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Differential Subordination Results for a Family of Sakaguchi Type Functions

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Abstract

The object of the present work is to introduce and study a new family $G(\eta, m, n; h)$ of analytic functions defined by Sakaguchi type functions in the open unit disk. We obtain some subordination results for this family.

1. Introduction and Preliminaries

Let $\mathcal{A}$ denote the family of functions f of the form:

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j, \quad (z \in U), \tag{1.1}$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C}: |z| < 1\}$.

A function $f \in \mathcal{A}$ is called starlike of order α in U if and only if

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (0 \leq \alpha < 1\,; z \in U).$$

Denote the family of all starlike functions of order α in U by $S^*(\alpha)$.

A function $f \in \mathcal{A}$ is called starlike with respect to symmetrical point, if (see [9])

$$Re\left\{\frac{zf'(z)}{f(z) - f(-z)}\right\} > 0, \quad z \in U.$$

The set of all such functions is denoted by S_s^*.

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Frasin [2] introduced and studied the family $S(\gamma, m, n)$ consisting of functions $f \in \mathcal{A}$ which satisfy the condition

$$Re\left\{\frac{(m-n)zf'(z)}{f(mz)-f(nz)}\right\} > \gamma,$$

for some $0 \leq \gamma < 1,\ m, n\ \in \mathbb{C}$ with $m \neq n, |m| \leq 1, |n| \leq 1$ and for $z \in U$.

We note that the family $\mathrm{S}(\gamma, 1, n)$ was studied by Owa et al. (see [6]), while the family $\mathrm{S}(\gamma, 1, -1) \equiv S_s(\gamma)$ was considered by Sakaguchi (see [9]) and is called the Sakaguchi function of order γ. Also $\mathrm{S}(0,1,-1) \equiv S_s$ is the family of starlike functions with respect to symmetrical points in U and $\mathrm{S}(\gamma, 1,0) \equiv S_s(\gamma)$ is the family of starlike functions of order $\gamma, 0 \leq \gamma < 1$.

The Hadamard product (or convolution) $(f_1 * f_2)(z)$ of two functions

$$f_q(z) = z + \sum_{j=2}^{\infty} a_{n,q} z^j\ \in \mathcal{A} \quad (q = 1,2)$$

is given by

$$(f_1 * f_2)(z) = z + \sum_{j=2}^{\infty} a_{j,1} a_{j,2} z^j\ .$$

A function $f \in \mathcal{A}$ is said to be prestarlike of order α in U if

$$\frac{z}{(1-z)^{2(1-\alpha)}} * f(z) \in S^*(\alpha) \quad (\alpha < 1).$$

Denote the family of all prestarlike functions of order α in U by $\Re(\alpha)$.

Clearly a function $f \in \mathcal{A}$ is in the family $\Re(0)$ if and only if f is convex univalent in U and $\Re\left(\frac{1}{2}\right) = \mathrm{S}^*\left(\frac{1}{2}\right)$.

Let H be the family of functions h with $h(0) = 1$, which are analytic and convex univalent in U.

For two functions f and g analytic in U, we say that the function f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)\ (z \in U)$, if there exists a Schwarz function $w(z)$ analytic in U with $w(0) = 0$ and $|w(z)| < 1\ (z \in U)$ such that $f(z) = g(w(z))$. In particular, if the function g is univalent in U, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

To prove our main results, we will require the following Lemmas.

Lemma 1.1 [5]. *Let g be analytic in U and let h be analytic and convex univalent in U with $h(0) = g(0)$. If*

$$g(z) + \frac{1}{\mu} z g'(z) \prec h(z), \tag{1.2}$$

where $Re\, \mu \geq 0$ and $\mu \neq 0$, then

$$g(z) \prec \tilde{h}(z) = \mu z^{-\mu} \int_0^z t^{\mu-1} h(t) dt \prec h(z)$$

and $\tilde{h}(z)$ is the best dominant of (1.2).

