Omega_3}{6z(-1)t^2 + 2(s^2 - 2rt)}$$

where

$\Omega_3 = (n + 3)(n + 2)z^{n+1}t((-W_2 + rW_1 + sW_0)G_{2n+2} + (rW_2 - r^2W_1 - (t + rs)W_0)G_{2n+1} + (sW_2 - (t+rs)W_1 + (rt-s^2)W_0)G_{2n}) + (n+2)(n+1)z^n((-rW_2 + (r^2 + s)W_1 - tW_0)G_{2n+2} + ((r^2 + s)W_2 - (r^3 + 2rs + t)W_1)G_{2n+1} + t(-W_2 + 2sW_0 + r^2W_0)G_{2n}) + (n+1)nz^{n-1}(-W_1G_{2n+2} + (rW_1 - W_2)G_{2n+1} - tW_0G_{2n}) + 2(-sW_2 + (rs + t)W_1 + (s^2 - rt)W_0)$

and for $z = b$ *we get*

$$\sum_{k=0}^{n} z^k W_{2k} = \frac{\Omega_4}{3z^2(-1)t^2 + 2z(s^2 - 2rt) + (-(r^2 + 2s))}$$

where

$\Omega_4 = (n + 3)z^{n+2}t((-W_2 + rW_1 + sW_0)G_{2n+2} + (rW_2 - r^2W_1 - (t + rs)W_0)G_{2n+1} + (sW_2 - (t + rs)W_1 + (rt - s^2)W_0)G_{2n}) + (n + 2)z^{n+1}((-rW_2 +$

$(r^2+s)W_1 - tW_0)G_{2n+2} + ((r^2+s)W_2 - (r^3+2rs+t)W_1)G_{2n+1} + t(-W_2 + 2sW_0 + r^2W_0)G_{2n}) + (n+1)z^n(-W_1G_{2n+2} + (rW_1 - W_2)G_{2n+1} - tW_0G_{2n}) + 2z(-sW_2 + (rs+t)W_1 + (s^2-rt)W_0) + (W_2 - (r^2+2s)W_0)$.

(iv) *If* $z^3(-1)t^2 + z^2(s^2-2rt) + z(-(r^2+2s)) + 1 = u(z-a)^3 = 0$ *for some* $u, a \in \mathbb{C}$ *with* $u \neq 0$, *i.e.,* $z = a$, *then*

$$\sum_{k=0}^{n} z^k W_{2k} = \frac{\Omega_5}{6(-1)t^2}$$

where

$\Omega_5 = (n+3)(n+2)(n+1)z^n t((-W_2 + rW_1 + sW_0)G_{2n+2} + (rW_2 - r^2W_1 - (t+rs)W_0)G_{2n+1} + (sW_2 - (t+rs)W_1 + (rt-s^2)W_0)G_{2n}) + (n+2)(n+1)nz^{n-1}((-rW_2 + (r^2+s)W_1 - tW_0)G_{2n+2} + ((r^2+s)W_2 - (r^3+2rs+t)W_1) G_{2n+1} + t(-W_2 + 2sW_0 + r^2W_0)G_{2n}) + (n+1)n(n-1)z^{n-2}(-W_1G_{2n+2} + (rW_1 - W_2)G_{2n+1} - tW_0G_{2n})$.

(c) $(m = 2, j = 1)$.

(i) *If* $z^3(-1)t^2 + z^2(-2rt+s^2) + z(-1)(r^2+2s) + 1 \neq 0$ *then*

$$\sum_{k=0}^{n} z^k W_{2k+1} = \frac{\Omega_1}{z^3(-1)t^2 + z^2(-2rt+s^2) + z(-1)(r^2+2s) + 1}$$

where

$\Omega_1 = z^{n+3}t^2(-W_0G_{2n+2} + (rW_0 - W_1)G_{2n+1} + (-W_2 + rW_1 + sW_0)G_{2n}) + z^{n+2}((sW_2 - (rs+t)W_1 - rtW_0)G_{2n+2} + (-(rs+t)W_2 + s(r^2+s)W_1 + t(s+r^2)W_0)G_{2n+1} + t(-rW_2 + (r^2+s)W_1 - tW_0)G_{2n}) + z^{n+1}(-W_2G_{2n+2} - (sW_1 + tW_0)G_{2n+1} - tW_1G_{2n}) + z^2t(W_2 - rW_1 - sW_0) + z(rW_2 - (r^2+s)W_1 + tW_0) + W_1$.

(ii) *If* $z^3(-1)t^2 + z^2(-2rt+s^2) + z(-1)(r^2+2s) + 1 = u(z-a)(z-b)(z-c) = 0$ *for some* $u, a, b, c \in \mathbb{C}$ *with* $u \neq 0$ *and* $a \neq b \neq c$, *i.e.,* $z = a$ *or* $z = b$ *or* $z = c$ *then*

$$\sum_{k=0}^{n} z^k W_{2k+1} = \frac{\Omega_2}{3z^2(-1)t^2 + 2z(-2rt+s^2) + (-1)(r^2+2s)}$$

where

$\Omega_2 = (n+3)z^{n+2}t^2(-W_0G_{2n+2} + (rW_0 - W_1)G_{2n+1} + (-W_2 + rW_1 + sW_0)G_{2n}) + (n+2)z^{n+1}((sW_2 - (rs+t)W_1 - rtW_0)G_{2n+2} + (-(rs+t)W_2 + s(r^2+s)W_1 + t(s+r^2)W_0)G_{2n+1} + t(-rW_2 + (r^2+s)W_1 - tW_0)G_{2n}) + (n+1)z^n(-W_2G_{2n+2} - (sW_1 + tW_0)G_{2n+1} - tW_1G_{2n}) + 2zt(W_2 - rW_1 - sW_0) + (rW_2 - (r^2+s)W_1 + tW_0)$.

(iii) *If* $z^3(-1)t^2 + z^2(-2rt + s^2) + z(-1)(r^2 + 2s) + 1 = u(z-a)^2(z-b) = 0$ *for some* $u, a, b \in \mathbb{C}$ *with* $u \neq 0$ *and* $a \neq b$, *i.e.,* $z = a$ *or* $z = b$ *then for* $z = a$ *we get*

$$\sum_{k=0}^{n} z^k W_{2k+1} = \frac{\Omega_3}{6z(-1)t^2 + 2(-2rt + s^2)}$$

where

$\Omega_3 = (n+3)(n+2)z^{n+1}t^2(-W_0G_{2n+2} + (rW_0 - W_1)G_{2n+1} + (-W_2 + rW_1 + sW_0)G_{2n}) + (n+2)(n+1)z^n((sW_2 - (rs+t)W_1 - rtW_0)G_{2n+2} + (-(rs+t)W_2 + s(r^2+s)W_1 + t(s+r^2)W_0)G_{2n+1} + t(-rW_2 + (r^2+s)W_1 - tW_0)G_{2n}) + (n+1)nz^{n-1}(-W_2G_{2n+2} - (sW_1 + tW_0)G_{2n+1} - tW_1G_{2n}) + 2t(W_2 - rW_1 - sW_0)$

and for $z = b$ *we get*

$$\sum_{k=0}^{n} z^k W_{2k+1} = \frac{\Omega_4}{3z^2(-1)t^2 + 2z(-2rt + s^2) + (-1)(r^2 + 2s)}$$

where

$\Omega_4 = (n+3)z^{n+2}t^2(-W_0G_{2n+2} + (rW_0 - W_1)G_{2n+1} + (-W_2 + rW_1 + sW_0)G_{2n}) + (n+2)z^{n+1}((sW_2 - (rs+t)W_1 - rtW_0)G_{2n+2} + (-(rs+t)W_2 + s(r^2+s)W_1 + t(s+r^2)W_0)G_{2n+1} + t(-rW_2 + (r^2+s)W_1 - tW_0)G_{2n}) + (n+1)z^n(-W_2G_{2n+2} - (sW_1 + tW_0)G_{2n+1} - tW_1G_{2n}) + 2zt(W_2 - rW_1 - sW_0) + (rW_2 - (r^2+s)W_1 + tW_0)$.

(iv) *If* $z^3(-1)t^2 + z^2(-2rt + s^2) + z(-1)(r^2 + 2s) + 1 = u(z-a)^3 = 0$ *for some* $u, a \in \mathbb{C}$ *with* $u \neq 0$, *i.e.,* $z = a$, *then*

$$\sum_{k=0}^{n} z^k W_{2k+1} = \frac{\Omega_5}{6(-1)t^2}$$

where

$\Omega_5 = (n+3)(n+2)(n+1)z^nt^2(-W_0G_{2n+2} + (rW_0 - W_1)G_{2n+1} + (-W_2 + rW_1 + sW_0)G_{2n}) + (n+2)(n+1)nz^{n-1}((sW_2 - (rs+t)W_1 - rtW_0)G_{2n+2} + (-(rs+t)W_2 + s(r^2+s)W_1 + t(s+r^2)W_0)G_{2n+1} + t(-rW_2 + (r^2+s)W_1 - tW_0)G_{2n}) + (n+1)n(n-1)z^{n-2}(-W_2G_{2n+2} - (sW_1 + tW_0)G_{2n+1} - tW_1G_{2n})$.

(d) $(m = -1, j = 0)$.

(i) *If* $z^3(-1) + z^2r + zs + t \neq 0$ *then*

$$\sum_{k=0}^{n} z^k W_{-k} = \frac{\Omega_1}{z^3(-1) + z^2r + zs + t}$$

where

$\Omega_1 = z^{n+3}(-W_0G_{-n+1} + (rW_0 - W_1)G_{-n} + (-W_2 + rW_1 + sW_0)G_{-n-1}) + z^{n+2}((-W_1 + rW_0)G_{-n+1} + (-W_2 + 2W_1r - W_0r^2)G_{-n} + (rW_2 - r^2W_1 - (rs + t)W_0)G_{-n-1}) + z^{n+1}((-W_2 + rW_1 + sW_0)G_{-n+1} - (-rW_2 + r^2W_1 + (t + rs)W_0)G_{-n} + (sW_2 - (t + rs)W_1 + (rt - s^2)W_0)G_{-n-1}) + z^2W_1 + z(W_2 - rW_1) + tW_0.$

(ii) *If* $z^3(-1) + z^2r + zs + t = u(z-a)(z-b)(z-c) = 0$ *for some* $u, a, b, c \in \mathbb{C}$ *with* $u \neq 0$ *and* $a \neq b \neq c$, *i.e.,* $z = a$ *or* $z = b$ *or* $z = c$ *then*

$$\sum_{k=0}^{n} z^k W_{-k} = \frac{\Omega_2}{3z^2(-1) + 2zr + s}$$

where

$\Omega_2 = (n+3)z^{n+2}(-W_0G_{-n+1} + (rW_0 - W_1)G_{-n} + (-W_2 + rW_1 + sW_0)G_{-n-1}) + (n+2)z^{n+1}((-W_1 + rW_0)G_{-n+1} + (-W_2 + 2W_1r - W_0r^2)G_{-n} + (rW_2 - r^2W_1 - (rs + t)W_0)G_{-n-1}) + (n+1)z^n((-W_2 + rW_1 + sW_0)G_{-n+1} - (-rW_2 + r^2W_1 + (t + rs)W_0)G_{-n} + (sW_2 - (t + rs)W_1 + (rt - s^2)W_0)G_{-n-1}) + 2zW_1 + (W_2 - rW_1).$

