Earthline Journal of Mathematical Sciences
E-ISSN: 2581-8147
Volume 12, Number 2, 2023, Pages 243-253
https://doi.org/10.34198/ejms.12223.243253

(P_g^*)-modules

Thaar Younis Ghawi

Department of Mathematics, College of Education, University of Al-Qadisiyah, Iraq
e-mail: thar.younis@qu.edu.iq

Abstract

The aim of this article is to investigate the notion of (P_g^*)-modules. We looked at some of these modules properties and characterizations. Moreover, relationships between a (P_g^*)-module and other modules are also discussed.

1. Introduction

Unless otherwise mentioned, all rings will have unit elements and all modules will be right unitary, in this work. We will use $T \subseteq M$, $T \leq M$ and $T \leq^{\oplus} M$ to signify that T is a subset, a submodule and a direct summand of M. We will denote the class of all unital right modules over a ring $\mathfrak{R}$ with the symbol Mod-$\mathfrak{R}$. Assume that $\mathfrak{R}$ is a ring and $M \in$Mod-$\mathfrak{R}$. A nonzero submodule $T \leq M$ is called to be large in M, denoted as $T \trianglelefteq M$, if $T \cap E \neq 0$ for any nonzero submodule $E \leq M$ [3]. Dually, if $T + E \neq M$ for any proper submodule E of M, then a submodule $T \neq M$ is said to be small (in M) and denoted as $T \ll M$. The Jacobson radical of M is defined as the sum of all small submodules of M, denoted as $Rad(M)$. If $E = M$ with $M = T + E$ for every $E \trianglelefteq M$, then $T \leq M$ is called g-small in M, denoted as $T \ll_g M$, see [12]. If every proper submodule of M is g-small, then M called generalized hollow [5]. Obviously, the subclass g-small is generalized of small. Defined Zhou and Zhang [12] the generalized radical of $M \in$ Mod-$\mathfrak{R}$ as follows:

$$Rad_g(M) = \cap\{T \trianglelefteq M \,|\, T \text{ is maximal in } M\} = \sum\{T \,\big|\, T \ll_g M\}.$$

If there exists submodules T and $\grave{T}$ of $M \in$ Mod-$\mathfrak{R}$ such that $M = T \oplus \grave{T}$, $T \leq A$ and $A \cap \grave{T} \ll_g M$, for any submodule $A \leq M$, then M is called g-lifting [8]. In the case of

Received: April 5, 2023; Accepted: April 30, 2023; Published: May 6, 2023

2020 Mathematics Subject Classification: Primary: 16D10, 16D70, 16D80; Secondary: 16D99.

Keywords and phrases: g-small submodule, g-lifting module, (P_g^*)-module.

submodules L_1 and L_2 of M with $L_1+L_2=M$, L_2 is referred to as a g-radical supplement of L_1 if, $L_1 \cap L_2 \subseteq Rad_g(L_2)$. If each submodule of $M \in$ Mod-$\mathfrak{R}$ has a g-radical supplement, then M is called g-radical supplemented see [6]. If each submodule of M has a g-radical supplement which is a direct summand of M, Ghawi [2] calls it $\oplus$-g-radical supplemented. He also introduced the definition of (P_g^*)-modules in the same article, but this was not the author's main concern in the paper. If for any $T \leq M$, there is a direct summand H of M such that $H \leq T$ and $T/H \subseteq Rad_g(M/H)$, then $M \in$ Mod-$\mathfrak{R}$ is said to have (P_g^*) property or, (P_g^*)-module.

In this paper, the detailed study of the notion of (P_g^*)-modules are our interest. Various properties and characterizations of (P_g^*)-modules are obtained in Section 2. We show that direct summands of a (P_g^*)-module are also (P_g^*)-modules. We have the outcome that the factor module of (P_g^*)-module is also (P_g^*)-module. Considering a direct sum of (P_g^*)-modules, we indicate that if $L_1$is semisimple and L_2 a (P_g^*)-module which are relatively projective, then $M = L_1 \oplus L_2$ is a (P_g^*)-module. Some connections between a class of (P_g^*)-modules and some other kinds of modules are discussed, such as g-lifting, $\oplus$-g-radical supplemented and g-radical supplemented modules, in Section 3. We refer the reader to [3] and [11] for unexplained concepts and notations in this work.