Lemma 1.2 [8]. *Let $\alpha < 1, f \in S^*(\alpha)$ and $g \in \Re(\alpha)$. Then, for any analytic function F in U,*

$$\frac{g * (fF)}{g * f}(U) \subset \overline{co}\big(F(U)\big),$$

where $\overline{co}\big(F(U)\big)$ denotes the closed convex hull of $F(U)$.

Such type of study was carried out by various authors for another classes, like, Liu [3,4], Prajapat and Raina [7], Yang et al. [12], Atshan and Wanas [1], Wanas and Majeed [10] and Wanas and Pall-Szabo [11].

2. Main Results

Definition 2.1. A function $f \in \mathcal{A}$ is said to be in the class $G(\eta, m, n; h)$ if it satisfies the subordination condition:

$$(1-\eta)\left(\frac{f(mz) - f(nz)}{(m-n)z}\right) + \eta\left(\frac{f'(mz) - f'(nz)}{(m-n)}\right) \prec h(z),$$

where $\eta \in \mathbb{C}$ and $h \in H$.

Theorem 2.1. *Let $0 \leq \eta < \xi$. Then $G(\xi, m, n; h) \subset G(\eta, m, n; h)$.*

Proof. Let $0 \leq \eta < \xi$ and $f \in G(\xi, m, n; h)$.

Suppose that

$$g(z) = \frac{f(mz) - f(nz)}{(m-n)z}. \tag{2.1}$$

Then, the function g is analytic in U with $g(0) = 1$.

Since $f \in G(\xi, m, n; h)$, we have

$$(1-\xi)\left(\frac{f(mz)-f(nz)}{(m-n)z}\right)+\xi\left(\frac{f'(mz)-f'(nz)}{m-n}\right) \prec h(z), \tag{2.2}$$

From (2.1) and (2.2), we get

$$(1-\xi)\left(\frac{f(mz)-f(nz)}{(m-n)z}\right)+\xi\left(\frac{f'(mz)-f'(nz)}{m-n}\right) = g(z)+\xi z g'(z) \prec h(z). \tag{2.3}$$

An application of Lemma 1.1, we obtain

$$g(z) \prec h(z). \tag{2.4}$$

Noting that $0 \leq \frac{\eta}{\xi} < 1$ and that h is convex univalent in U, it follows from (2.1), (2.3) and (2.4) that

$$(1-\eta)\left(\frac{f(mz)-f(nz)}{(m-n)z}\right)+\eta\left(\frac{f'(mz)-f'(nz)}{m-n}\right)$$

$$=\frac{\eta}{\xi}\left((1-\xi)\left(\frac{f(mz)-f(nz)}{(m-n)z}\right)+\xi\left(\frac{f'(mz)-f'(nz)}{m-n}\right)\right)+\left(1-\frac{\eta}{\xi}\right)g(z) \prec h(z).$$

Therefore $f \in G(\eta, m, n; h)$ and we obtain the result.

Theorem 2.2. *Let* $f \in G(\eta, m, n; h), g \in \mathcal{A}$ *and*

$$Re\left\{\frac{g(z)}{z}\right\} > \frac{1}{2}\,. \tag{2.5}$$

Then

$$f * g \in G(\eta, m, n; h).$$

Proof. Let $f \in G(\eta, m, n; h)$ and $g \in \mathcal{A}$. Then, we have

$$(1-\eta)\left(\frac{(f*g)(mz)-(f*g)(nz)}{(m-n)z}\right)+\eta\left(\frac{(f*g)'(mz)-(f*g)'(nz)}{m-n}\right)$$

$$=(1-\eta)\left(\frac{g(z)}{z}\right)*\left(\frac{f(mz)-f(nz)}{(m-n)z}\right)+\eta\left(\frac{g(z)}{z}\right)*\left(\frac{f'(mz)-f'(nz)}{m-n}\right)$$

$$=\left(\frac{g(z)}{z}\right)*\psi(z), \tag{2.6}$$

where

$$\psi(z) = (1-\eta)\left(\frac{f(mz)-f(nz)}{(m-n)z}\right) + \eta\left(\frac{f'(mz)-f'(nz)}{m-n}\right) \prec h(z). \tag{2.7}$$