(iii) *If* $z^3(-1) + z^2r + zs + t = u(z-a)^2(z-b) = 0$ *for some* $u, a, b \in \mathbb{C}$ *with* $u \neq 0$ *and* $a \neq b$, *i.e.,* $z = a$ *or* $z = b$ *then for* $z = a$ *we get*

$$\sum_{k=0}^{n} z^k W_{-k} = \frac{\Omega_3}{6z(-1) + 2r}$$

where

$\Omega_3 = (n+3)(n+2)z^{n+1}(-W_0G_{-n+1} + (rW_0 - W_1)G_{-n} + (-W_2 + rW_1 + sW_0) G_{-n-1}) + (n+2)(n+1)z^n((-W_1 + rW_0)G_{-n+1} + (-W_2 + 2W_1r - W_0r^2) G_{-n} + (rW_2 - r^2W_1 - (rs + t)W_0)G_{-n-1}) + (n+1)nz^{n-1}((-W_2 + rW_1 + sW_0)G_{-n+1} - (-rW_2 + r^2W_1 + (t + rs)W_0)G_{-n} + (sW_2 - (t + rs)W_1 + (rt - s^2)W_0)G_{-n-1}) + 2W_1$

and for $z = b$ *we get*

$$\sum_{k=0}^{n} z^k W_{-k} = \frac{\Omega_4}{3z^2(-1) + 2zr + s}$$

where

$\Omega_4 = (n+3)z^{n+2}(-W_0G_{-n+1} + (rW_0 - W_1)G_{-n} + (-W_2 + rW_1 + sW_0)G_{-n-1}) + (n+2)z^{n+1}((-W_1 + rW_0)G_{-n+1} + (-W_2 + 2W_1r - W_0r^2)G_{-n} + (rW_2 - r^2W_1 - (rs + t)W_0)G_{-n-1}) + (n+1)z^n((-W_2 + rW_1 + sW_0)G_{-n+1} - (-rW_2 + r^2W_1 + (t + rs)W_0)G_{-n} + (sW_2 - (t + rs)W_1 + (rt - s^2)W_0)G_{-n-1}) + 2zW_1 + (W_2 - rW_1).$

(iv) *If* $z^3(-1)+z^2r+zs+t=u(z-a)^3=0$ *for some* $u,a\in\mathbb{C}$ *with* $u\neq 0$, *i.e.,* $z=a$, *then*

$$\sum_{k=0}^{n} z^k W_{-k} = \frac{\Omega_5}{6(-1)}$$

where

$\Omega_5=(n+3)(n+2)(n+1)z^n(-W_0G_{-n+1}+(rW_0-W_1)G_{-n}+(-W_2+rW_1+sW_0)G_{-n-1})+(n+2)(n+1)nz^{n-1}((-W_1+rW_0)G_{-n+1}+(-W_2+2W_1r-W_0r^2)G_{-n}+(rW_2-r^2W_1-(rs+t)W_0)G_{-n-1})+(n+1)n(n-1)z^{n-2}((-W_2+rW_1+sW_0)G_{-n+1}-(-rW_2+r^2W_1+(t+rs)W_0)G_{-n}+(sW_2-(t+rs)W_1+(rt-s^2)W_0)G_{-n-1})$.

(e) $(m=-2, j=0)$.

(i) *If* $z^3(-1)+z^2(2s+r^2)+z(2rt-s^2)+t^2\neq 0$ *then*

$$\sum_{k=0}^{n} z^k W_{-2k} = \frac{\Omega_1}{z^3(-1)+z^2(2s+r^2)+z(2rt-s^2)+t^2}$$

where

$\Omega_1=z^{n+3}(-1)(W_1G_{-2n}+(W_2-rW_1)G_{-2n-1}+tW_0G_{-2n-2})+z^{n+2}((-rW_2+(r^2+s)W_1-tW_0)G_{-2n}+((r^2+s)W_2-(r^3+2sr+t)W_1)G_{-2n-1}+t(-W_2+(r^2+2s)W_0)G_{-2n-2})+z^{n+1}t((-W_2+rW_1+sW_0)G_{-2n}+(rW_2-r^2W_1-(t+rs)W_0)G_{-2n-1}+(sW_2-(rs+t)W_1+(rt-s^2)W_0)G_{-2n-2})+z^2W_2+z(-sW_2+(rs+t)W_1+rtW_0)+t^2W_0$.

(ii) *If* $z^3(-1)+z^2(2s+r^2)+z(2rt-s^2)+t^2=u(z-a)(z-b)(z-c)=0$ *for some* $u,a,b,c\in\mathbb{C}$ *with* $u\neq 0$ *and* $a\neq b\neq c$, *i.e.,* $z=a$ *or* $z=b$ *or* $z=c$ *then*

$$\sum_{k=0}^{n} z^k W_{-2k} = \frac{\Omega_2}{3z^2(-1)+2z(2s+r^2)+(2rt-s^2)}$$

where

$\Omega_2=(n+3)z^{n+2}(-1)(W_1G_{-2n}+(W_2-rW_1)G_{-2n-1}+tW_0G_{-2n-2})+(n+2)z^{n+1}((-rW_2+(r^2+s)W_1-tW_0)G_{-2n}+((r^2+s)W_2-(r^3+2sr+t)W_1)G_{-2n-1}+t(-W_2+(r^2+2s)W_0)G_{-2n-2})+(n+1)z^nt((-W_2+rW_1+sW_0)G_{-2n}+(rW_2-r^2W_1-(t+rs)W_0)G_{-2n-1}+(sW_2-(rs+t)W_1+(rt-s^2)W_0)G_{-2n-2})+2zW_2+(-sW_2+(rs+t)W_1+rtW_0)$.

(iii) *If* $z^3(-1)+z^2(2s+r^2)+z(2rt-s^2)+t^2=u(z-a)^2(z-b)=0$ *for some* $u,a,b\in\mathbb{C}$ *with* $u\neq 0$ *and* $a\neq b$*, i.e.,* $z=a$ *or* $z=b$ *then for* $z=a$ *we get*

$$\sum_{k=0}^{n} z^k W_{-2k}=\frac{\Omega_3}{6z(-1)+2(2s+r^2)}$$

where

$\Omega_3=(n+3)(n+2)z^{n+1}(-1)(W_1G_{-2n}+(W_2-rW_1)G_{-2n-1}+tW_0G_{-2n-2})+(n+2)(n+1)z^n((-rW_2+(r^2+s)W_1-tW_0)G_{-2n}+((r^2+s)W_2-(r^3+2sr+t)W_1)G_{-2n-1}+t(-W_2+(r^2+2s)W_0)G_{-2n-2})+(n+1)nz^{n-1}t((-W_2+rW_1+sW_0)G_{-2n}+(rW_2-r^2W_1-(t+rs)W_0)G_{-2n-1}+(sW_2-(rs+t)W_1+(rt-s^2)W_0)G_{-2n-2})+2W_2$

and for $z=b$ *we get*

$$\sum_{k=0}^{n} z^k W_{-2k}=\frac{\Omega_4}{3z^2(-1)+2z(2s+r^2)+(2rt-s^2)}$$

where

$\Omega_4=(n+3)z^{n+2}(-1)(W_1G_{-2n}+(W_2-rW_1)G_{-2n-1}+tW_0G_{-2n-2})+(n+2)z^{n+1}((-rW_2+(r^2+s)W_1-tW_0)G_{-2n}+((r^2+s)W_2-(r^3+2sr+t)W_1)G_{-2n-1}+t(-W_2+(r^2+2s)W_0)G_{-2n-2})+(n+1)z^nt((-W_2+rW_1+sW_0)G_{-2n}+(rW_2-r^2W_1-(t+rs)W_0)G_{-2n-1}+(sW_2-(rs+t)W_1+(rt-s^2)W_0)G_{-2n-2})+2zW_2+(-sW_2+(rs+t)W_1+rtW_0).$

(iv) *If* $z^3(-1)+z^2(2s+r^2)+z(2rt-s^2)+t^2=u(z-a)^3=0$ *for some* $u,a\in\mathbb{C}$ *with* $u\neq 0$*, i.e.,* $z=a$*, then*

$$\sum_{k=0}^{n} z^k W_{-2k}=\frac{\Omega_5}{6(-1)}$$

where

$\Omega_5=(n+3)(n+2)(n+1)z^n(-1)(W_1G_{-2n}+(W_2-rW_1)G_{-2n-1}+tW_0G_{-2n-2})+(n+2)(n+1)nz^{n-1}((-rW_2+(r^2+s)W_1-tW_0)G_{-2n}+((r^2+s)W_2-(r^3+2sr+t)W_1)G_{-2n-1}+t(-W_2+(r^2+2s)W_0)G_{-2n-2})+(n+1)n(n-1)z^{n-2}t((-W_2+rW_1+sW_0)G_{-2n}+(rW_2-r^2W_1-(t+rs)W_0)G_{-2n-1}+(sW_2-(rs+t)W_1+(rt-s^2)W_0)G_{-2n-2}).$

(f) $(m=-2,j=1)$.

(i) *If* $z^3(-1)+z^2(2s+r^2)+z(2rt-s^2)+t^2\neq 0$ *then*

$$\sum_{k=0}^{n} z^k W_{-2k+1}=\frac{\Omega_1}{z^3(-1)+z^2(2s+r^2)+z(2rt-s^2)+t^2}$$

where

$\Omega_1=z^{n+3}(-1)(W_2G_{-2n}+(sW_1+tW_0)G_{-2n-1}+tW_1G_{-2n-2})+z^{n+2}((sW_2-(rs+t)W_1-rtW_0)G_{-2n}+(-(rs+t)W_2+s(r^2+s)W_1+t(r^2+s)W_0)G_{-2n-1}+t(-W_2r+(r^2+s)W_1-tW_0)G_{-2n-2})+z^{n+1}(-1)t^2(W_0G_{-2n}+(W_1-rW_0)G_{-2n-1}-(-W_2+rW_1+sW_0)G_{-2n-2})+z^2(rW_2+sW_1+tW_0)+z(tW_2+rtW_1-s^2W_1-tsW_0)+t^2W_1.$

(ii) *If* $z^3(-1)+z^2(2s+r^2)+z(2rt-s^2)+t^2=u(z-a)(z-b)(z-c)=0$ *for some* $u,a,b,c\in\mathbb{C}$ *with* $u\neq 0$ *and* $a\neq b\neq c$*, i.e.,* $z=a$ *or* $z=b$ *or* $z=c$ *then*

$$\sum_{k=0}^{n} z^k W_{-2k+1}=\frac{\Omega_2}{3z^2(-1)+2z(2s+r^2)+(2rt-s^2)}$$

where $\Omega_2=(n+3)z^{n+2}(-1)(W_2G_{-2n}+(sW_1+tW_0)G_{-2n-1}+tW_1G_{-2n-2})+(n+2)z^{n+1}((sW_2-(rs+t)W_1-rtW_0)G_{-2n}+(-(rs+t)W_2+s(r^2+s)W_1+t(r^2+s)W_0)G_{-2n-1}+t(-W_2r+(r^2+s)W_1-tW_0)G_{-2n-2})+(n+1)z^n(-1)t^2(W_0G_{-2n}+(W_1-rW_0)G_{-2n-1}-(-W_2+rW_1+sW_0)G_{-2n-2})+2z(rW_2+sW_1+tW_0)+(tW_2+rtW_1-s^2W_1-tsW_0).$

(iii) *If* $z^3(-1)+z^2(2s+r^2)+z(2rt-s^2)+t^2=u(z-a)^2(z-b)=0$ *for some* $u,a,b\in\mathbb{C}$ *with* $u\neq 0$ *and* $a\neq b$*, i.e.,* $z=a$ *or* $z=b$ *then for* $z=a$ *we get*

$$\sum_{k=0}^{n} z^k W_{-2k+1}=\frac{\Omega_3}{6z(-1)+2(2s+r^2)}$$

where

$\Omega_3=(n+3)(n+2)z^{n+1}(-1)(W_2G_{-2n}+(sW_1+tW_0)G_{-2n-1}+tW_1G_{-2n-2})+(n+2)(n+1)z^n((sW_2-(rs+t)W_1-rtW_0)G_{-2n}+(-(rs+t)W_2+s(r^2+s)W_1+t(r^2+s)W_0)G_{-2n-1}+t(-W_2r+(r^2+s)W_1-tW_0)G_{-2n-2})+(n+1)nz^{n-1}(-1)t^2(W_0G_{-2n}+(W_1-rW_0)G_{-2n-1}-(-W_2+rW_1+sW_0)G_{-2n-2})+2(rW_2+sW_1+tW_0)$

and for $z=b$ *we get*

$$\sum_{k=0}^{n} z^k W_{-2k+1}=\frac{\Omega_4}{3z^2(-1)+2z(2s+r^2)+(2rt-s^2)}$$

where

$\Omega_4 = (n+3)z^{n+2}(-1)(W_2G_{-2n} + (sW_1 + tW_0)G_{-2n-1} + tW_1G_{-2n-2}) + (n+2)z^{n+1}((sW_2 - (rs+t)W_1 - rtW_0)G_{-2n} + (-(rs+t)W_2 + s(r^2+s)W_1 + t(r^2+s)W_0)G_{-2n-1} + t(-W_2r + (r^2+s)W_1 - tW_0)G_{-2n-2}) + (n+1)z^n(-1)t^2(W_0G_{-2n} + (W_1 - rW_0)G_{-2n-1} - (-W_2 + rW_1 + sW_0)G_{-2n-2}) + 2z(rW_2 + sW_1 + tW_0) + (tW_2 + rtW_1 - s^2W_1 - tsW_0)$.