2. (P_g^*)-modules

We will start with the next main definition which is established in [2, p.12].

Definition 2.1. $M \in$ Mod-$\mathfrak{R}$ is said to have (P_g^*) property or a (P_g^*)-module if for any $T \leq M$, there is a $H \leq^{\oplus} M$ such that $H \leq T$ and $T/H \subseteq Rad_g(M/H)$.

Consider the following consequence.

Proposition 2.2. *Let $M \in$ Mod-$\mathfrak{R}$ have (P_g^*) property and $T \leq M$. Then there is a direct summand $X \leq M$ and a submodule Y of M such that $X \leq T$, $T = X + Y$ and $Y \subseteq Rad_g(M)$.*

Proof. If M is a (P_g^*)-module and $T \leq M$, then there exists a decomposition $M = L_1 \oplus L_2$, $L_1 \leq T$ and $T/L_1 \subseteq Rad_g(M/L_1)$. Then $T = T \cap M = T \cap (L_1 + L_2) = L_1 + (T \cap L_2)$. Put $Y = T \cap L_2$, so that $T = L_1 + Y$. Since $M/L_1 \cong L_2$, we deduce that $\varphi: M/L_1 \to L_2$ is an $\mathfrak{R}$-isomorphism. Since $T/L_1 \subseteq Rad_g(M/L_1)$, we have that $T \cap L_2 = \varphi(T/L_1) \subseteq \varphi(Rad_g(M/L_1)) \subseteq Rad_g(L_2)$, hence $Y \subseteq Rad_g(M)$. □

We will present a characteristic for (P_g^*)-modules in the following.

Theorem 2.3. *Let* $M \in$ Mod-$\mathfrak{R}$. *Then the following are equivalent.*

(1) M *have* (P_g^*) *property.*

(2) *For each* $L \leq M$, *there is a decomposition* $M = L_1 \oplus L_2$, $L_1 \leq L$ *and* $L \cap L_2 \subseteq Rad_g(L_2)$.

Proof. (1) $\Longrightarrow$ (2) Let $L \leq M$, so by (1), there is a direct summand L_1 of M such that $L_1 \leq L$ and $L/L_1 \subseteq Rad_g(M/L_1)$. Thus $M = L_1 \oplus L_2$ for some $L_2 \leq M$. We deduce that $L = L_1 \oplus (L \cap L_2)$. As $M/L_1 \cong M_2$, we have $\alpha: M/L_1 \to L_2$ is an $\mathfrak{R}$-isomorphism. Since $L/L_1 \subseteq Rad_g(M/L_1)$, we deduce that $L \cap L_2 = \alpha((L_1 \oplus (L \cap L_2))/L_1) = \alpha(L/L_1) \subseteq \alpha(Rad_g(M/L_1)) \subseteq Rad_g(L_2)$.

(2) $\Longrightarrow$ (1) Assume $L \leq M$. Then there exists submodules L_1 and L_2 of $M \in$Mod-$\mathfrak{R}$ such that $M = L_1 \oplus L_2$, $L_1 \leq L$ and $L \cap L_2 \subseteq Rad_g(L_2)$. We have that $L = L_1 \oplus (L \cap L_2)$. As $L_2 \cong M/L_1$, so there exists an $\mathfrak{R}$-isomorphism $\varphi: L_2 \to M/L_1$. Since $L \cap L_2 \subseteq Rad_g(L_2)$, we have $L/L_1 = (L_1 \oplus (L \cap L_2))/L_1 = \varphi(L \cap L_2) \subseteq \varphi(Rad_g(L_2)) \subseteq Rad_g(M/L_1)$. So M have (P_g^*) property. □

Proposition 2.4. *Let* $M \in$Mod-$\mathfrak{R}$ *have* (P_g^*) *property. For any* $T \leq M$, *there exists a g-radical supplement* B *in* M *such that* $T \cap B \leq^{\oplus} T$.

Proof. Let $T \leq M$. Since M is a (P_g^*)-module, Theorem 2.3 implies $M = A \oplus B = T + B$, $A \leq T$ and $T \cap B \subseteq Rad_g(B)$, this means $B \leq M$ is a g-radical supplement of T. Also, we deduce that $T = T \cap M = T \cap (A \oplus B) = A \oplus (T \cap B)$, as required. □

The following lemma must be proved.

