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Applications of (p,q)-Gegenbauer Polynomials on a Family of Bi-univalent Functions

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Abstract

In this work, we investigate the (p,q)-Gegenbauer polynomials for a class of analytic and bi-univalent functions defined in the open unit disk, with respect to subordination. We give an elementary proof to establish some estimates for the coefficient bounds for functions in the new class. We conclude the study by giving a result of the Fekete-Szegö theorem. A corollary was given to show some results of some subclasses of our new class.

1 Preliminaries and Definitions

In this paper, let the unit disk be denoted by $\varDelta := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Let $\mathcal{A}$ be the class of analytic (or holomorphic) functions in $\varDelta$. Also let $\mathcal{S}$ be the class

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of functions in $\mathcal{A}$ that are also univalent in Δ and normalized by the conditions: $f(0) = f'(0) - 1 = 0$. Functions in $\mathcal{S}$ can therefore be represented as

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \Delta. \tag{1.1}$$

An important subclass of $\mathcal{S}$ in this study is the class $\mathcal{S}^\star(s,t,\delta)$ consisting of functions satisfying the conditions

$$\mathcal{R}e\, \frac{(s-t)zf'(z)}{f(sz)-f(tz)} > \delta \in [0,1), \quad f(z) \neq 0,\ s,t \in \mathbb{C},\ s \neq t,\ |t| \leqq 1,\ z \in \Delta.$$

Frasin [7] introduced the class $\mathcal{S}^\star(s,t,\delta)$ while the classes $\mathcal{S}^\star(1,t,\delta)$, $\mathcal{S}^\star(1,-1,\delta)$, $\mathcal{S}^\star(1,-1,0) \equiv \mathcal{S}_s$, $\mathcal{S}^\star(1,0,\delta) \equiv \mathcal{S}^\star(\delta)$ and $\mathcal{S}^\star(1,0,0) \equiv \mathcal{S}^\star$ were introduced by Owa et al. [17], Sakaguchi [27], Sakaguchi [27], Robertson [26] and Alexander [1] respectively. The classes $\mathcal{S}_s$ and $\mathcal{S}^\star$ are the well-known classes of starlike functions with respect to symmetrical points in Δ and starlike functions in Δ.

The Koebe one-quarter theorem (see [30]) declares that the image domain of every function $f \in \mathcal{S}$ contains a disk of radius $1/4$. This means that every function $f \in \mathcal{S}$ has an inverse function f^{-1} which can be defined by

$$f^{-1}(f(z)) = z,\ z \in \Delta$$

and

$$f(f^{-1}(w)) = w,\ w : |w| < r_0(f),\ r_0(f) \geqq 1/4$$

therefore

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots = F(w). \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be *bi-univalent* in Δ if both f and f^{-1} are univalent in Δ. Let $\mathcal{B}$ denote the class of analytic and bi-univalent functions in Δ. In 1967, Lewin [14] introduced the class of bi-univalent functions and declared that every bi-univalent function has upper bound $|a_2| < 1.51$. Some other results, examples, properties, definitions and some historical background, are archived in [10–13, 16, 21, 25, 28, 29].

Suppose $c(z)$ is an analytic function, then

$$\nabla := \left\{ c(z) : c(z) = \sum_{n=1}^{\infty} c_n z^n,\ c(0) = 0,\ |c(z)| < 1,\ z \in \Delta \right\} \tag{1.3}$$

is called the class of Schwarz functions. Let $\prec$ denote subordination, so if $j, J \in \mathcal{A}$, $j \prec J$ if $j(z) = J(c(z))$ and $z \in \Delta$. However, if J is a univalent function, then for $z \in \Delta$, $j(z) \prec J(z)$ *if, and only if,* $j(0) = J(0)$ and $j(\Delta) \subset J(\Delta)$.