(iv) *If* $z^3(-1) + z^2(2s+r^2) + z(2rt - s^2) + t^2 = u(z-a)^3 = 0$ *for some* $u, a \in \mathbb{C}$ *with* $u \neq 0$*, i.e.,* $z = a$*, then*

$$\sum_{k=0}^{n} z^k W_{-2k+1} = \frac{\Omega_5}{6(-1)}$$

where

$\Omega_5 = (n+3)(n+2)(n+1)z^n(-1)(W_2G_{-2n} + (sW_1 + tW_0)G_{-2n-1} + tW_1G_{-2n-2}) + (n+2)(n+1)nz^{n-1}((sW_2 - (rs+t)W_1 - rtW_0)G_{-2n} + (-(rs+t)W_2 + s(r^2+s)W_1 + t(r^2+s)W_0)G_{-2n-1} + t(-W_2r + (r^2+s)W_1 - tW_0)G_{-2n-2}) + (n+1)n(n-1)z^{n-2}(-1)t^2(W_0G_{-2n} + (W_1 - rW_0)G_{-2n-1} - (-W_2 + rW_1 + sW_0)G_{-2n-2})$.

10.3 The Sum Formula $\sum_{k=0}^{n} z^k W_{mk+j}$ of Generalized Tribonacci Polynomials in Terms of Generalized Tribonacci Polynomials and (r,s,t)-Tribonacci-Lucas Polynomials

By using Corollary 52 (b), we can give the sum formula $\sum_{k=0}^{n} z^k W_{mk+j}$ of generalized Tribonacci polynomials via matrix methods (in terms of elements of the sequence of generalized Tribonacci polynomials and (r,s,t)-Tribonacci-Lucas polynomials).

Theorem 65. *For all integers* m *and* j*, we have the followings sum formulas.*

(a) *If* $z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4 \neq 0$ *then*

$$\sum_{k=0}^{n} z^k W_{mk+j} = \frac{z^{n+3}\Theta_1 + z^{n+2}\Theta_2 + z^{n+1}\Theta_3 + z^2\Theta_4 + z\Theta_5 + \Theta_6}{z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4} = \frac{\Theta_W(z)}{\Gamma_W(z)} \quad (10.4)$$

where

$\Theta_W(z) = z^{n+3}\Theta_1 + z^{n+2}\Theta_2 + z^{n+1}\Theta_3 + z^2\Theta_4 + z\Theta_5 + \Theta_6,$

$z^{n+3}\Theta_1 = z^{n+3}((-W_jH_{m+2}^2 + (-W_{j+2} + sW_j)H_{m+1}^2 - tW_{j+1}H_m^2 + (W_{j+1} + rW_j)H_{m+2}H_{m+1} + (W_{j+2} - rW_{j+1})H_{m+2}H_m + (-sW_{j+1} + tW_j)H_{m+1}H_m) H_{m+mn+2} + ((-W_{j+1} + rW_j)H_{m+2}^2 - (t + rs)W_jH_{m+1}^2 + t(-W_{j+2} + rW_{j+1})H_m^2 + (W_{j+2} - r^2W_j)H_{m+2}H_{m+1} + (-rW_{j+2} + (r^2 + s)W_{j+1} + tW_j)H_{m+2}H_m + (-sW_{j+2} + (rs + t)W_{j+1} - rtW_j)H_{m+1}H_m)H_{m+mn+1} + ((-W_{j+2} + rW_{j+1} + sW_j)H_{m+2}^2 + (sW_{j+2} - (rs + t)W_{j+1} - s^2W_j)H_{m+1}^2 - W_jt^2H_m^2 + (rW_{j+2} - r^2W_{j+1} + (t - rs)W_j)H_{m+2}H_{m+1} + t(W_{j+1} - rW_j)H_{m+2}H_m + t(W_{j+2} - rW_{j+1} - 2sW_j)H_{m+1}H_m) H_{m+mn}),$

$z^{n+2}\Theta_2 = z^{n+2}(((-3W_{j+2} + 2rW_{j+1} + (r^2 + 4s)W_j)H_{m+2} + (2rW_{j+2} + (s - r^2)W_{j+1} - (r^3 + 4rs + 3t)W_j)H_{m+1} + (-(r^2 + 2s)W_{j+2} + (r^3 + 3rs + 6t)W_{j+1} - rtW_j) H_m)H_{m+mn+2} + ((2rW_{j+2} - (r^2 - s)W_{j+1} - (r^3 + 3t + 4rs)W_j)H_{m+2} + (-(r^2 - s)W_{j+2} - 3(t+rs)W_{j+1} + r(r^3 + 4rs + 5t)W_j)H_{m+1} + ((r^3 + 3rs + 6t)W_{j+2} - (r^4 + 4r^2s + 2s^2 + 7rt)W_{j+1} - 2stW_j)H_m)H_{m+mn+1} + (((r^2 + 4s)W_{j+2} - (r^3 + 4rs + 3t)W_{j+1} - (r^2s - 2rt + 4s^2)W_j)H_{m+2} + (-(r^3 + 4rs + 3t)W_{j+2} + (r^4 + 4r^2s + 5rt)W_{j+1} + (r^3s - r^2t + 4rs^2 + 4st)W_j)H_{m+1} + t(-rW_{j+2} - 2sW_{j+1} + (r^3 + 4rs + 6t)W_j)H_m)H_{m+mn}),$

$z^{n+1}\Theta_3 = z^{n+1}((2(3s + r^2)W_{j+2} - (2r^3 + 7rs + 9t)W_{j+1} - (r^2s - 3rt + 4s^2)W_j) H_{m+mn+2} + (-(2r^3 + 9t + 7rs)W_{j+2} + 2(r^4 + 4r^2s + 6rt + s^2)W_{j+1} + (r^3s - r^2t + 4rs^2 + 6st)W_j)H_{m+mn+1} + (-(r^2s - 3rt + 4s^2)W_{j+2} + (r^3s - r^2t + 4rs^2 + 6st)W_{j+1} + (-2r^3t + r^2s^2 + 4s^3 - 9t^2 - 10rst)W_j)H_{m+mn}),$

$z^2\Theta_4 = z^2((3W_{j+2} - 2rW_{j+1} - sW_j)H_{m+2}^2 + ((r^2 - s)W_{j+2} + 3(rs + t)W_{j+1} + (s^2 + rt)W_j)H_{m+1}^2 + t(rW_{j+2} + 2sW_{j+1} + 3tW_j)H_m^2 + (-4rW_{j+2} + 2(r^2 - s)W_{j+1} + (rs - 3t)W_j)H_{m+2}H_{m+1} + (-2sW_{j+2} + (rs - 3t)W_{j+1} + 2rtW_j)H_{m+2}H_m + ((rs - 3t)W_{j+2} + 2(s^2 + rt)W_{j+1} + 4stW_j)H_{m+1}H_m),$

$z\Theta_5 = z((-2(r^2 + 3s)W_{j+2} + (2r^3 + 7rs + 9t)W_{j+1} + (r^2s - 3rt + 4s^2)W_j)H_{m+2} + ((2r^3 + 7rs + 9t)W_{j+2} - 2(r^4 + 4r^2s + 6rt + s^2)W_{j+1} - (r^3s + 4rs^2 - r^2t + 6st)W_j) H_{m+1} + ((r^2s + 4s^2 - 3rt)W_{j+2} - (4rs^2 + r^3s - r^2t + 6st)W_{j+1} - 2t(r^3 + 4rs + 9t)W_j) H_m),$

$\Theta_6 = (4r^3t - r^2s^2 - 4s^3 + 27t^2 + 18rst)W_j,$

and

$\Gamma_W(z) = \Gamma_1(z) + \Gamma_2(z) + \Gamma_3(z) + \Gamma_4(z),$

$z^3\Gamma_1 = z^3(-t^m(4r^3t - r^2s^2 + 18rst - 4s^3 + 27t^2)),$

$z^2\Gamma_2 = z^2((r^2 + 3s)H_{m+2}^2 + (r^4 + 4r^2s + s^2 + 6rt)H_{m+1}^2 + t(r^3 + 4rs + 9t)H_m^2 -$

$(2r^3 + 7rs + 9t)H_{m+2}H_{m+1} - (r^2s + 4s^2 - 3rt)H_{m+2}H_m + rs(r^2 + 4s)H_{m+1}H_m - t(r^2 - 6s)H_{m+1}H_m),$

$z\Gamma_3 = z(-4r^3t + r^2s^2 + 4s^3 - 27t^2 - 18rst)H_m,$

$\Gamma_4 = 4r^3t - r^2s^2 + 18rst - 4s^3 + 27t^2.$

(b) *If* $z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4 = u(z-a)(z-b)(z-c) = 0$ *for some* $u, a, b, c \in \mathbb{C}$ *with* $u \neq 0$ *and* $a \neq b \neq c$, *i.e.,* $z = a$ *or* $z = b$ *or* $z = c$ *then*

$$\sum_{k=0}^{n} z^k W_{mk+j} = \frac{(n+3)z^{n+2}\Theta_1 + (n+2)z^{n+1}\Theta_2 + (n+1)z^n\Theta_3 + 2z\Theta_4 + \Theta_5}{3z^2\Gamma_1 + 2z\Gamma_2 + \Gamma_3}.$$

(c) *If* $z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4 = u(z-a)^2(z-b) = 0$ *for some* $u, a, b \in \mathbb{C}$ *with* $u \neq 0$ *and* $a \neq b$, *i.e.,* $z = a$ *or* $z = b$ *then for* $z = a$ *we get*

$$\sum_{k=0}^{n} z^k W_{mk+j} = \frac{(n+3)(n+2)z^{n+1}\Theta_1 + (n+2)(n+1)z^n\Theta_2 + (n+1)nz^{n-1}\Theta_3 + 2\Theta_4}{6z\Gamma_1 + 2\Gamma_2},$$

and for $z = b$ *we get*

$$\sum_{k=0}^{n} z^k W_{mk+j} = \frac{(n+3)z^{n+2}\Theta_1 + (n+2)z^{n+1}\Theta_2 + (n+1)z^n\Theta_3 + 2z\Theta_4 + \Theta_5}{3z^2\Gamma_1 + 2z\Gamma_2 + \Gamma_3}.$$

(d) *If* $z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4 = u(z-a)^3 = 0$ *for some* $u, a \in \mathbb{C}$ *with* $u \neq 0$, *i.e.,* $z = a$, *then*

$$\sum_{k=0}^{n} z^k W_{mk+j} = \frac{(n+3)(n+2)(n+1)z^n\Theta_1 + (n+2)(n+1)nz^{n-1}\Theta_2 + (n+1)n(n-1)z^{n-2}\Theta_3}{6\Gamma_1}.$$

Proof. We only prove (a). The proof of (b), (c) and (d) are as in Theorem 61 (b), (c) and (d), respectively.

Proof of (a). We use the same method as in Theorem 61 by setting

$$\begin{aligned} M &= zA^m = z\begin{pmatrix} G_{m+1} & sG_m + tG_{m-1} & tG_m \\ G_m & sG_{m-1} + tG_{m-2} & tG_{m-1} \\ G_{m-1} & sG_{m-2} + tG_{m-3} & tG_{m-2} \end{pmatrix} \\ &= z \times \frac{1}{4r^3t - r^2s^2 - 4s^3 + 27t^2 + 18rst}\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \end{aligned}$$

in (10.1), where $b_{11}, b_{21}, b_{31}, b_{12}, b_{22}, b_{32}, b_{13}, b_{23}, b_{33}$ are as in Corollary 52 (b) (by replacing n with m in $b_{11}, b_{21}, b_{31}, b_{12}, b_{22}, b_{32}, b_{13}, b_{23}, b_{33}$). Also, use the identities

$$\begin{aligned}
H_{m+3} &= rH_{m+2} + sH_{m+1} + tH_m \\
H_{m-1} &= \frac{1}{t}(H_{m+2} - rH_{m+1} - sH_m) \\
H_{m-2} &= \frac{1}{t^2}(-sH_{m+2} + (rs+t)H_{m+1} + (s^2 - rt)H_m) \\
H_{m-3} &= \frac{1}{t^3}((s^2 - rt)H_{m+2} + (r^2t - rs^2 - st)H_{m+1} + (-s^3 + t^2 + 2rst)H_m)
\end{aligned}$$

in the formula of A^m and as well as in $b_{11}, b_{21}, b_{31}, b_{12}, b_{22}, b_{32}, b_{13}, b_{23}, b_{33}$.