Lemma 2.5. *Let* $M \in$ Mod-$\mathfrak{R}$ *and* $X \leq T \leq^{\oplus} M$. *If* $X \subseteq Rad_g(M)$, *then* $X \subseteq Rad_g(T)$.

Proof. Let $x \in X$. Then $x \in Rad_g(M)$ and so $x\mathfrak{R} \ll_g M$ by [5, Lemma 5]. Since $x\mathfrak{R} \leq T \leq^{\oplus} M$, [2, Lemma 2.12] imply $x\mathfrak{R} \ll_g T$, then $x\mathfrak{R} \subseteq Rad_g(T)$, and $x \in Rad_g(T)$. So, $X \subseteq Rad_g(T)$. □

Proposition 2.6. *For* $M \in$ Mod-$\mathfrak{R}$, *we consider the following*:

(1) M *have* (P_g^*) *property.*

(2) *There is a decomposition* $M = L_1 \oplus L_2$, $L_1 \leq T$ *and* $T \cap L_2 \subseteq Rad_g(M)$, *for each* $T \leq M$,

(3) *Each submodule* T *of* M *can be written as* $T = T_1 \oplus T_2$, $T_1 \leq^{\oplus} M$ *and* $T_2 \subseteq Rad_g(M)$.

Then (1) $\Leftrightarrow$ (2) $\Longrightarrow$ (3).

Proof. (1) $\Longrightarrow$ (2) It is directly follows by Theorem 2.3.

(2) $\Longrightarrow$ (1) Assume that $T \leq M$. From (2), there exists submodules L_1 and L_2 of M such that $M = L_1 \oplus L_2$, $L_1 \leq T$ and $T \cap L_2 \subseteq Rad_g(M)$. Since $T \cap L_2 \leq L_2 \leq^{\oplus} M$, Lemma 2.5 implies $T \cap L_2 \subseteq Rad_g(L_2)$, thus (1) holds, by Theorem 2.3.

(2) $\Longrightarrow$ (3) Let $T \leq M$. There is a decomposition $M = L_1 \oplus L_2$, $L_1 \leq T$ and $T \cap L_2 \subseteq Rad_g(M)$. We can conclude that $T = L_1 \oplus X$, $L_1 \leq^{\oplus} M$ and $X \subseteq Rad_g(M)$ by putting $X = T \cap L_2$. □

The corollary that follows is obvious.

Corollary 2.7. *Each generalized hollow module have* (P_g^*) *property.*

A submodule $3\mathbb{Z}_{24}$ is a direct summand in $\mathbb{Z}$-module $\mathbb{Z}_{24}$ (not g-small). While, all other proper submodules of $\mathbb{Z}_{24}$, on the other hand, are g-small as $\mathbb{Z}$-module. This indicates that $\mathbb{Z}$-module $\mathbb{Z}_{24}$ is a (P_g^*)-module, but not generalized hollow.

The following is established in [4, Lemma 2.3].

Lemma 2.8. *Let* $M \in$ Mod-$\mathfrak{R}$ *and* $T \leq M$ *such that* M/T *projective. If* $D \leq^{\oplus} M$ *with* $M = D + T$, *then* $D \cap T \leq^{\oplus} M$.

Proposition 2.9. *Let* $M \in$Mod-$\mathfrak{R}$ *have* (P_g^*) *property and* $T \leq M$. *If* M/T *is projective, then* T *have* (P_g^*) *property.*

Proof. Suppose $D \leq T$, there exists submodules L_1 and L_2 of M such that $M = L_1 \oplus L_2$, $L_1 \leq D$ and $D \cap L_2 \subseteq Rad_g(M)$. Thus, $M = T + L_2$, and so $T \cap L_2 \leq^{\oplus} M$, by Lemma 2.8. Also, we have $T = L_1 \oplus (T \cap L_2)$ and $D \cap (T \cap L_2) = D \cap L_2 \subseteq Rad_g(T \cap L_2)$, by Lemma 2.5. □

If $g(T) \subseteq T$ for all $g \in End(M)$, then a submodule T of an $\mathfrak{R}$-module M is called fully invariant. Recall [7] that $M \in$ Mod-$\mathfrak{R}$ is called duo if every submodule of M is fully invariant. And T is called to be weak distributive if $T = (T \cap L_1) + (T \cap L_2)$ for all submodules L_1, L_2 of M with $L_1 + L_2 = M$. If all its submodules of $M \in$ Mod-$\mathfrak{R}$ are weak distributive, the module M is called to be weakly distributive, as shown in [1].