Lately, orthogonal polynomials have been a focal point of studies in the field of geometric function theory. Some works in this direction can be found in [3–6, 9, 10, 18, 20–22, 24, 28]. The Chebyshev polynomials of the second kind is the natural generalization of Chebyshev polynomials of the first kind. It can be used in different areas of mathematics such as in theory of approximation, linear algebra, discrete analysis, representation theory and physics. For $n \in \{2, 3, 4, \ldots\}$, $0 < q < p \leqq 1$ and a variable s, the generating function of (p,q)-Chebyshev polynomials of the second kind is defined by

$$\mathcal{C}_{p,q}(z,x) = \frac{1}{1 - xpz\eta_p - xqz\eta_q - spqz^2\eta_{p,q}} = \sum_{n=0}^{\infty} \mathcal{U}_n(x,s,p,q) z^n, \quad z \in \Delta \tag{1.4}$$

where $\eta_q f(z) = f(qz)$ is the Fibonacci operator defined by Mason [15] and in a similar manner, Kizilatecs et al. [10] defined the operator $\eta_{p,q} f(z) = f(pqz)$. The recurrence relation for the (p,q)-Chebyshev polynomials of the second kind is defined by

$$\mathcal{U}_n(x,s,p,q) = (p^n + q^n) x \mathcal{U}_{n-1}(x,s,p,q) + (pq)^{n-1} s \mathcal{U}_{n-2}(x,s,p,q) \tag{1.5}$$

with initial values

$$\mathcal{U}_0(x,s,p,q) = 1 \quad \text{and} \quad \mathcal{U}_1(x,s,p,q) = (p+q)x.$$

Remark 1.1. A careful observation shows that the recurrence relation in (1.5) has the following special cases.

1. $\mathcal{U}_n(x/2,s,p,q) = \mathcal{F}_n(x,s,p,q)$ is the (p,q)-Fibonacci polynomial;

2. $\mathcal{U}_n(x,-1,1,1)=\mathcal{U}_n(x)$ is the second kind Chebyshev polynomial;
3. $\mathcal{U}_n(x/2,1,1)=\mathcal{F}_{n+1}(x)$ is the Fibonacci polynomials;
4. $\mathcal{U}_n(1/2,1,1)=\mathcal{F}_{n+1}$ is the Fibonacci numbers;
5. $\mathcal{U}_n(x,1,1,1)=\mathcal{P}_{n+1}(x)$ is the Pell polynomials,
6. $\mathcal{U}_n(1,1,1,1)=\mathcal{P}_{n+1}$ is the Pell numbers;
7. $\mathcal{U}_n(1/2,2y,1,1)=\mathcal{J}_{n+1}(y)$ is the Jacobsthal polynomials and
8. $\mathcal{U}_n(1/2,2,1,1)=\mathcal{J}_{n+1}$ Jacobsthal numbers.

Let α be a nonzero real constant, the generating function for the Gegenbauer polynomials is defined by

$$\mathcal{G}_\alpha(x,z)=\frac{1}{(1-2xz+z^2)^\alpha} \tag{1.6}$$

where $x\in[-1,1]$ and $z\in\Delta$. For a fixed x, the function $\mathcal{G}_\alpha$ is analytic in Δ so it can be expanded in a Taylor's series as

$$\mathcal{G}_\alpha(x,z)=\sum_{n=0}^{\infty}\mathcal{V}_n^\alpha(x)\,z^n \tag{1.7}$$

where $\mathcal{V}_n^\alpha(x)$ is known as the Gegenbauer polynomials of degree n. Obviously, $\mathcal{G}_\alpha$ generates nothing when $\alpha=0$, therefore the generating function of the Gegenbauer polynomials is set to

$$\mathcal{G}_0(x,z)=1-\log(1-2xz+z^2)=\sum_{n=0}^{\infty}\mathcal{V}_n^0(x)z^n. \tag{1.8}$$

The Gegenbauer polynomials can as well be defined by the relation

$$\mathcal{V}_n^\alpha(x)=\frac{1}{n}[2x(n+\alpha-1)\mathcal{V}_{n-1}^\alpha(x)-(n+2\alpha-2)\mathcal{V}_{n-2}^\alpha(x)] \tag{1.9}$$

which produce some initial values expressed as

$$\left.\begin{aligned}&\mathcal{V}_0^\alpha(x)=1,\\&\mathcal{V}_1^\alpha(x)=2\alpha x,\\&\mathcal{V}_2^\alpha(x)=2\alpha(1+\alpha)x^2-\alpha.\end{aligned}\right\} \tag{1.10}$$

Observe that from (1.10), if $\alpha = 1$, then we get the second kind Chebyshev polynomials and if $\alpha = \frac{1}{2}$, then we get the Legendre polynomials. See [2, 19, 23, 31, 32] for some details.