After some calculations, we see that

$$\begin{aligned}
\det(I - zA^m) &= \frac{1}{4r^3t - r^2s^2 + 18rst - 4s^3 + 27t^2}\Gamma_W(z) \\
&= \frac{1}{4r^3t - r^2s^2 + 18rst - 4s^3 + 27t^2}(z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4)
\end{aligned}$$

and the 3rd row and 1st column entry of matrix $(I - z^{n+1}A^{mn+m})Adj(I - zA^m)f_W(j)$ is equal to

$$\frac{1}{4r^3t - r^2s^2 + 18rst - 4s^3 + 27t^2}(z^{n+3}\Theta_1 + z^{n+2}\Theta_2 + z^{n+1}\Theta_3 + z^2\Theta_4 + z\Theta_5 + \Theta_6)$$

where $\Gamma_W(z) = z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4, z^3\Gamma_1, z^2\Gamma_2, z\Gamma_3, \Gamma_4,$ and $\Theta_W(z) = z^{n+3}\Theta_1 + z^{n+2}\Theta_2 + z^{n+1}\Theta_3 + z^2\Theta_4 + z\Theta_5 + \Theta_6, z^{n+3}\Theta_1, z^{n+2}\Theta_2, z^{n+1}\Theta_3, z^2\Theta_4, z\Theta_5, \Theta_6,$ are as in the statement of (a) of Theorem.

Note that, since $\det(I - zA^m)(\sum_{k=0}^{n} z^k f_W(mk + j) = (I - z^{n+1}A^{mn+m})Adj(I - zA^m)f_W(j)$, i.e., matrices over the either side is equal, the 3rd row and 1st column entry of matrix of matrix $\det(I - zA^m)(\sum_{k=0}^{n} z^k f_W(mk + j))$ is equal to the 3rd row and 1st column entry of matrix $(I - z^{n+1}A^{mn+m})Adj(I - zA^m)f_W(j)$. So, to complete the proof, we will just compare the linear combination of the 3rd row and 1st column entries of the matrices. Then, we get

$$\begin{aligned}
& \frac{1}{4r^3t - r^2s^2 + 18rst - 4s^3 + 27t^2}(z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4)\sum_{k=0}^{n} z^k W_{mk+j} \\
= \; & \frac{1}{4r^3t - r^2s^2 + 18rst - 4s^3 + 27t^2}(z^{n+3}\Theta_1 + z^{n+2}\Theta_2 + z^{n+1}\Theta_3 + z^2\Theta_4 + z\Theta_5 + \Theta_6)
\end{aligned}$$

and so

$$\sum_{k=0}^{n} z^k W_{mk+j} = \frac{z^{n+3}\Theta_1 + z^{n+2}\Theta_2 + z^{n+1}\Theta_3 + z^2\Theta_4 + z\Theta_5 + \Theta_6}{z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4} = \frac{\Theta_W(z)}{\Gamma_W(z)}.$$

□

11 Generating Function of Generalized Tribonacci Polynomials

In this section, we present generating function of the sequence W_{mn+j} and its special cases.

11.1 Generating Function of Generalized Tribonacci Polynomials via Generalized Tribonacci Polynomials

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} W_{mn+j} z^n$ of the sequence W_{mn+j} (in terms of elements of the sequence of generalized Tribonacci polynomials).

Lemma 66. *Assume that* $|z| < \min\{|\alpha|^{-m}, |\beta|^{-m}, |\gamma|^{-m}\}$. *Suppose that* $f_{W_{mn+j}}(z) = \sum_{n=0}^{\infty} W_{mn+j} z^n$ *is the ordinary generating function of the generalized Tribonacci (sequence of) polynomials* $\{W_{mn+j}\}$. *Then,* $\sum_{n=0}^{\infty} W_{mn+j} z^n$ *is given by*

$$\sum_{n=0}^{\infty} W_{mn+j} z^n = \frac{z^2\Theta_4 + z\Theta_5 + \Theta_6}{z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4}$$

where (as in Theorem 61 (a))

$z^2\Theta_4 = z^2((W_0W_{j+2} + (W_1 - rW_0)W_{j+1} + (W_2 - rW_1 - sW_0)W_j)W_{m+2}^2 + ((W_2 - sW_0)W_{j+2} + (tW_0 + rsW_0)W_{j+1} + (s^2W_0 + (rs+t)W_1 - sW_2)W_j)W_{m+1}^2 + t(W_1W_{j+2} + (W_2 - rW_1)W_{j+1} + tW_0W_j)W_m^2 + (-(W_1 + rW_0)W_{j+2} + (r^2W_0 - W_2)W_{j+1} + (-rW_2 + r^2W_1 + (rs-t)W_0)W_j)W_{m+1}W_{m+2} + ((rW_1 - W_2)W_{j+2} + (rW_2 - (s+r^2)W_1 - tW_0)W_{j+1} + t(rW_0 - W_1)W_j)W_{m+2}W_m + ((sW_1 - tW_0)W_{j+2} + (sW_2 - (rs+t)W_1 + rtW_0)W_{j+1} + t(-W_2 + rW_1 + 2sW_0)W_j)W_{m+1}W_m)$,

$z\Theta_5 = z(((W_1^2 - W_0W_2)W_{j+2} + (tW_0^2 - W_1W_2 + rW_0W_2 + sW_0W_1)W_{j+1} + (-2W_2^2 - r^2W_1^2 - trW_0^2 + 3rW_1W_2 + 2sW_0W_2 + (2t - sr)W_0W_1)W_j)W_{m+2} + ((tW_0^2 - W_1W_2 +$

$rW_0W_2 + sW_0W_1)W_{j+2} + (W_2^2 - (r^2+s)W_0W_2 - (rs+t)W_0W_1)W_{j+1} + (rW_2^2 - 2(t + rs)W_1^2 - 2stW_0^2 + (2s-r^2)W_1W_2 + (2t-rs)W_0W_2 - (2s^2+rt)W_0W_1)W_j)W_{m+1} + ((W_2^2 - sW_1^2 - rW_1W_2 - tW_0W_1)W_{j+2} + (-rW_2^2 + (rs+t)W_1^2 + r^2W_1W_2 - tW_0W_2 + rtW_0W_1) W_{j+1} + t(-rW_1^2 - 2tW_0^2 + 2W_1W_2 - rW_0W_2 - 2sW_0W_1)W_j)W_m - rtW_0^2W_{j+1}W_{m+1}),$

$\Theta_6 = (W_2^3 + (t+rs)W_1^3 + t^2W_0^3 - 2rW_1W_2^2 - sW_0W_2^2 + (r^2-s)W_1^2W_2 + (s^2+rt) W_0W_1^2 + 2stW_0^2W_1 + rtW_0^2W_2 + (rs-3t)W_0W_1W_2)W_j,$

and

$z^3\Gamma_1 = z^3(-t^m(W_2^3 + (t+rs)W_1^3 + t^2W_0^3 + (r^2-s)W_1^2W_2 - 2rW_1W_2^2 - sW_0W_2^2 + rtW_0^2W_2 + (s^2+rt)W_0W_1^2 + 2stW_0^2W_1 + (rs-3t)W_0W_1W_2)),$

$z^2\Gamma_2 = z^2((3W_2 - 2rW_1 - sW_0)W_{m+2}^2 + ((r^2-s)W_2 + (3rs+3t)W_1 + (s^2+rt)W_0) W_{m+1}^2 + t(rW_2 + 2sW_1 + 3tW_0)W_m^2 + (-4rW_2 + 2(r^2-s)W_1 + (rs-3t)W_0)W_{m+2}W_{m+1} + (-2sW_2 + (rs-3t)W_1 + 2rtW_0)W_{m+2}W_m + ((rs-3t)W_2 + 2(s^2+rt)W_1 + 4stW_0)W_{m+1}W_m),$

$z\Gamma_3 = z((-3W_2^2 + (s-r^2)W_1^2 - trW_0^2 + 4rW_1W_2 + 2sW_0W_2 + (3t-sr)W_0W_1)W_{m+2} + (2rW_2^2 - (3rs+3t)W_1^2 - 2stW_0^2 + (2s-2r^2)W_1W_2 + (3t-rs)W_0W_2 - 2(s^2+rt)W_0W_1) W_{m+1} + (sW_2^2 - (s^2+rt)W_1^2 - 3t^2W_0^2 + (3t-rs)W_1W_2 - 2rtW_0W_2 - 4stW_0W_1)W_m),$

$\Gamma_4 = W_2^3 + (t+rs)W_1^3 + t^2W_0^3 - 2rW_1W_2^2 + (r^2-s)W_1^2W_2 - sW_0W_2^2 + (s^2+rt)W_0W_1^2 + rtW_0^2W_2 + 2stW_0^2W_1 + (rs-3t)W_0W_1W_2.$

Proof. Proof. Use Theorem 61 (a) and Theorem 6. □

Now, we consider special cases of Lemma 66.

Corollary 67. *The ordinary generating functions of the sequences* W_n, W_{2n}, W_{2n+1}, W_{-n}, W_{-2n}, W_{-2n+1} *are given as follows:*

(a) $(m = 1, j = 0, |z| < \min\{|\alpha|^{-1}, |\beta|^{-1}, |\gamma|^{-1}\})$.

$$\sum_{n=0}^{\infty} W_n z^n = \frac{z^2(W_2 - rW_1 - sW_0) + z(W_1 - rW_0) + W_0}{z^3(-t) + z^2(-1)s + z(-1)r + 1}.$$

(b) $(m = 2, j = 0, |z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}\})$.

$$\sum_{n=0}^{\infty} W_{2n} z^n = \frac{z^2(-sW_2 + (t+rs)W_1 + (s^2-rt)W_0) + z(W_2 - (r^2+2s)W_0) + W_0}{z^3(-t^2) + z^2(-2rt + s^2) + z(-1)(2s + r^2) + 1}.$$

(c) $(m = 2, j = 1, |z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}\})$.

$$\sum_{n=0}^{\infty} W_{2n+1} z^n = \frac{z^2t(W_2 - rW_1 - sW_0) + z(rW_2 - (r^2+s)W_1 + tW_0) + W_1}{z^3(-t^2) + z^2(-2rt + s^2) + z(-1)(2s + r^2) + 1}.$$

(d) $(m=-1, j=0, |z|<\min\{|\alpha|, |\beta|, |\gamma|\})$.

$$\sum_{n=0}^{\infty} W_{-n}z^n = \frac{z^2W_1 + z(W_2 - rW_1) + tW_0}{z^3(-1) + z^2r + zs + t}.$$

(e) $(m=-2, j=0, |z|<\min\{|\alpha|^2, |\beta|^2, |\gamma|^2\})$.

$$\sum_{n=0}^{\infty} W_{-2n}z^n = \frac{z^2W_2 + z(-sW_2 + (t+rs)W_1 + rtW_0) + t^2W_0}{z^3(-1) + z^2(r^2+2s) + z(2rt - s^2) + t^2}.$$

(f) $(m=-2, j=1, |z|<\min\{|\alpha|^2, |\beta|^2, |\gamma|^2\})$.

$$\sum_{n=0}^{\infty} W_{-2n+1}z^n = \frac{z^2(rW_2 + sW_1 + tW_0) + z(tW_2 + (rt - s^2)W_1 - stW_0) + t^2W_1}{z^3(-1) + z^2(r^2+2s) + z(2rt - s^2) + t^2}.$$

Proof. Proof. Use Lemma 66 (or Theorem 62). □

Lemma 66 gives the following results as particular examples (generating functions of (r,s,t)-Tribonacci and (r,s,t)-Tribonacci-Lucas polynomials).