Lemma 2.10. *Let T be a fully invariant submodule of $M = L_1 \oplus L_2$ for some submodules L_1 and L_2 of M. Then $M/T = ((L_1 + T)/T) \oplus ((L_2 + T)/T)$.*

Proof. See [10, Lemma 3.3]. □

However, we arrive at the following conclusion.

Proposition 2.11. *Let $M \in$ Mod-$\mathfrak{R}$ have (P_g^*) property. Then*

(1) *M/H have (P_g^*) property, for each fully invariant submodule H of M.*

(2) *M/H have (P_g^*) property, for each weak distributive submodule H of M.*

Proof. (1) Let H be a fully invariant submodule of M and $U/H \leq M/H$, where $H \leq U$. Since $U \leq M$, there exists submodules L_1 and L_2 of M such that $M = L_1 \oplus L_2$, $L_1 \leq U$ and $U \cap L_2 \subseteq Rad_g(L_2)$. By Lemma 2.10, $M/H = ((L_1 + H)/H) \oplus ((L_2 + H)/H)$. As $L_1 \leq U$, so $(L_1 + H)/H \leq U/H$. Now, define the natural $\mathfrak{R}$-epimorphism map $\pi: L_2 \to (L_2 + H)/H$. As $U \cap L_2 \subseteq Rad_g(L_2)$, then $\pi(U \cap L_2) \subseteq \pi(Rad_g(L_2)) \subseteq Rad_g((L_2 + H)/H)$, but $(U/H) \cap ((L_2 + H)/H) = \pi(U \cap L_2)$, we deduce that $(U/H) \cap ((L_2 + H)/H) \subseteq Rad_g((L_2 + H)/H)$. Hence M/H is a (P_g^*)-module.

(2) Let H be a weak distributive submodule of M and $U/H \leq M/H$, where $H \leq U$. Then there is a decomposition $M = L_1 \oplus L_2, L_1 \leq U$ and $U \cap L_2 \subseteq Rad_g(L_2)$. We deduce that $H = (H \cap L_1) + (H \cap L_2)$. Also, $M/H = (L_1 + H)/H + (L_2 + H)/H$. We conclude that $(L_1 + H) \cap (L_2 + H) = \left(L_1 + (H \cap L_2)\right) \cap (L_2 + H) = (L_1 \cap L_2) + (H \cap L_2) + H = H$, that implies $(L_1 + H)/H \cap (L_2 + H)/H = 0$. Therefore, $((L_1 + H)/H) \oplus ((L_2 + H)/H) = M/H$. Then we continue with the same steps to proof (1). □

Corollary 2.12. *For a duo (or, a weakly distributive) module have (P_g^*) property, every factor module have (P_g^*) property.*

Corollary 2.13. *If $M \in$ Mod-$\mathfrak{R}$ have (P_g^*) property, then so is $M/Rad_g(M)$.*

Proof. We have $Rad_g(M)$ is a fully invariant submodule of M by [12, Corollary 2.11], hence the result is follows by Proposition 2.11(1). □

The following proposition shows that the property (P_g^*) for modules is inherited by its direct summands.

Proposition 2.14. *A direct summand of a* (P_g^*)*-module is so a* (P_g^*)*-module.*

Proof. Let $D \leq^{\oplus} M$ and M is a (P_g^*)-module. If $X \leq D$, then there exists submodules L_1 and L_2 of M with $L_1 \leq X$ and $X \cap L_2 \subseteq Rad_g(L_2)$, where $M = L_1 \oplus L_2$. We have $D = L_1 \oplus (D \cap L_2)$. It is easily to see that $D \cap L_2 \leq^{\oplus} L_2$. From $X \cap (D \cap L_2) \leq X \cap L_2 \subseteq Rad_g(L_2)$, Lemma 2.5 implies $X \cap (D \cap L_2) \subseteq Rad_g(D \cap L_2)$. The proof is now complete. □

Proposition 2.15. *Let $M = \oplus_{i\in I} L_i$ be a duo module. Then L_i is a* (P_g^*)*-module for $i \in I$, if and only if M is a* (P_g^*)*-module.*