It is interesting to know that (1.9) can be generalized by the recurrence relation

$$\mathcal{V}_n^{\alpha}(x, s, p, q) = \frac{1}{n}[(p^n + q^n)x(n + \alpha - 1)\mathcal{V}_{n-1}^{\alpha}(x, s, p, q) + (pq)^{n-1}s(n + 2\alpha - 2)\mathcal{V}_{n-2}^{\alpha}(x, s, p, q)] \quad (1.11)$$

where $0 < q < p \leqq 1$, s is an arbitrary variable and the initial values are given by

$$\left.\begin{array}{l} \mathcal{V}_0^{\alpha}(x, s, p, q) = 1 \\ \mathcal{V}_1^{\alpha}(x, s, p, q) = \alpha(p + q)x \\ \mathcal{V}_2^{\alpha}(x, s, p, q) = \frac{1}{2}[\alpha(1 + \alpha)(p^2 + q^2)(p + q)x^2 + 2\alpha pqs]. \end{array}\right\} \quad (1.12)$$

We remark that (1.12) are the (p, q)-Gegenbauer polynomials from which for $\alpha = 1$, we get the (p, q)-Chebyshev polynomials and for $\alpha = \frac{1}{2}$, we get the (p, q)-Legendre polynomials. Further, a careful variation of the involving parameters show that we will get the listed polynomials in Remark 1.1.

2 Associated Lemmas

Let $c(z)$ be as defined in (1.3), then the following lemmas hold to prove our results.

Lemma 2.1 ([30]). *Let $c(z) \in \nabla$, then $|c_n| \leqq 1$ $\forall n \in \mathbb{N}$. Equality occurs for functions $c(z) = e^{i\vartheta}z^n$ $(\vartheta \in [0, 2\pi))$.*

Lemma 2.2 ([8]). *Let $c(z) \in \nabla$, then for $\sigma \in \mathbb{C}$,*

$$|c_2 + \sigma c_1^2| \leqq \max\{1; |\sigma|\}.$$

Equality holds for functions $c(z) = z$ or $c(z) = z^2$.

3 New Class of Bi-univalent Functions

Let $0 < q < p \leqq 1$, $s, t \in \mathbb{C}$ $(s \neq t, |t| \leqq 1)$ and $x \in (\frac{1}{2}, 1]$, then a function $f \in \mathcal{B}$ is a member of class $\mathcal{BS}(s,t,\mathcal{G})$ if it satisfies the subordination conditions

$$\frac{(s-t)zf'(z)}{f(sz)-f(tz)} \prec \mathcal{G}_{p,q}(x,z), \quad z \in \Delta \tag{3.1}$$

and

$$\frac{(s-t)wF'(w)}{F(sw)-F(tw)} \prec \mathcal{G}_{p,q}(x,w), \quad w \in \Delta. \tag{3.2}$$

where $F(w) = f^{-1}(w)$ is as defined in (1.4) and $\mathcal{G}_{p,q}(x,z)$ is the generating function of the (p,q)-Gegenbauer polynomials in (1.12).

Remark 3.1. The following are subclasses of class $\mathcal{BS}(s,t,\mathcal{G})$.

1. If we set $\alpha = 1$, then class $\mathcal{BS}(s,t,\mathcal{G})$ becomes class $\mathcal{BS}(s,t,\mathcal{C}_{p,q})$ which consists of Sakaguchi type bi-starlike functions that are subordinate to (p,q)-Chebyshev function and defined by the conditions

$$\frac{(s-t)zf'(z)}{f(sz)-f(tz)} \prec \mathcal{C}_{p,q}(x,z) \quad \text{and} \quad \frac{(s-t)wF'(w)}{F(sw)-F(tw)} \prec \mathcal{C}_{p,q}(x,w), \quad w,z \in \Delta. \tag{3.3}$$

   This class was earlier studied in [9].