Corollary 68. *Assume that* $|z| < \min\{|\alpha|^{-m}, |\beta|^{-m}, |\gamma|^{-m}\}$. *Generating functions of* (r,s,t)*-Tribonacci and* (r,s,t)*-Tribonacci-Lucas polynomials are given, respectively, as follows:*

(a)

$$\sum_{n=0}^{\infty} G_{mn+j}z^n = \frac{z^2\Theta_4 + z\Theta_5 + \Theta_6}{z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4}$$

where (as Theorem 61 (a))

$z^2\Theta_4 = z^2(G_{m+2}^2G_{j+1} + rG_{m+1}^2G_{j+2} + tG_{m+1}^2G_j + tG_m^2G_{j+2} - G_{m+1}G_{m+2}G_{j+2} + sG_mG_{m+1}G_{j+2} - rG_{m+1}G_{m+2}G_{j+1} - sG_mG_{m+2}G_{j+1} - tG_mG_{m+1}G_{j+1} - tG_mG_{m+2}G_j)$,

$z\Theta_5 = z(G_{m+2}G_{j+2} - rG_{m+1}G_{j+2} - rG_{m+2}G_{j+1} - sG_mG_{j+2} + r^2G_{m+1}G_{j+1} - 2tG_jG_{m+1} + (t+rs)G_mG_{j+1} + rtG_mG_j)$,

$\Theta_6 = tG_j$,

and

$z^3\Gamma_1 = z^3(-t^{m+1})$,

$z^2\Gamma_2 = z^2(rG_{m+2}^2 + (r^3+2rs+3t)G_{m+1}^2 + t(r^2+2s)G_m^2 - 2(r^2+s)G_{m+2}G_{m+1} - (3t+rs)G_{m+2}G_m + (r^2s+2s^2-rt)G_{m+1}G_m)$,

$z\Gamma_3 = z(sG_{m+2} - (rs+3t)G_{m+1} - (s^2-2rt)G_m)$,

$\Gamma_4 = t$.

(b)

$$\sum_{n=0}^{\infty} H_{mn+j} z^n = \frac{z^2\Theta_4 + z\Theta_5 + \Theta_6}{z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4}$$

where (as Theorem 61 (a))

$z^2\Theta_4 = z^2((3H_{j+2}-2rH_{j+1}-sH_j)H_{m+2}^2 + ((r^2-s)H_{j+2}+3(rs+t)H_{j+1}+(s^2+rt)H_j) H_{m+1}^2 + t(rH_{j+2}+2sH_{j+1}+3tH_j)H_m^2 + (-4rH_{j+2}+2(r^2-s)H_{j+1}+(rs-3t)H_j) H_{m+2}H_{m+1} + (-2sH_{j+2}+(rs-3t)H_{j+1}+2rtH_j)H_{m+2}H_m + ((rs-3t)H_{j+2}+2(s^2+rt)H_{j+1}+4stH_j)H_{m+1}H_m)$,

$z\Theta_5 = z((-2(r^2+3s)H_{j+2} + (2r^3+7rs+9t)H_{j+1} + (r^2s+4s^2-3rt)H_j)H_{m+2} + +((2r^3+7rs+9t)H_{j+2} - (2r^4+8r^2s+3rt+2s^2)H_{j+1} - (r^3s+4rs^2-r^2t+6st) H_j)H_{m+1} + ((r^2s-3rt+4s^2)H_{j+2} - (r^3s+4rs^2-r^2t+6st)H_{j+1} - 2t(r^3+4rs+9t) H_j)H_m - 9rtH_{j+1}H_{m+1})$,

$\Theta_6 = (4r^3t - r^2s^2 - 4s^3 + 27t^2 + 18rst)H_j$,

and

$z^3\Gamma_1 = z^3(-t^m(4r^3t - r^2s^2 + 18rst - 4s^3 + 27t^2))$,

$z^2\Gamma_2 = z^2((r^2+3s)H_{m+2}^2 + (r^4+4r^2s+s^2+6rt)H_{m+1}^2 + t(r^3+4rs+9t)H_m^2 - (2r^3+7rs+9t)H_{m+2}H_{m+1} - (r^2s-3rt+4s^2)H_{m+2}H_m + (r^3s+4rs^2-r^2t+6st)H_{m+1}H_m)$,

$z\Gamma_3 = z(-4r^3t + r^2s^2 + 4s^3 - 27t^2 - 18rst)H_m$,

$\Gamma_4 = 4r^3t - r^2s^2 - 4s^3 + 27t^2 + 18rst$.

Proof. In Lemma 66, take $W_n = G_n$ with $G_0 = 0, G_1 = 1, G_2 = r$ and $W_n = H_n$ with $H_0 = 3, H_1 = r, H_2 = 2s + r^2$, respectively. □

Now, we consider special cases of Corollay 67 (or Corollary 68). (Note that we have already presented the case $m = 1, j = 0$ in Corollary 11)

Corollary 69. *The ordinary generating functions of the sequences G_n, G_{2n}, G_{2n+1}, G_{-n}, G_{-2n}, G_{-2n+1} and H_n, H_{2n}, H_{2n+1}, H_{-n}, H_{-2n}, H_{-2n+1} are given as follows:*

(a) $(m=1, j=0, |z| < \min\{|\alpha|^{-1}, |\beta|^{-1}, |\gamma|^{-1}\})$.

$$\begin{aligned}\sum_{n=0}^{\infty} G_n z^n &= \frac{z}{1-rz-sz^2-tz^3},\\ \sum_{n=0}^{\infty} H_n z^n &= \frac{3-2rz-sz^2}{1-rz-sz^2-tz^3}.\end{aligned}$$

(b) $(m=2, j=0, |z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}\})$.

$$\begin{aligned}\sum_{n=0}^{\infty} G_{2n} z^n &= \frac{rz+tz^2}{1-(2s+r^2)z-(2rt-s^2)z^2-t^2z^3},\\ \sum_{n=0}^{\infty} H_{2n} z^n &= \frac{3-2\left(2s+r^2\right)z-(2rt-s^2)z^2}{1-(2s+r^2)z-(2rt-s^2)z^2-t^2z^3}.\end{aligned}$$

(c) $(m=2, j=1, |z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, |\gamma|^{-2}\})$.

$$\begin{aligned}\sum_{n=0}^{\infty} G_{2n+1} z^n &= \frac{1-sz}{1-(2s+r^2)z-(2rt-s^2)z^2-t^2z^3},\\ \sum_{n=0}^{\infty} H_{2n+1} z^n &= \frac{r+(3t+rs)z-stz^2}{1-(2s+r^2)z-(2rt-s^2)z^2-t^2z^3}.\end{aligned}$$

(d) $(m=-1, j=0, |z| < \min\{|\alpha|, |\beta|, |\gamma|\})$.

$$\begin{aligned}\sum_{n=0}^{\infty} G_{-n} z^n &= \frac{z^2}{t+sz+rz^2-z^3},\\ \sum_{n=0}^{\infty} H_{-n} z^n &= \frac{3t+2sz+rz^2}{t+sz+rz^2-z^3}.\end{aligned}$$

(e) $(m=-2, j=0, |z| < \min\{|\alpha|^{2}, |\beta|^{2}, |\gamma|^{2}\})$.

$$\begin{aligned}\sum_{n=0}^{\infty} G_{-2n} z^n &= \frac{tz+rz^2}{t^2+(2rt-s^2)z+(r^2+2s)z^2-z^3},\\ \sum_{n=0}^{\infty} H_{-2n} z^n &= \frac{3t^2+(4rt-2s^2)z+(r^2+2s)z^2}{t^2+(2rt-s^2)z+(r^2+2s)z^2-z^3}.\end{aligned}$$

(f) $(m=-2, j=1, |z| < \min\{|\alpha|^2, |\beta|^2, |\gamma|^2\})$.

$$\begin{aligned}\sum_{n=0}^{\infty} G_{-2n+1}z^n &= \frac{t^2+(2rt-s^2)z+(r^2+s)z^2}{t^2+(2rt-s^2)z+(r^2+2s)z^2-z^3},\\ \sum_{n=0}^{\infty} H_{-2n+1}z^n &= \frac{rt^2+(-rs^2+2r^2t-st)z+(r^3+3rs+3t)z^2}{t^2+(2rt-s^2)z+(r^2+2s)z^2-z^3}.\end{aligned}$$

Proof. Use Corollay 67 (or Corollary 68). □

11.2 Generating Function of Generalized Tribonacci Polynomials via Generalized Tribonacci Polynomials and (r,s,t)-Tribonacci Polynomials

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} W_{mn+j}z^n$ of the sequence W_{mn+j} (in terms of elements of the sequence of generalized Tribonacci polynomials and (r,s,t)-Tribonacci polynomials).

Lemma 70. *Assume that* $|z| < \min\{|\alpha|^{-m}, |\beta|^{-m}, |\gamma|^{-m}\}$. *Suppose that* $f_{W_{mn+j}}(z) = \sum_{n=0}^{\infty} W_{mn+j}z^n$ *is the ordinary generating function of the generalized Tribonacci (sequence of) polynomials* $\{W_{mn+j}\}$. *Then,* $\sum_{n=0}^{\infty} W_{mn+j}z^n$ *is given by*

$$\sum_{n=0}^{\infty} W_{mn+j}z^n = \frac{z^2\Theta_4+z\Theta_5+\Theta_6}{z^3\Gamma_1+z^2\Gamma_2+z\Gamma_3+\Gamma_4}$$

where (as inTheorem 63 (a))

$z^2\Theta_4 = z^2(G_{m+2}^2W_{j+1} + (rW_{j+2} + tW_j)G_{m+1}^2 + tG_m^2W_{j+2} - (rW_{j+1} + W_{j+2})G_{m+1}G_{m+2} - (sW_{j+1}+tW_j)G_mG_{m+2} + (sW_{j+2}-tW_{j+1})G_mG_{m+1})$,

$z\Theta_5 = z((W_{j+2}-rW_{j+1})G_{m+2} + (-rW_{j+2}+r^2W_{j+1}-2tW_j)G_{m+1} + (-sW_{j+2} + (tW_{j+1}+rsW_{j+1}) + rtW_j)G_m)$,

$\Theta_6 = tW_j$,

and

$z^3\Gamma_1 = z^3(-t^{m+1})$,

$z^2\Gamma_2 = z^2(rG_{m+2}^2 + (r^3+2rs+3t)G_{m+1}^2 + (r^2t+2st)G_m^2 - 2(r^2+s)G_{m+1}G_{m+2} - (rs+3t)G_mG_{m+2} + (r^2s+2s^2-rt)G_mG_{m+1})$,

$z\Gamma_3 = z(sG_{m+2} - (rs+3t)G_{m+1} + (2rt-s^2)G_m)$,

$\Gamma_4 = t$.

Proof. Use Theorem 63 (a). □

Lemma 70 gives the following result as particular example (generating functions of (r,s,t)-Tribonacci-Lucas polynomials).

Corollary 71. *Assume that* $|z| < \min\{|\alpha|^{-m}, |\beta|^{-m}, |\gamma|^{-m}\}$. *Generating function of* (r,s,t)*-Tribonacci-Lucas polynomials is given, as follows:*

$$\sum_{n=0}^{\infty} H_{mn+j} z^n = \frac{z^2\Theta_4 + z\Theta_5 + \Theta_6}{z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4}$$

where (as in Theorem 63 (a))

$z^2\Theta_4 = z^2(G_{m+2}^2 H_{j+1} + (rH_{j+2} + tH_j)G_{m+1}^2 + tG_m^2 H_{j+2} - (rH_{j+1} + H_{j+2})G_{m+1}G_{m+2} - (sH_{j+1} + tH_j)G_m G_{m+2} + (sH_{j+2} - tH_{j+1})G_m G_{m+1})$,

$z\Theta_5 = z((H_{j+2} - rH_{j+1})G_{m+2} + (-rH_{j+2} + r^2 H_{j+1} - 2tH_j)G_{m+1} + (-sH_{j+2} + (tH_{j+1} + rsH_{j+1}) + rtH_j)G_m)$,

$\Theta_6 = tH_j$,

and

$z^3\Gamma_1 = z^3(-t^{m+1})$,

$z^2\Gamma_2 = z^2(rG_{m+2}^2 + (r^3 + 2rs + 3t)G_{m+1}^2 + (r^2 t + 2st)G_m^2 - 2(r^2 + s)G_{m+1}G_{m+2} - (rs + 3t)G_m G_{m+2} + (r^2 s + 2s^2 - rt)G_m G_{m+1})$,

$z\Gamma_3 = z(sG_{m+2} - (rs + 3t)G_{m+1} + (2rt - s^2)G_m)$,

$\Gamma_4 = t$.