Proof. Let L_i be a (P_g^*)-module for $i \in I$, and let T be a submodule of $M = \oplus_{i\in I} L_i$. As T is fully invariant, then by [7, Lemma 2.1] $T = \oplus_{i\in I}(T \cap L_i)$. Since $T \cap L_i \leq L_i$ for $i \in I$, there exists decompositions $L_i = H_i \oplus \grave{H}_i$ such that $H_i \leq T \cap L_i$ and $(T \cap L_i) \cap \grave{H}_i = T \cap \grave{H}_i \subseteq Rad_g(\grave{H}_i)$. We have that $M = (\oplus_{i\in I} H_i) \oplus (\oplus_{i\in I} \grave{H}_i)$, $\oplus_{i\in I} H_i \leq \oplus_{i\in I}(T \cap L_i) = T$ and $T \cap \left(\oplus_{i\in I} \grave{H}_i\right) = \oplus_{i\in I}(T \cap \grave{H}_i) \subseteq \oplus_{i\in I}(Rad_g(\grave{H}_i)) = Rad_g(\oplus_{i\in I} \grave{H}_i)$ by [9, Corollary 2.3]. Thus M is a (P_g^*)-module. Conversely, is follows directly from Proposition 2.14. □

Theorem 2.16. *Let L_1 be a semisimple module and L_2 have* (P_g^*) *property which are relatively projective. Then $M = L_1 \oplus L_2$ is a* (P_g^*)*-module.*

Proof. Let $(0 \neq)X \leq M$, and let $T = L_1 \cap (X + L_2)$. We have two cases:

Case (i) If $T \neq 0$. Since $T \leq L_1$, there is a submodule $T_1$of L_1 such that $L_1 = T \oplus T_1$, and hence $M = T \oplus T_1 \oplus L_2 = X + (L_2 \oplus T_1)$. Thus T is $L_2 \oplus T_1$-projective. By [11, 41.14], there is $L \leq X$ such that $M = L \oplus (L_2 \oplus T_1)$. We may assume $X \cap (L_2 \oplus T_1) \neq 0$. It is easy to see that $X \cap (K + T_1) = K \cap (X + T_1)$ for any $K \leq L_2$. Specially, $X \cap (L_2 + T_1) = L_2 \cap (X + T_1)$. Thus, $X = L \oplus \left(X \cap (L_2 \oplus T_1)\right) = L \oplus \left(L_2 \cap (X + T_1)\right)$. As L_2 is a (P_g^*)-module, there is a decomposition $L_2 = U_1 \oplus U_2$ such that $U_1 \leq L_2 \cap (X + T_1)$ and $U_2 \cap (X + T_1) \subseteq Rad_g(U_2)$. We conclude that $M = (L \oplus U_1) \oplus (U_2 \oplus T_1)$. We have $L \oplus U_1 \leq X$ and $X \cap (U_2 \oplus T_1) = U_2 \cap (X + T_1) \subseteq Rad_g(M)$. From $U_2 \oplus T_1 \leq^{\oplus} M$, we deduce that $X \cap (U_2 \oplus T_1) \subseteq Rad_g(U_2 \oplus T_1)$ by Lemma 2.5.

Case (ii) If $T = 0$, we get $X \leq L_2$. Since L_2 is a (P_g^*)-module, there is a submodule $U_1 \leq X$, $L_2 = U_1 \oplus U_2$ and $X \cap U_2 \subseteq Rad_g(U_2)$ for a submodule $U_2 \leq L_2$. Thus,

$M = U_1 \oplus (L_1 \oplus U_2)$ and $X \cap (L_1 \oplus U_2) = X \cap U_2 \subseteq Rad_g(M)$. Again by Lemma 2.5, we deduce that $X \cap (L_1 \oplus U_2) \subseteq Rad_g(L_1 \oplus U_2)$, and the proof is now complete. □

Lemma 2.17. *Let $M \in$ Mod-$\mathfrak{R}$ have (P_g^*) property and $T \leq M$. Then T is semisimple, whenever $T \cap Rad_g(M) = 0$.*