2. If we set $\alpha = 1 = s$ and $t = 0$, then class $\mathcal{BS}(s,t,\mathcal{G})$ becomes class $\mathcal{BS}(\mathcal{C}_{p,q})$ which consists of bi-starlike functions that are subordinate to (p,q)-Chebyshev function and defined by the conditions

$$\frac{zf'(z)}{f(z)} \prec \mathcal{C}_{p,q}(x,z) \quad \text{and} \quad \frac{wF'(w)}{F(w)} \prec \mathcal{C}_{p,q}(x,w) \quad z,w \in \Delta.$$

3. If we set $\alpha = p = q = s = 1$ and $t = 0$, then class $\mathcal{BS}(s,t,\mathcal{G})$ becomes class $\mathcal{BS}(\mathcal{C})$ which consists of bi-starlike functions that are subordinate to Chebyshev function and defined by the conditions

$$\frac{zf'(z)}{f(z)} \prec \mathcal{C}(x,z) \quad \text{and} \quad \frac{wF'(w)}{F(w)} \prec \mathcal{C}(x,w) \quad z,w \in \Delta.$$

In this work, we use the (p,q)-Gegenbauer polynomials to define two new classes of analytic-bi-univalent functions that are associated with them. The initial coefficient estimates were afterward established for the two classes.

4 Main Results

In what follows, let all the parameters be as declared in Section 3 unless otherwise mentioned. Thus, the established results are as follows.

Theorem 4.1. *Let $f \in \mathcal{B}$ be a member of $\mathcal{BS}(s,t,\mathcal{G})$. Then*

$$|a_2| \leqq \frac{\alpha\sqrt{2}(p+q)x\sqrt{(p+q)x}}{\sqrt{|2\alpha(3-2s-2t+st)(p+q)^2x^2-(2-s-t)^2[(\alpha+1)(p^2+q^2)(p+q)x^2+2pqs]|}}$$

$$|a_3| \leqq \alpha(p+q)x\left[\frac{1}{(3-s^2-t^2+st)}+\frac{\alpha(p+q)x}{(2-s-t)^2}\right]$$

and for $\mu \in \mathbb{R}$ we get

$$|a_3-\mu a_2^2| \leqq \begin{cases} \frac{(p+q)x}{(3-2s-2t+st)} & if \quad |\mu-1| \leqq \lambda \\ \frac{(p+q)^3x^3|\mu-1|}{(3-2s-2t+st)(p^2+q^2)x^2-(2-s-t)^2[(p^2+q^2)x^2(p+q)+pqs]} & if \quad |\mu-1| \geqq \lambda \end{cases}$$

where

$$\lambda = \frac{(3-2s-2t+st)-(2-s-t)^2\left(\frac{(p^2+q^2)}{(p+q)}+\frac{pqs}{(p+q)^2x^2}\right)}{(3-s^2-t^2-st)}.$$

Proof. Let $f \in \mathcal{BS}(s,t,\mathcal{G})$, then there exists the analytic functions

$$c(z) = c_1z + c_2z^2 + c_3z^3 + \cdots, \; d(w) = d_1w + d_2w^2 + d_3w^3 + \cdots \in \nabla \tag{4.1}$$

such that $c(0) = 0 = d(0)$, $|c(z)|, |d(w)| < 1$ and

$$\frac{(s-t)zf'(z)}{f(sz)-f(tz)} = \mathcal{G}_{p,q}(x, c(z)), \quad z \in \Delta \tag{4.2}$$

and

$$\frac{(s-t)wF'(w)}{F(sw)-F(tw)} = \mathcal{G}_{p,q}(x, d(w)), \quad w \in \Delta. \tag{4.3}$$

Using (1.12) and (4.1) in (4.2) and (4.3) we get

$$\frac{(s-t)zf'(z)}{f(sz)-f(tz)} = \mathcal{V}_0(x,s,p,q) + \mathcal{V}_1(x,s,p,q)c(z) + \mathcal{V}_2(x,s,p,q)c^2(z) + \cdots, \quad (4.4)$$

and

$$\frac{(s-t)wF'(w)}{F(sw)-F(tw)} = \mathcal{V}_0(x,s,p,q) + \mathcal{V}_1(x,s,p,q)d(w) + \mathcal{V}_2(x,s,p,q)d^2(w) + \cdots. \quad (4.5)$$