Proof. In Lemma 70, take $W_n = H_n$ with $H_0 = 3, H_1 = r, H_2 = 2s + r^2$. □

Note that if we take $W_n = G_n$ with $G_0 = 0, G_1 = 1, G_2 = r$ in Lemma 70, then we get Corollary 68 (a).

11.3 Generating Function of Generalized Tribonacci Polynomials via Generalized Tribonacci Polynomials and (r,s,t)-Tribonacci-Lucas Polynomials

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} W_{mn+j} z^n$ of the sequence W_{mn+j} (in terms of elements of the sequence of generalized Tribonacci polynomials and (r,s,t)-Tribonacci-Lucas polynomials).

Lemma 72. *Assume that* $|z| < \min\{|\alpha|^{-m}, |\beta|^{-m}, |\gamma|^{-m}\}$. *Suppose that* $f_{W_{mn+j}}(z) = \sum_{n=0}^{\infty} W_{mn+j} z^n$ *is the ordinary generating function of the generalized Tribonacci (sequence of) polynomials* $\{W_{mn+j}\}$. *Then,* $\sum_{n=0}^{\infty} W_{mn+j} z^n$ *is given by*

$$\sum_{n=0}^{\infty} W_{mn+j} z^n = \frac{z^2\Theta_4 + z\Theta_5 + \Theta_6}{z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4}$$

where (as in Theorem 65 (a))

$z^2\Theta_4 = z^2((3W_{j+2} - 2rW_{j+1} - sW_j)H_{m+2}^2 + ((r^2-s)W_{j+2} + 3(rs+t)W_{j+1} + (s^2+rt)W_j) H_{m+1}^2 + t(rW_{j+2} + 2sW_{j+1} + 3tW_j)H_m^2 + (-4rW_{j+2} + 2(r^2 - s)W_{j+1} + (rs - 3t)W_j) H_{m+2}H_{m+1} + (-2sW_{j+2} + (rs - 3t)W_{j+1} + 2rtW_j)H_{m+2}H_m + ((rs - 3t)W_{j+2} + 2(s^2 + rt)W_{j+1} + 4stW_j)H_{m+1}H_m)$,

$z\Theta_5 = z((-2(r^2 + 3s)W_{j+2} + (2r^3 + 7rs + 9t)W_{j+1} + (r^2s - 3rt + 4s^2)W_j)H_{m+2} + ((2r^3 + 7rs + 9t)W_{j+2} - 2(r^4 + 4r^2s + 6rt + s^2)W_{j+1} - (r^3s + 4rs^2 - r^2t + 6st)W_j)H_{m+1} + ((r^2s + 4s^2 - 3rt)W_{j+2} - (4rs^2 + r^3s - r^2t + 6st)W_{j+1} - 2t(r^3 + 4rs + 9t)W_j)H_m)$,

$\Theta_6 = (4r^3t - r^2s^2 - 4s^3 + 27t^2 + 18rst)W_j$,

and

$z^3\Gamma_1 = z^3(-t^m(4r^3t - r^2s^2 + 18rst - 4s^3 + 27t^2))$,

$z^2\Gamma_2 = z^2((r^2+3s)H_{m+2}^2 + (r^4+4r^2s+s^2+6rt)H_{m+1}^2 + t(r^3+4rs+9t)H_m^2 - (2r^3+7rs+9t)H_{m+2}H_{m+1} - (r^2s+4s^2-3rt)H_{m+2}H_m + rs(r^2+4s)H_{m+1}H_m - t(r^2-6s)H_{m+1}H_m)$,

$z\Gamma_3 = z(-4r^3t + r^2s^2 + 4s^3 - 27t^2 - 18rst)H_m$,

$\Gamma_4 = 4r^3t - r^2s^2 + 18rst - 4s^3 + 27t^2$.

Proof. Use Theorem 65 (a). □

Lemma 72 gives the following result as particular example (generating function of (r, s, t)-Tribonacci polynomials).

Corollary 73. *Assume that* $|z| < \min\{|\alpha|^{-m}, |\beta|^{-m}, |\gamma|^{-m}\}$. *Generating function of* (r, s, t)*-Tribonacci polynomials is given, as follows:*

$$\sum_{n=0}^{\infty} G_{mn+j} z^n = \frac{z^2\Theta_4 + z\Theta_5 + \Theta_6}{z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4}$$

where (as in Theorem 65 (a))

$z^2\Theta_4 = z^2((3G_{j+2} - 2rG_{j+1} - sG_j)H_{m+2}^2 + ((r^2 - s)G_{j+2} + 3(rs + t)G_{j+1} + (s^2 + rt)G_j)H_{m+1}^2 + t(rG_{j+2} + 2sG_{j+1} + 3tG_j)H_m^2 + (-4rG_{j+2} + 2(r^2 - s)G_{j+1} + (rs - 3t)G_j)$

$H_{m+2}H_{m+1} + (-2sG_{j+2} + (rs-3t)G_{j+1} + 2rtG_j)H_{m+2}H_m + ((rs-3t)G_{j+2} + 2(s^2 + rt)G_{j+1} + 4stG_j)H_{m+1}H_m)$,

$z\Theta_5 = z((-2(r^2+3s)G_{j+2} + (2r^3+7rs+9t)G_{j+1} + (r^2s-3rt+4s^2)G_j)H_{m+2} + ((2r^3+7rs+9t)G_{j+2} - 2(r^4+4r^2s+6rt+s^2)G_{j+1} - (r^3s+4rs^2-r^2t+6st)G_j)H_{m+1} + ((r^2s+4s^2-3rt)G_{j+2} - (4rs^2+r^3s-r^2t+6st)G_{j+1} - 2t(r^3+4rs+9t)G_j)H_m)$,

$\Theta_6 = (4r^3t - r^2s^2 - 4s^3 + 27t^2 + 18rst)G_j$,

and

$z^3\Gamma_1 = z^3(-t^m(4r^3t - r^2s^2 + 18rst - 4s^3 + 27t^2))$,

$z^2\Gamma_2 = z^2((r^2+3s)H_{m+2}^2 + (r^4+4r^2s+s^2+6rt)H_{m+1}^2 + t(r^3+4rs+9t)H_m^2 - (2r^3+7rs+9t)H_{m+2}H_{m+1} - (r^2s+4s^2-3rt)H_{m+2}H_m + rs(r^2+4s)H_{m+1}H_m - t(r^2-6s)H_{m+1}H_m)$,

$z\Gamma_3 = z(-4r^3t + r^2s^2 + 4s^3 - 27t^2 - 18rst)H_m$,

$\Gamma_4 = 4r^3t - r^2s^2 + 18rst - 4s^3 + 27t^2$.

Proof. In Lemma 72, take $W_n = G_n$ with $G_0 = 0, G_1 = 1, G_2 = r$. □

Note that if we take $W_n = H_n$ with $H_0 = 3, H_1 = r, H_2 = 2s + r^2$ in Lemma 72, then we get Corollary 68 (b).

12 A Special Case of Generating Function of Generalized Tribonacci Polynomials: The Case $r = s = t = 1$

In this section, we present a special case of the ordinary generating function of generalized Tribonacci polynomials.

12.1 Generating Function of Generalized Tribonacci Numbers: The Case $r = 1, s = 1, t = 1$

In this subsection, we consider the case $r = 1, s = 1, t = 1$. A generalized Tribonacci sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2)\}_{n\geq 0}$ is defined by the third-order recurrence relations

$$W_n = W_{n-1} + W_{n-2} + W_{n-3} \tag{12.1}$$

with the initial values $W_0 = c_0, W_1 = c_1, W_2 = c_2$ not all being zero.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -W_{-(n-1)} - W_{-(n-2)} + W_{-(n-3)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (12.1) holds for all integer n.Binet formula of generalized Tribonacci numbers can be given as

$$W_n = \frac{b_1 \alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{b_2 \beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{b_3 \gamma^n}{(\gamma-\alpha)(\gamma-\beta)} \tag{12.2}$$

where

$$\begin{aligned} b_1 &= W_2 - (\beta+\gamma)W_1 + \beta\gamma W_0, & (12.3)\\ b_2 &= W_2 - (\alpha+\gamma)W_1 + \alpha\gamma W_0, & (12.4)\\ b_3 &= W_2 - (\alpha+\beta)W_1 + \alpha\beta W_0. & (12.5) \end{aligned}$$

Here, α, β and γ are the roots of the cubic equation

$$x^3 - x^2 - x - 1 = 0.$$

Moreover

$$\begin{aligned} \alpha &= \frac{1 + \sqrt[3]{19+3\sqrt{33}} + \sqrt[3]{19-3\sqrt{33}}}{3}, \\ \beta &= \frac{1 + \omega\sqrt[3]{19+3\sqrt{33}} + \omega^2\sqrt[3]{19-3\sqrt{33}}}{3}, \\ \gamma &= \frac{1 + \omega^2\sqrt[3]{19+3\sqrt{33}} + \omega\sqrt[3]{19-3\sqrt{33}}}{3}, \end{aligned}$$

where

$$\omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3).$$

Two special cases of the sequence $\{W_n\}$ are the well known Tribonacci sequence $\{T_n\}_{n\geq 0}$ and Tribonacci-Lucas (Tribonacci-Lucas-Lucas) sequence $\{K_n\}_{n\geq 0}$. Tribonacci sequence$\{T_n\}_{n\geq 0}$, Tribonacci-Lucas sequence $\{K_n\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations

$$\begin{aligned} T_{n+3} &= T_{n+2} + T_{n+1} + T_n, \qquad T_0 = 0, T_1 = 1, T_2 = 1, & (12.6)\\ K_{n+3} &= K_{n+2} + K_{n+1} + K_n, \qquad K_0 = 3, K_1 = 1, K_2 = 3, & (12.7) \end{aligned}$$

The sequences $\{T_n\}_{n\geq 0}$, $\{K_n\}_{n\geq 0}$, can be extended to negative subscripts by defining

$$\begin{aligned} T_{-n} &= -T_{-(n-1)} - T_{-(n-2)} + T_{-(n-3)}, & (12.8)\\ K_{-n} &= -K_{-(n-1)} - K_{-(n-2)} + K_{-(n-3)}, & (12.9) \end{aligned}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (12.6)-(12.7) hold for all integer n.

For all integers n, Tribonacci and Tribonacci-Lucas numbers can be expressed using Binet's formulas as

$$\begin{array}{rcl} T_n & = & \dfrac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)}+\dfrac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)}+\dfrac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}, \\ K_n & = & \alpha^n+\beta^n+\gamma^n, \end{array}$$

respectively. Here, $G_n = T_n$ and $H_n = K_n$.

Next, we give the ordinary generating function $\sum\limits_{n=0}^{\infty} W_{mn+j}z^n$ of the generalized Tribonacci numbers $\{W_{mn+j}\}$.