From (2.5), note that the function $\frac{g(z)}{z}$ has the Herglotz representation

$$\frac{g(z)}{z} = \int_{|x|=1} \frac{d\mu(x)}{1-xz} \quad (z \in U), \tag{2.8}$$

where $\mu(x)$ is a probability measure defined on the unit circle $|x| = 1$ and

$$\int_{|x|=1} d\mu(x) = 1.$$

Since h is convex univalent in U, it follows from (2.6) to (2.8) that

$$(1-\eta)\left(\frac{(f*g)(mz)-(f*g)(nz)}{(m-n)z}\right) + \eta\left(\frac{(f*g)'(mz)-(f*g)'(nz)}{m-n}\right)$$

$$= \int_{|x|=1} \psi(xz)\, d\mu(x) \prec h(z).$$

Therefore, $f * g \in G(\eta, m, n; h)$.

Theorem 2.3. *Let* $f \in G(\eta, m, n; h)$ *and let* $g \in \mathcal{A}$ *be prestarlike of order* $\alpha(\alpha < 1)$. *Then*

$$f * g \in G(\eta, m, n; h).$$

Proof. Let $f \in G(\eta, m, n; h)$ and $g \in \mathcal{A}$. Then, we have

$$(1-\eta)\left(\frac{f(mz)-f(nz)}{(m-n)z}\right) + \eta\left(\frac{f'(mz)-f'(nz)}{m-n}\right) \prec h(z). \tag{2.9}$$

Hence

$$(1-\eta)\left(\frac{(f*g)(mz)-(f*g)(nz)}{(m-n)z}\right) + \eta\left(\frac{(f*g)'(mz)-(f*g)'(nz)}{m-n}\right)$$

$$= (1-\eta)\left(\frac{g(z)}{z}\right) * \left(\frac{f(mz)-f(nz)}{(m-n)z}\right) + \eta\left(\frac{g(z)}{z}\right) * \left(\frac{f'(mz)-f'(nz)}{m-n}\right)$$

$$= \frac{g(z) * (z\psi(z))}{g(z) * z} \quad (z \in U), \tag{2.10}$$

where $\psi(z)$ is defined as in (2.7).

Since h is convex univalent in U, $\psi(z) \prec h(z)$, $g(z) \in \Re(\alpha)$ and $z \in S^*(\alpha)$, $(\alpha < 1)$, it follows from (2.10) and Lemma 1.2, we obtain the result.

Theorem 2.4. *Let* $f \in G(\eta, m, n; h)$ *be defined as in* (1.1). *Then*

$$k(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) dt, \quad (Re(c) > -1)$$

is also in the class $G(\eta, m, n; h)$.

Proof. Let $f \in G(\eta; h)$ be defined as in (1.1). Then, we have

$$(1-\eta)\left(\frac{f(mz) - f(nz)}{(m-n)z}\right) + \eta\left(\frac{f'(mz) - f'(nz)}{m-n}\right) \prec h(z). \tag{2.11}$$

Note that

$$k(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) dt = z + \sum_{j=2}^{\infty} \frac{c+1}{c+n} a_j z^j. \tag{2.12}$$

We find from (2.12) that $k \in \mathcal{A}$ and

$$f(z) = \frac{ck(z) + zk'(z)}{c+1}. \tag{2.13}$$

Define the function p by

$$p(z) = (1-\eta)\left(\frac{k(mz) - k(nz)}{(m-n)z}\right) + \eta\left(\frac{k'(mz) - k'(nz)}{m-n}\right). \tag{2.14}$$