Also, using (1.1) and some simplifications in (4.2) we get

$$\begin{aligned} 1 + (2-s-t)a_2z + \Big[(3-s^2-t^2-st)a_3 - (s^2+t^2-2s-2t+2st)a_2^2\Big]z^2 + \cdots \\ = 1 + \mathcal{V}_1^\alpha(x,s,p,q)c_1z + [\mathcal{V}_1^\alpha(x,s,p,q)c_2 + \mathcal{V}_2^\alpha(x,s,p,q)c_1^2]z^2 + \cdots \quad (4.6) \end{aligned}$$

and using (1.1), (1.2) and some simplifications in (4.3) we get

$$\begin{aligned} 1 - (2-s-t)a_2w + \Big[(6-s^2-t^2-2s-2t)a_2^2 - (3-s^2-t^2-st)a_3\Big]w^2 + \cdots \\ = 1 + \mathcal{V}_1^\alpha(x,s,p,q)d_1w + [\mathcal{V}_1^\alpha(x,s,p,q)d_2 + \mathcal{V}_2^\alpha(x,s,p,q)d_1^2]w^2 + \cdots. \quad (4.7) \end{aligned}$$

In view of the corresponding equations in (4.6) and (4.7) we get

$$(2-s-t)a_2 = \mathcal{V}_1^\alpha(x,s,p,q)c_1 \quad (4.8)$$

$$(3-s^2-t^2-st)a_3-(s^2+t^2-2s-2t+2st)a_2^2 = \mathcal{V}_1^\alpha(x,s,p,q)c_2+\mathcal{V}_2^\alpha(x,s,p,q)c_1^2 \quad (4.9)$$

$$-(2-s-t)a_2 = \mathcal{V}_1^\alpha(x,s,p,q)d_1 \quad (4.10)$$

and

$$(6-s^2-t^2-2s-2t)a_2^2 - (3-s^2-t^2-st)a_3 = \mathcal{V}_1^\alpha(x,s,p,q)d_2 + \mathcal{V}_2^\alpha(x,s,p,q)d_1^2. \quad (4.11)$$

Now if we add (4.8) and (4.10) we will get

$$\mathcal{V}_1^\alpha(x,s,p,q)c_1 + \mathcal{V}_1^\alpha(x,s,p,q)d_1 = 0 \Longrightarrow c_1 = -d_1 \quad (\text{and } c^2 = d^2). \quad (4.12)$$

Also, if we add the squares of (4.8) and (4.10), we will get

$$2(2-s-t)^2a_2^2 = [\mathcal{V}_1^\alpha(x,s,p,q)]^2(c_1^2+d_1^2) \quad (4.13)$$

or

$$\frac{2(2-s-t)^2}{[\mathcal{V}_1^\alpha(x,s,p,q)]^2}a_2^2 = c_1^2 + d_1^2. \tag{4.14}$$

Likewise if we add (4.9) and (4.11), we will get

$$(6-4s-4t-2st)a_2^2 = \mathcal{V}_1^\alpha(x,s,p,q)(c_2+d_2) + \mathcal{V}_2^\alpha(x,s,p,q)(c_1^2+d_1^2) \tag{4.15}$$

and putting (4.14) into (4.15) simplifies to

$$a_2^2 = \frac{[\mathcal{V}_1^\alpha(x,s,p,q)]^3(c_2+d_2)}{\{2(3-2s-2t-st)[\mathcal{V}_1^\alpha(x,s,p,q)]^2 - 2(2-s-t)^2\mathcal{V}_2^\alpha(x,s,p,q)\}} \tag{4.16}$$

so that using (1.12), taking modulus of both sides and applying Lemma 2.1 give the required result.

Now if we subtract (4.11) from (4.9), we get

$$a_3 = \frac{\mathcal{V}_1^\alpha(x,s,p,q)(c_2-d_2)}{2(3-s^2-t^2-st)} + \frac{[\mathcal{V}_1^\alpha(x,s,p,q)]^2(c_1^2+d_1^2)}{2(2-s-t)^2} \tag{4.17}$$

and using (1.12), we get

$$a_3 = \frac{\alpha(p+q)x(c_2-d_2)}{2(3-s^2-t^2-st)} + \frac{\alpha^2(p+q)^2x^2(c_1^2+d_1^2)}{2(2-s-t)^2} \tag{4.18}$$

so that taking modulus of both sides and applying Lemma 2.1 give the required result.