Lemma 74. *Assume that* $|z| < \min\{|\alpha|^{-m}, |\beta|^{-m}, |\gamma|^{-m}\}$. *Suppose that* $f_{W_{mn+j}}(z) = \sum\limits_{n=0}^{\infty} W_{mn+j}z^n$ *is the ordinary generating function of the generalized Tribonacci (sequence of) polynomials* $\{W_{mn+j}\}$. *Then,* $\sum\limits_{n=0}^{\infty} W_{mn+j}z^n$ *is given by*

$$\sum_{n=0}^{\infty} W_{mn+j}z^n = \frac{z^2\Theta_4+z\Theta_5+\Theta_6}{z^3\Gamma_1+z^2\Gamma_2+z\Gamma_3+\Gamma_4}$$

where

$z^2\Theta_4 = z^2((W_0W_{j+2}+(W_1-W_0)W_{j+1}+(W_2-W_1-W_0)W_j)W_{m+2}^2+((W_2-W_0)W_{j+2}+(W_0+W_0)W_{j+1}+(W_0+2W_1-W_2)W_j)W_{m+1}^2+(W_1W_{j+2}+(W_2-W_1)W_{j+1}+W_0W_j)W_m^2+(-(W_1+W_0)W_{j+2}+(W_0-W_2)W_{j+1}+(-W_2+W_1)W_j)W_{m+1}W_{m+2}+((W_1-W_2)W_{j+2}+(W_2-2W_1-W_0)W_{j+1}+(W_0-W_1)W_j)W_{m+2}W_m+((W_1-W_0)W_{j+2}+(W_2-2W_1+W_0)W_{j+1}+(-W_2+W_1+2W_0)W_j)W_{m+1}W_m)$,

$z\Theta_5 = z(((W_1^2-W_0W_2)W_{j+2}+(W_0^2-W_1W_2+W_0W_2+W_0W_1)W_{j+1}+(-2W_2^2-W_1^2-W_0^2+3W_1W_2+2W_0W_2+W_0W_1)W_j)W_{m+2}+((W_0^2-W_1W_2+W_0W_2+W_0W_1)W_{j+2}+(W_2^2-2W_0W_2-2W_0W_1)W_{j+1}+(W_2^2-4W_1^2-2W_0^2+W_1W_2+W_0W_2-3W_0W_1)W_j)W_{m+1}+((W_2^2-W_1^2-W_1W_2-W_0W_1)W_{j+2}+(-W_2^2+2W_1^2+W_1W_2-W_0W_2+W_0W_1)W_{j+1}+(-W_1^2-2W_0^2+2W_1W_2-W_0W_2-2W_0W_1)W_j)W_m-W_0^2W_{j+1}W_{m+1})$,

$\Theta_6 = (W_2^3+2W_1^3+W_0^3-2W_1W_2^2-W_0W_2^2+2W_0W_1^2+W_0^2W_2+2W_0^2W_1-2W_0W_1W_2)W_j$,

and

$z^3\Gamma_1 = z^3(-1)(W_2^3+2W_1^3+W_0^3-2W_1W_2^2-W_0W_2^2+2W_0W_1^2+W_0^2W_2+2W_0^2W_1-2W_0W_1W_2)$,

$z^2\Gamma_2 = z^2((3W_2-2W_1-W_0)W_{m+2}^2+(6W_1+2W_0)W_{m+1}^2+(W_2+2W_1+3W_0)W_m^2+(-4W_2-2W_0)W_{m+2}W_{m+1}+(-2W_2-2W_1+2W_0)W_{m+2}W_m+(-2W_2+4W_1+4W_0)W_{m+1}W_m)$,

$z\Gamma_3 = z((-3W_2^2 - W_0^2 + 4W_1W_2 + 2W_0W_2 + 2W_0W_1)W_{m+2} + (2W_2^2 - 6W_1^2 - 2W_0^2 + 2W_0W_2 - 4W_0W_1)W_{m+1} + (W_2^2 - 2W_1^2 - 3W_0^2 + 2W_1W_2 - 2W_0W_2 - 4W_0W_1)W_m),$

$\Gamma_4 = W_2^3 + 2W_1^3 + W_0^3 - 2W_1W_2^2 - W_0W_2^2 + 2W_0W_1^2 + W_0^2W_2 + 2W_0^2W_1 - 2W_0W_1W_2.$

Proof. Set $r = 1, s = 1, t = 1, G_n = T_n$ and $H = K_n$ in Lemma 66. □

Now, we consider special cases of the last Lemma.

Corollary 75. *The ordinary generating functions of the sequences* W_n, W_{2n}, W_{2n+1}, W_{-n}, W_{-2n}, W_{-2n+1} *are given as follows:*

(a) $(m = 1, j = 0, |z| < |\alpha|^{-1} \simeq 0.543689)$.

$$\sum_{n=0}^{\infty} W_n z^n = \frac{z^2(W_2 - W_1 - W_0) + z(W_1 - W_0) + W_0}{z^3(-1) + z^2(-1) + z(-1) + 1}.$$

(b) $(m = 2, j = 0, |z| < |\alpha|^{-2} \simeq 0.295597)$.

$$\sum_{n=0}^{\infty} W_{2n} z^n = \frac{z^2(-W_2 + 2W_1) + z(W_2 - 3W_0) + W_0}{z^3(-1) + z^2(-1) + 3z(-1) + 1}.$$

(c) $(m = 2, j = 1, |z| < |\alpha|^{-2} \simeq 0.295597)$.

$$\sum_{n=0}^{\infty} W_{2n+1} z^n = \frac{z^2(W_2 - W_1 - W_0) + z(W_2 - 2W_1 + W_0) + W_1}{z^3(-1) + z^2(-1) + 3z(-1) + 1}.$$

(d) $(m = -1, j = 0, |z| < |\beta| = |\gamma| \simeq 0.737352\,)$.

$$\sum_{n=0}^{\infty} W_{-n} z^n = \frac{z^2 W_1 + z(W_2 - W_1) + W_0}{z^3(-1) + z^2 + z + 1}.$$

(e) $(m = -2, j = 0, |z| < |\beta|^2 = |\gamma|^2 \simeq 0.543689)$.

$$\sum_{n=0}^{\infty} W_{-2n} z^n = \frac{z^2 W_2 + z(-W_2 + 2W_1 + W_0) + W_0}{z^3(-1) + 3z^2 + z + 1}.$$

(f) $(m = -2, j = 1, |z| < |\beta|^2 = |\gamma|^2 \simeq 0.543689)$.

$$\sum_{n=0}^{\infty} W_{-2n+1} z^n = \frac{z^2(W_2 + W_1 + W_0) + z(W_2 - W_0) + W_1}{z^3(-1) + 3z^2 + z + 1}.$$

The last Lemma gives the following results as particular examples (generating functions of Tribonacci and Tribonacci-Lucas numbers).

Corollary 76. *Assume that* $|z| < \min\{|\alpha|^{-m}, |\beta|^{-m}, |\gamma|^{-m}\}$. *Generating functions of Tribonacci and Tribonacci-Lucas numbers are given, respectively, as follows:*

(a)

$$\sum_{n=0}^{\infty} T_{mn+j} z^n = \frac{z^2\Theta_4 + z\Theta_5 + \Theta_6}{z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4}$$

where

$z^2\Theta_4 = z^2(T_{m+2}^2 T_{j+1} + T_{m+1}^2 T_{j+2} + T_{m+1}^2 T_j + T_m^2 T_{j+2} - T_{m+1}T_{m+2}T_{j+2} + T_m T_{m+1} T_{j+2} - T_{m+1}T_{m+2}T_{j+1} - T_m T_{m+2} T_{j+1} - T_m T_{m+1} T_{j+1} - T_m T_{m+2} T_j)$,

$z\Theta_5 = z(T_{m+2}T_{j+2} - T_{m+1}T_{j+2} - T_{m+2}T_{j+1} - T_m T_{j+2} + T_{m+1}T_{j+1} - 2T_j T_{m+1} + 2T_m T_{j+1} + T_m T_j)$,

$\Theta_6 = T_j$,

and

$z^3\Gamma_1 = z^3(-1)$,

$z^2\Gamma_2 = z^2(T_{m+2}^2 + 6T_{m+1}^2 + 3T_m^2 - 4T_{m+2}T_{m+1} - 4T_{m+2}T_m + 2T_{m+1}T_m)$,

$z\Gamma_3 = z(T_{m+2} - 4T_{m+1} + T_m)$,

$\Gamma_4 = 1$.

(b)

$$\sum_{n=0}^{\infty} K_{mn+j} z^n = \frac{z^2\Theta_4 + z\Theta_5 + \Theta_6}{z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4}$$

where

$z^2\Theta_4 = z^2((3K_{j+2} - 2K_{j+1} - K_j)K_{m+2}^2 + (6K_{j+1} + 2K_j)K_{m+1}^2 + (K_{j+2} + 2K_{j+1} + 3K_j)K_m^2 + (-4K_{j+2} - 2K_j)K_{m+2}K_{m+1} + (-2K_{j+2} - 2K_{j+1} + 2K_j)K_{m+2}K_m + (-2K_{j+2} + 4K_{j+1} + 4K_j)K_{m+1}K_m)$,

$z\Theta_5 = z((-8K_{j+2} + 18K_{j+1} + 2K_j)K_{m+2} + (18K_{j+2} - 15K_{j+1} - 10K_j)K_{m+1} + (2K_{j+2} - 10K_{j+1} - 28K_j)K_m - 9K_{j+1}K_{m+1})$,

$\Theta_6 = 44K_j$,

and

$z^3\Gamma_1 = z^3(-44)$,

$z^2\Gamma_2 = z^2(4K_{m+2}^2 + 12K_{m+1}^2 + 14K_m^2 - 18K_{m+2}K_{m+1} - 2K_{m+2}K_m + 10K_{m+1}K_m),$

$z\Gamma_3 = z(-44)K_m,$

$\Gamma_4 = 44.$

Now, we consider special cases of the last two corollaries.

Corollary 77. *The ordinary generating functions of the sequences T_n, T_{2n}, T_{2n+1}, T_{-n}, T_{-2n}, T_{-2n+1} and K_n, K_{2n}, K_{2n+1}, K_{-n}, K_{-2n}, K_{-2n+1} are given as follows:*

(a) $(m = 1, j = 0, |z| < |\alpha|^{-1} \simeq 0.543689)$.

$$\sum_{n=0}^{\infty} T_n z^n = \frac{z}{1 - z - z^2 - z^3},$$
$$\sum_{n=0}^{\infty} K_n z^n = \frac{3 - 2z - z^2}{1 - z - z^2 - z^3}.$$

(b) $(m = 2, j = 0, |z| < |\alpha|^{-2} \simeq 0.295597)$.

$$\sum_{n=0}^{\infty} T_{2n} z^n = \frac{z + z^2}{1 - 3z - z^2 - z^3},$$
$$\sum_{n=0}^{\infty} K_{2n} z^n = \frac{3 - 6z - z^2}{1 - 3z - z^2 - z^3}.$$

(c) $(m = 2, j = 1, |z| < |\alpha|^{-2} \simeq 0.295597)$.

$$\sum_{n=0}^{\infty} T_{2n+1} z^n = \frac{1 - z}{1 - 3z - z^2 - z^3},$$
$$\sum_{n=0}^{\infty} K_{2n+1} z^n = \frac{1 + 4z - z^2}{1 - 3z - z^2 - z^3}.$$

(d) $(m = -1, j = 0, |z| < |\beta| = |\gamma| \simeq 0.737352)$.

$$\sum_{n=0}^{\infty} T_{-n} z^n = \frac{z^2}{1 + z + z^2 - z^3},$$
$$\sum_{n=0}^{\infty} K_{-n} z^n = \frac{3 + 2z + z^2}{1 + z + z^2 - z^3}.$$

(e) $(m=-2, j=0, |z|<|\beta|^2=|\gamma|^2 \simeq 0.543689)$.

$$\begin{aligned}\sum_{n=0}^{\infty} T_{-2n} z^n &= \frac{z+z^2}{1+z+3z^2-z^3}, \\ \sum_{n=0}^{\infty} K_{-2n} z^n &= \frac{3+2z+3z^2}{1+z+3z^2-z^3}.\end{aligned}$$

(f) $(m=-2, j=1, |z|<|\beta|^2=|\gamma|^2 \simeq 0.543689)$.

$$\begin{aligned}\sum_{n=0}^{\infty} T_{-2n+1} z^n &= \frac{1+z+2z^2}{1+z+3z^2-z^3}, \\ \sum_{n=0}^{\infty} K_{-2n+1} z^n &= \frac{1+7z^2}{1+z+3z^2-z^3}.\end{aligned}$$

From the last corollary, we obtain the following results for Tribonacci and Tribonacci-Lucas numbers.

Corollary 78. *Infinite sums of* T_n, T_{2n}, T_{2n+1}, T_{-n}, T_{-2n}, T_{-2n+1} *and* K_n, K_{2n}, K_{2n+1}, K_{-n}, K_{-2n}, K_{-2n+1} *are given as follows:*

(a) $z=\frac{1}{2}$.

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{T_n}{2^n} &= 4, \\ \sum_{n=0}^{\infty} \frac{K_n}{2^n} &= 14.\end{aligned}$$

(b) $z=\frac{1}{4}$.