Proof. Let $E \leq T$. Since M is a (P_g^*)-module, there exists submodules L_1 and L_2 of M such that $M = L_1 \oplus L_2$, $L_1 \leq E$ and $E \cap L_2 \subseteq Rad_g(L_2)$, and then $E \cap L_2 \subseteq Rad_g(M)$. We deduce that $E = E \cap M = E \cap (L_1 \oplus L_2) = L_1 \oplus (E \cap L_2)$. Because $E \cap L_2 \subseteq T \cap Rad_g(M) = 0$, we have $E = L_1$ and E a direct summand of M. Therefore $E \leq^{\oplus} T$ and T a semisimple. □

Theorem 2.18. *Let $M \in$ Mod-$\mathfrak{R}$ have (P_g^*) property. Then M has a decomposition $M = L_1 \oplus L_2$, where L_1 is semisimple and $Rad_g(L_2) \trianglelefteq L_2$.*

Proof. Since $Rad_g(M) \leq M$, there is a submodule T of M with $T \oplus Rad_g(M)$ is large in M. As $T \cap Rad_g(M) = 0$, Lemma 2.17 implies T is semisimple. Since M have (P_g^*) property, there exists submodules L_1 and L_2 of M such that $M = L_1 \oplus L_2$, $L_1 \leq T$ and $T \cap L_2 \subseteq Rad_g(L_2)$. As $T \cap L_2 \subseteq T \cap Rad_g(M) = 0$, so that $M = T \oplus L_2$. Clearly, $Rad_g(T) = 0$. Thus $Rad_g(M) = Rad_g(L_2)$, this means $T \oplus Rad_g(L_2) \trianglelefteq T \oplus L_2$, and hence $Rad_g(L_2) \trianglelefteq L_2$. □

3. (P_g^*)-module and Other Related Concepts

Our purpose throughout this section is to demonstrate some relations between the concept of (P_g^*)-module and other types of modules.

Proposition 3.1. *Let $M \in$ Mod-$\mathfrak{R}$ be a module. Consider the following*:

(1) *M is semisimple.*

(2) *M has (P_g^*) property.*

(3) *Each direct summand of M is $\oplus$-g-radical supplemented.*

(4) *M is $\oplus$-g-radical supplemented.*

(5) *M is g-radical supplemented.*

Then (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) $\Longrightarrow$ (5). *Moreover*, (5) $\Longrightarrow$ (1) *if* $Rad_g(M) = 0$.

Proof. (1) $\Longrightarrow$ (2) and (3) $\Longrightarrow$ (4) $\Longrightarrow$ (5) Clear.

(2) $\Longrightarrow$ (3) According to Proposition 2.14.

(5) $\Longrightarrow$ (1) Let $T \leq M$ with $Rad_g(M) = 0$. By (5), there exists $E \leq M$ such that $M = T + E$ and $T \cap E \subseteq Rad_g(E)$. We conclude that $T \cap E = 0$, from $Rad_g(E) \subseteq Rad_g(M)$. Thus, $M = T \oplus E$, and (1) holds. □

Proposition 3.2. *If* $M \in$ Mod-$\mathfrak{R}$ *such that* $Rad_g(M) = M$, *then* M *have* (P_g^*) *property.*

Proof. It is easy to check. □

Example 3.3. For any prime number p, and a positive integer $n > 1$. The $\mathbb{Z}$-module $\mathbb{Z}_{p^n}$ is generalized hollow, thus it is a (P_g^*)-module, but $Rad_g(\mathbb{Z}_{p^n}) = p\mathbb{Z}_{p^n} \neq \mathbb{Z}_{p^n}$. In general, this indicates that the reverse of Proposition 3.2 does not true.

Proposition 3.4. *Let* $M \in$Mod-$\mathfrak{R}$ *be indecomposable* (*non-cyclic*). *If* M *have* (P_g^*) *property*, *then* $Rad_g(M) = M$.

Proof. Let $m \in M$. Since M have (P_g^*) property, there exists submodules L_1 and L_2 of M such that $M = L_1 \oplus L_2$, $L_1 \leq m\mathfrak{R}$ and $m\mathfrak{R} \cap L_2 \subseteq Rad_g(L_2)$. Hence, either $L_1 = M$ or $L_1 = 0$. If $L_1 = M$, then $M = m\mathfrak{R}$ which is a contradiction. Thus, $L_1 = 0$ and $L_2 = M$. We deduce that $m \in m\mathfrak{R} \subseteq Rad_g(M)$. The proof is now complete. □

As an application example of Proposition 3.4, we know that $Rad_g(\mathbb{Q}) = \mathbb{Q}$, in fact $\mathbb{Q}$ has (P_g^*) property as $\mathbb{Z}$-module, and it is indecomposable and non-cyclic.