Let $\mu \in \mathbb{R}$, then in view of (4.16) and (4.17) and noting that

$$a_3 - \mu a_2^2 = (1-\mu)a_2^2 + (a_3 - a_2^2),$$

then we get

$$\begin{aligned} a_3 - \mu a_2^2 =& \frac{(1-\mu)[\mathcal{V}_1^\alpha(x,s,p,q)]^3(c_2+d_2)}{(3-2s-2t+st)[\mathcal{V}_1^\alpha(x,s,p,q)]^2 - (2-s-t)\mathcal{V}_2^\alpha(x,s,p,q)} \\ &+ \frac{\mathcal{V}_1^\alpha(x,s,p,q)(c_2-d_2)}{(3-s^2-t^2-st)} + \frac{[\mathcal{V}_1^\alpha(x,s,p,q)]^2(c_1^2+d_1^2)}{(2-s-t)^2} \\ &- \frac{[\mathcal{V}_1^\alpha(x,s,p,q)]^3(c_2+d_2)}{(3-2s-2t+st)[\mathcal{V}_1^\alpha(x,s,p,q)]^2 - (2-s-t)\mathcal{V}_2^\alpha(x,s,p,q)}. \end{aligned}$$

$$\begin{aligned}a_3-\mu a_2^2 =& \frac{\mathcal{V}_1^{\alpha}(x,s,p,q)}{2}\Bigg[y(\mu)(c_2+d_2)+\frac{(c_2-d_2)}{3-s^2-t^2-st}+\frac{\mathcal{V}_1^{\alpha}(x,s,p,q)(c_1^2+d_1^2)}{(2-s-t)^2}\\ &-\frac{[\mathcal{V}_1^{\alpha}(x,s,p,q)]^2(c_2+d_2)}{(3-2s-2t+st)[\mathcal{V}_1^{\alpha}(x,s,p,q)]^2-(2-s-t)^2\mathcal{V}_2^{\alpha}(x,s,p,q)}\Bigg]\\ =&\frac{\mathcal{V}_1^{\alpha}(x,s,p,q)}{2}\Bigg[\left(y(\mu)+\frac{1}{(3-s^2-t^2-st)}\right)c_2\\ &+\left(y(\mu)-\frac{1}{(3-s^2-t^2-st)}\right)d_2\Bigg]\end{aligned}$$

where

$$y(\mu)=\frac{[\mathcal{V}_1^{\alpha}(x,s,p,q)]^2(1-\mu)}{(3-s-t+st)[\mathcal{V}_1^{\alpha}(x,s,p,q)]^2-(2-s-t)^2\mathcal{V}_2^{\alpha}(x,s,p,q)}.$$

so that the application of triangle inequality gives the required result. □

A special case of Theorem 4.1 is given in the following Corollary.

Corollary 4.2. *Let $f\in\mathcal{B}$ be in the class $\mathcal{BS}(s,t,\mathcal{G})$. If $p=q=1$ and $s=-1$, then*

$$|a_2|\leqq\frac{4\alpha x\sqrt{x}}{\sqrt{|8\alpha(5-3t)x^2-2(3-t)^2[2(\alpha+1)x^2-1]|}},$$

$$|a_3|\leqq 2\alpha x\left[\frac{1}{(2-t^2-t)}+\frac{2\alpha x}{(3-t)^2}\right]$$

and for $\mu\in\mathbb{R}$ we get

$$|a_3-\mu a_2^2|\leqq\begin{cases}\frac{2x}{(3-2s-2t+st)} & if \quad |\mu-1|\leqq\lambda\\ \frac{8x^3|\mu-1|}{(3-2s-2t+st)2x^2-(2-s-t)^2[4x^2-1]} & if \quad |\mu-1|\geqq\lambda\end{cases}$$

where

$$\lambda=\frac{(3-2s-2t+st)-(2-s-t)^2(4x^2-1)}{4x^2(3-s^2-t^2-st)}.$$

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