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{T_{2n}}{4^n} &= \frac{20}{11}, \\ \sum_{n=0}^{\infty} \frac{K_{2n}}{4^n} &= \frac{92}{11}.\end{aligned}$$

(c) $z = \frac{1}{4}$.

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{T_{2n+1}}{4^n} &= \frac{48}{11}, \\ \sum_{n=0}^{\infty} \frac{K_{2n+1}}{4^n} &= \frac{124}{11}.\end{aligned}$$

(d) $z = \frac{1}{2}$.

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{T_{-n}}{2^n} &= \frac{2}{13}, \\ \sum_{n=0}^{\infty} \frac{K_{-n}}{2^n} &= \frac{34}{13}.\end{aligned}$$

(e) $z = \frac{1}{2}$.

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{T_{-2n}}{2^n} &= \frac{6}{17}, \\ \sum_{n=0}^{\infty} \frac{K_{-2n}}{2^n} &= \frac{38}{17}.\end{aligned}$$

(f) $z = \frac{1}{2}$.

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{T_{-2n+1}}{2^n} &= \frac{16}{17}, \\ \sum_{n=0}^{\infty} \frac{K_{-2n+1}}{2^n} &= \frac{22}{17}.\end{aligned}$$

13 Some Remarks

When we defined the generalized Tribonacci polynomials in (1.1), we supposed that W_0, W_1, W_2 are arbitrary complex (or real) polynomials with real coefficients and r, s and t are polynomials with real coefficients with $t \neq 0$. However, if we take W_0, W_1, W_2, r, s and t are arbitrary complex or real functions (with real coefficients) and z are arbitrary complex or real number (function), then we can apply to the results obtained in the previous sections (when we check the proofs, we see that proofs work for these W_0, W_1, W_2, r, s, t and z). Now, we present some special cases of W_0, W_1, W_2, r, s, t and z as examples of functions.

13.1 The Case $r = x + \sin x,\ s = x + \cos x,\ t = 1,\ x \in \mathbb{R}$

If we set

$$\begin{aligned} r &= x + \sin x, \\ s &= x + \cos x, \\ t &= 1, \\ x &\in \mathbb{R}, \end{aligned}$$

in (1.1) then we get,

$$W_{n+3} = (x + \sin x)W_{n+2} + (x + \cos x)W_{n+1} + W_n$$

with $W_0 = a(x)$, $W_1 = b(x)$, $W_2 = c(x)$ and

$$\begin{aligned} G_{n+3} &= (x + \sin x)G_{n+2} + (x + \cos x)G_{n+1} + G_n, \\ G_0 &= 0, G_1 = 1, G_2 = x + \sin x, \\ H_{n+3} &= (x + \sin x)H_{n+2} + (x + \cos x)H_{n+1} + H_n, \\ H_0 &= 3, H_1 = x + \sin x, H_2 = 2(x + \cos x) + (x + \sin x)^2. \end{aligned}$$

We can apply to the results of previous sections for the functions r, s, t, z. For example, for all integers n, we get, by Theorem 51 (a),

$$\begin{pmatrix} x + \sin x & x + \cos x & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} G_{n+1} & (x + \cos x)G_n + G_{n-1} & G_n \\ G_n & (x + \cos x)G_{n-1} + G_{n-2} & G_{n-1} \\ G_{n-1} & (x + \cos x)G_{n-2} + G_{n-3} & G_{n-2} \end{pmatrix},$$

for all $x \in \mathbb{R}$.

We now apply to Theorem 62 for the case $r = x + \sin x$, $s = x + \cos x$, $t = 1$, $x \in \mathbb{R}$.

Theorem 79. *We have the following sum formulas.*

(a) $(m = 1, j = 0)$. *If* $z^3(-1) + z^2(-1)(x + \cos x) + z(-1)(x + \sin x) + 1 \neq 0$ *then*

$$\sum_{k=0}^{n} z^k G_k = \frac{-z^{n+3}G_n + z^{n+2}((x + \sin x)G_{n+1} - G_{n+2}) - z^{n+1}G_{n+1} + z}{-z^3 - z^2(x + \cos x) - z(x + \sin x) + 1}.$$

(b) $(m=1, j=0)$. *If* $z^3(-1)+z^2(-1)(x+\cos x)+z(-1)(x+\sin x)+1\neq 0$ *then*

$$\sum_{k=0}^{n} z^k H_k = \frac{\Omega}{-z^3-z^2(x+\cos x)-z(x+\sin x)+1}.$$

where

$\Omega = -z^{n+3}H_n + z^{n+2}((x+\sin x)H_{n+1}-H_{n+2}) - z^{n+1}H_{n+1} - z^2(x+\cos x) - 2z(x+\sin x)+3.$

If we set $z=1$ in the last theorem, we get the following Corollary.

Corollary 80. *We have the following sum formulas.*

(a) $(m=1, j=0)$. *If* $-2x-\cos x-\sin x\neq 0$ *then*

$$\sum_{k=0}^{n} G_k = \frac{-G_{n+2}-G_n+(-1+x+\sin x)G_{n+1}+1}{-2x-\cos x-\sin x}.$$

(b) $(m=1, j=0)$. *If* $-2x-\cos x-\sin x\neq 0$ *then*

$$\sum_{k=0}^{n} H_k = \frac{-H_{n+2}+(-1+x+\sin x)H_{n+1}-H_n-2\sin x-\cos x-3x+3}{-2x-\cos x-\sin x}.$$

If we set $z=e^{3ix}=\cos 3x+i\sin 3x$ (for $x\in\mathbb{R}$) in the last theorem, we get the following Corollary.

Corollary 81. *We have the following sum formulas.*

(a) $(m=1, j=0)$. *If* $-e^{9ix}-e^{6ix}(x+\cos x)-e^{3ix}(x+\sin x)+1\neq 0$ *then*

$$\sum_{k=0}^{n} e^{3ikx} G_k = \frac{-e^{3i(n+3)x}G_n+e^{3i(n+2)x}((x+\sin x)G_{n+1}-G_{n+2})-e^{3i(n+1)x}G_{n+1}+e^{3ix}}{-e^{9ix}-e^{6ix}(x+\cos x)-e^{3ix}(x+\sin x)+1}.$$

(b) $(m=1, j=0)$. *If* $-e^{9ix}-e^{6ix}(x+\cos x)-e^{3ix}(x+\sin x)+1\neq 0$ *then*

$$\sum_{k=0}^{n} e^{3ikx} H_k = \frac{\Omega}{-e^{9ix}-e^{6ix}(x+\cos x)-e^{3ix}(x+\sin x)+1}$$

where

$\Omega = -e^{3i(n+3)x}H_n + e^{3i(n+2)x}((x+\sin x)H_{n+1}-H_{n+2}) - e^{3i(n+1)x}H_{n+1} - e^{6ix}(x+\cos x) - 2e^{3ix}(x+\sin x)+3.$

13.2 The Case $r = 1,\ s = 1,\ t = 1$

For the case $r = 1,\ s = 1,\ t = 1$,we now apply to Corollary 77 for specific z.

Corollary 82. *We have the following infinite sum formulas for T_n, T_{2n}, T_{2n+1}, T_{-n}, T_{-2n}, T_{-2n+1} and K_n, K_{2n}, K_{2n+1}, K_{-n}, K_{-2n}, K_{-2n+1}:*

(a) $(m = 1, j = 0, z = \frac{\cos x}{2}, |z| = \left|\frac{\cos x}{2}\right| < |\alpha|^{-1} \simeq 0.543689)$. *We can define two functions $f, g : \mathbb{R} \to \mathbb{R}$ by*

$$
\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} \left(\frac{\cos x}{2}\right)^n T_n = \frac{8\cos x}{14 - (2+\cos x)\cos 2x - 9\cos x}, \\
g(x) &= \sum_{n=0}^{\infty} \left(\frac{\cos x}{2}\right)^n K_n = \frac{46 - 16\cos x - 2\cos 2x}{14 - (2+\cos x)\cos 2x - 9\cos x}.
\end{aligned}
$$

(b) $(m = 2, j = 0, z = \frac{\sin x}{4}, |z| = \left|\frac{\sin x}{4}\right| < |\alpha|^{-2} \simeq 0.295597)$. *We can define two functions $f, g : \mathbb{R} \to \mathbb{R}$ by*

$$
\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} \left(\frac{\sin x}{4}\right)^n T_{2n} = \frac{64\sin x - 8\cos 2x + 8}{8\cos 2x - 195\sin x + \sin 3x + 248}, \\
g(x) &= \sum_{n=0}^{\infty} \left(\frac{\sin x}{4}\right)^n K_{2n} = \frac{8\cos 2x - 384\sin x + 760}{8\cos 2x - 195\sin x + \sin 3x + 248}.
\end{aligned}
$$

(c) $(m = 2, j = 1, z = \frac{\sin x + \cos x}{8}, |z| = \left|\frac{\sin x + \cos x}{8}\right| < |\alpha|^{-2} \simeq 0.295597)$. *We can define two functions $f, g : \mathbb{R} \to \mathbb{R}$ by*

$$
\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} \left(\frac{\sin x + \cos x}{8}\right)^n T_{2n+1} \\
&= \frac{128(\cos x + \sin x - 8)}{\sin 3x + 16\sin 2x + 387\sin x - \cos 3x + 387\cos x - 1008}, \\
g(x) &= \sum_{n=0}^{\infty} \left(\frac{\sin x + \cos x}{8}\right)^n K_{2n+1} \\
&= \frac{16(-32\cos x - 32\sin x + 2\cos x \sin x - 63)}{\sin 3x + 16\sin 2x + 387\sin x + 387\cos x - \cos 3x - 1008}.
\end{aligned}
$$

(d) $(m = -1, j = 0, z = \frac{\sin x}{2+3x^2}, |z| = \left|\frac{\sin x}{2+3x^2}\right| < |\beta| = |\gamma| \simeq 0.737352)$. *We can* *define two functions* $f, g : \mathbb{R} \to \mathbb{R}$ *by*

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \left(\frac{\sin x}{2+3x^2}\right)^n T_{-n} \\ &= \frac{-2(3x^2+2)(-1+\cos 2x)}{(36x^4+48x^2+13)\sin x + \sin 3x - 2(3x^2+2)\cos 2x + 108x^6 + 216x^4 + 150x^2 + 36}, \\ g(x) &= \sum_{n=0}^{\infty} \left(\frac{\sin x}{2+3x^2}\right)^n K_{-n} \\ &= \frac{2\left(3x^2+2\right)(4(3x^2+2)\sin x - \cos 2x + 54x^4 + 72x^2 + 25)}{(36x^4+48x^2+13)\sin x + \sin 3x - 2(3x^2+2)\cos 2x + 108x^6 + 216x^4 + 150x^2 + 36}. \end{aligned}$$

(e) $(m = -2, j = 0, z = \frac{e^{2ix}}{3} = \frac{\cos 2x + i \sin 2x}{3}, |z| = \left|\frac{e^{2ix}}{3}\right| = \frac{1}{3} < |\beta|^2 = |\gamma|^2 \simeq 0.543689)$. *We can define two functions* $f, g : \mathbb{R} \to \mathbb{C}$ *by*

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{e^{2inx}}{3^n} T_{-2n} = \frac{3e^{2ix}(e^{2ix}+3)}{27 + 9e^{2ix} + 9e^{4ix} - e^{6ix}}, \\ g(x) &= \sum_{n=0}^{\infty} \frac{e^{2inx}}{3^n} K_{-2n} = \frac{81 + 18e^{2ix} + 9e^{4ix}}{27 + 9e^{2ix} + 9e^{4ix} - e^{6ix}}. \end{aligned}$$

(f) $(m = -2, j = 1, z = \frac{1}{2+x^2}, |z| = \left|\frac{1}{2+x^2}\right| < |\beta|^2 = |\gamma|^2 \simeq 0.543689)$. *We can define two functions* $f, g : \mathbb{R} \to \mathbb{R}$ *by*

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} (2+x^2)^{-n} T_{-2n+1} = \frac{(x^2+2)((x^4+5x^2+8)}{x^6+7x^4+19x^2+17}, \\ g(x) &= \sum_{n=0}^{\infty} (2+x^2)^{-n} K_{-2n+1} = \frac{(x^2+2)(4x^2+x^4+11)}{x^6+7x^4+19x^2+17}. \end{aligned}$$

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