The following is immediately from Propositions 3.2 and 3.4.

Corollary 3.5. *Let* $M \in$ Mod-$\mathfrak{R}$ be *indecomposable* (*non-cyclic*). *Then* M *have* (P_g^*) *property if and only if* $Rad_g(M) = M$.

Proposition 3.6. *Every g-lifting module is a* (P_g^*)*-module. The reverse is true if, a module has a g-small generalized radical.*

Proof. The necessity is clear. Conversely, if $T \leq M$, so there is a decomposition $M = L_1 \oplus L_2$, $L_1 \leq T$ and $T \cap L_2 \subseteq Rad_g(L_2)$, and then $T \cap L_2 \subseteq Rad_g(M)$. As $Rad_g(M) \ll_g M$, we deduce that $T \cap L_2 \ll_g M$. From $T \cap L_2 \leq L_2 \leq^{\oplus} M$, [2, Lemma 2.12] implies $T \cap L_2 \ll_g L_2$. Therefore M is g-lifting. □

If M/T is finitely generated, then a submodule T of $M \in$ Mod-$\mathfrak{R}$ is called cofinite.

Proposition 3.7. *If* $M \in$ Mod-$\mathfrak{R}$ *is finitely generated, then the following are equivalent.*

(1) M *is g-lifting.*

(2) M *have* (P_g^*) *property.*

(3) *There is a decomposition* $M = L_1 \oplus L_2$ *such that* $L_1 \leq T$ *and* $T \cap L_2 \subseteq Rad_g(M)$, *for each cofinite submodule* T *of* M.

Proof. (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) Obvious.

(3) $\Longrightarrow$ (1) Let $T \leq M$. Since M is finitely generated, so is M/T, that is T is cofinite. By (3), then there exists submodules L_1 and L_2 of M such that $M = L_1 \oplus L_2$, $L_1 \leq T$ and $T \cap L_2 \subseteq Rad_g(M)$. From [2, Lemma 5.4] $Rad_g(M) \ll_g M$. So we have $T \cap L_2 \ll_g M$. Since $L_2 \leq^{\oplus} M$, hence $T \cap L_2 \ll_g L_2$ by [2, Lemma 2.12]. □

Corollary 3.8. *For a Noetherian* $M \in$ Mod-$\mathfrak{R}$, *the following are equivalent.*

(1) M *is g-lifting.*

(2) M *have* (P_g^*) *property.*

(3) *There is a decomposition* $M = L_1 \oplus L_2$ *such that* $L_1 \leq T$ *and* $T \cap L_2 \subseteq Rad_g(M)$, *for each cofinite submodule* T *of* M.

Proposition 3.9. *Let* $M \in$ Mod-$\mathfrak{R}$ *be nonzero indecomposable with* $Rad_g(M) \neq M$. *Then* M *is g-lifting if and only if it has* (P_g^*) *property.*

Proof. The necessity is clear. Conversely, let $E \trianglelefteq M$ with $Rad_g(M) + E = M$. So there exists a decomposition $M = L_1 \oplus L_2$, $L_1 \leq E$ and $E \cap L_2 \subseteq Rad_g(M)$. As M is indecomposable, either $L_2 = M$ or $L_2 = 0$. If $L_2 = M$ and $E \subseteq Rad_g(M)$, then $Rad_g(M) = M$, a contradiction. Thus, $L_1 = M$ and $L_2 = 0$. We deduce that $E = M$ and $Rad_g(M) \ll_g M$. Therefore, by Proposition 3.6, M is g-lifting. □

Theorem 3.10. C*onsider the following assertions for* $M \in$ Mod-$\mathfrak{R}$:

(1) M *have* (P_g^*) *property.*

(2) *Each direct summand of* M *is* $\oplus$-*g-radical supplemented.*

(3) M *is* $\oplus$-*g-radical supplemented.*

(4) M *is g-radical supplemented.*

Then $(1) \Longrightarrow (2) \Longrightarrow (3) \Longrightarrow (4)$. *If* $M \in$ Mod-$\mathfrak{R}$ *is projective, and every g-radical supplement submodule of* M *is a direct summand, then* $(4) \Longrightarrow (1)$.

Proof. $(1) \Longrightarrow (2)$ By Proposition 3.1.

$(2) \Longrightarrow (3) \Longrightarrow (4)$ Obvious.

$(4) \Longrightarrow (1)$ If $T \leq M$. According (4), M has a submodule E, $M = T + E$ and $T \cap E \subseteq Rad_g(E)$. By hypothesis, E is a direct summand of M, and so $M = A \oplus E$ for some $A \leq M$. Since $M = A \oplus E = T + E$ is projective, [11, 41.14] imply $M = \grave{E} \oplus E$ such that $\grave{E} \leq T$, and so (1) holds. □

Corollary 3.11. *Let* $M \in$ Mod-$\mathfrak{R}$ *be projective whose each g-radical supplement submodule is a direct summand of* M. *If* $Rad_g(M) \ll_g M$, *then the following five assertions are equivalent.*

(1) M *is g-lifting.*

(2) M *have* (P_g^*) *property.*

(3) *Each direct summand of* M *is* $\oplus$*-g-radical supplemented.*

(4) M *is* $\oplus$*-g-radical supplemented.*

(5) M *is g-radical supplemented.*

Proof. From Proposition 3.6 and Theorem 3.10. □

Corollary 3.12. *If* $M \in$ Mod-$\mathfrak{R}$ *is finitely generated projective whose each g-radical supplement submodule is a direct summand of* M. *Then the following are equivalent.*

(1) M *is g-lifting.*

(2) M *have* (P_g^*) *property.*

(3) *Each direct summand of* M *is* $\oplus$*-g-radical supplemented.*

(4) M *is* $\oplus$*-g-radical supplemented.*

(5) M *is g-radical supplemented.*

Proof. From [2, Lemma 5.4] and Corollary 3.11. □

Acknowledgments

The author would like to thank the referees for their valuable suggestions and comments.

References

[1] E. Büyükaşik and Y. Demirci, Weakly distributive modules. Applications to supplement submodules, *Proc. Indian Acad. Sci.* (*Math. Sci.*) 120 (2010), 525-534. https://doi.org/10.1007/s12044-010-0053-9

[2] T. Y. Ghawi, Some generalizations of g-lifting modules, *Journal of Al-Qadisiyah for Computer Science and Mathematics* 15(1) (2023), 109-121.

[3] K. R. Goodearl, *Ring Theory, Nonsingular Rings and Modules*, Dekker, New York, 1976.

[4] D. Keskin and W. Xue, Generalizations of lifting modules, *Acta Math. Hungar.* 91(3) (2001), 253-261.

[5] B. Koşar, C. Nebiyev and N. Sökmez, G-supplemented modules, *Ukrainian Math. J.* 67(6) (2015), 861-864.

[6] B. Koşar, C. Nebiyev and A. Pekin, A generalization of g-supplemented modules, *Miskolc Mathematical Notes* 20(1) (2019), 345-352. https://doi.org/10.18514/mmn.2019.2586

[7] A. C. Özcan, A. Harmanci and P. F. Smith, Duo modules, *Glasgow Math. J.* 48(3) (2006), 533-545. https://doi.org/10.1017/s0017089506003260

[8] T. C. Quynh and P. H. Tin, Some properties of *e*-supplemented and *e*-lifting modules, *Vietnam J. Math.* 41(3) (2013), 303-312. https://doi.org/10.1007/s10013-013-0022-6

[9] L. V. Thuyet and P. H. Tin, Some characterizations of modules via essentially small submodules, *Kyungpook Math. J.* 56 (2016), 1069-1083. https://doi.org/10.5666/kmj.2016.56.4.1069

[10] B. Ungor, S. Halicioglu and A. Harmancı, On a class of δ-supplemented modules, *Bull. Malays. Math. Sci. Soc.* 37(3) (2014), 703-717.

[11] R. Wisbauer, *Foundations of Module and Ring Theory*, Gordon and Breach, Reading, 1991.

[12] D. X. Zhou and X. R. Zhang, Small-essential submodules and Morita duality, *Southeast Asian Bull. Math.* 35 (2011), 1051-1062.