By using (2.13) and (2.14), we get

$$p(z) + \frac{1}{c+1} zp'(z) = \frac{c}{c+1} p(z) + \frac{1}{c+1}\big(zp'(z) + p(z)\big)$$

$$= (1-\eta)\left(\frac{\big(ck(mz) + zk'(mz)\big) - \big(ck(nz) + zk'(nz)\big)}{(m-n)z(c+1)}\right)$$

$$+\eta\left(\frac{\big(ck(mz) + zk'(mz)\big)' - \big(ck(nz) + zk'(nz)\big)'}{(m-n)(c+1)}\right)$$

$$= (1-\eta)\left(\frac{f(mz) - f(nz)}{(m-n)z}\right) + \eta\left(\frac{f'(mz) - f'(nz)}{m-n}\right). \tag{2.15}$$

From (2.11) and (2.15), we arrive at

$$p(z)+\frac{1}{c+1}zp'(z)\prec h(z),\quad (Re(c)>-1).$$

An application of Lemma 1.1, we obtain $p(z)\prec h(z)$. By (2.14), we get

$$(1-\eta)\left(\frac{k(mz)-k(nz)}{(m-n)z}\right)+\eta\left(\frac{k'(mz)-k'(nz)}{m-n}\right)\prec h(z).$$

Therefore, $k\in G(\eta,m,n;h)$.

Theorem 2.5. *Let* $\eta>0,\delta>0$ *and* $f\in G(\eta,m,n;\delta h+1-\delta)$. *If* $\delta\le\delta_0$, *where*

$$\delta_0=\frac{1}{2}\left(1-\frac{1}{\eta}\int_0^1\frac{u^{\frac{1}{\eta}-1}}{1+u}du\right)^{-1}, \tag{2.16}$$

then $f\in G(0,m,n;h)$. *The bound* δ_0 *is the sharp when* $h(z)=\frac{1}{1-z}$.

Proof. Suppose that

$$g(z)=\frac{f(mz)-f(nz)}{(m-n)z}. \tag{2.17}$$

Let $f\in G(\eta,m,n;\delta h+1-\delta)$ with $\eta>0$ and $\delta>0$. Then we have

$$g(z)+\eta zg'(z)=(1-\eta)\left(\frac{f(mz)-f(nz)}{(m-n)z}\right)+\eta\left(\frac{f'(mz)-f'(nz)}{m-n}\right)$$
$$\prec\delta h(z)+1-\delta.$$

An application of Lemma 1.1, we obtain

$$g(z)\prec\frac{\delta}{\eta}z^{-\frac{1}{\eta}}\int_0^z t^{\frac{1}{\eta}-1}h(t)dt+1-\delta=(h*\phi)(z), \tag{2.18}$$

where

$$\phi(z)=\frac{\delta}{\eta}z^{-\frac{1}{\eta}}\int_0^z\frac{t^{\frac{1}{\eta}-1}}{1-t}dt+1-\delta. \tag{2.19}$$

If $0<\delta\le\delta_0$, where $\delta_0>1$ is given by (2.16), then it follows from (2.19) that

$$Re\big(\phi(z)\big)=\frac{\delta}{\eta}\int_0^1 u^{\frac{1}{\eta}-1}Re\left(\frac{1}{1-uz}\right)du+1-\delta>\frac{\delta}{\eta}\int_0^1\frac{u^{\frac{1}{\eta}-1}}{1+u}du+1-\delta\ge\frac{1}{2}.$$

Now, by using the Herglotz representation for $\phi(z)$, from (2.17) and (2.18), we get

$$\frac{f(mz)-f(nz)}{(m-n)z} \prec (h*\phi)(z) \prec h(z).$$

Since h is convex univalent in U, $f \in G(0,m,n;h)$.

For $h(z)=\frac{1}{1-z}$ and $f \in \mathcal{A}$ defined by

$$\frac{f(mz)-f(nz)}{(m-n)z} = \frac{\delta}{\eta} z^{-\frac{1}{\eta}} \int_0^z \frac{t^{\frac{1}{\eta}-1}}{1-t} dt + 1 - \delta,$$

we have

$$(1-\eta)\left(\frac{f(mz)-f(nz)}{(m-n)z}\right) + \eta\left(\frac{f'(mz)-f'(nz)}{m-n}\right) = \delta h(z) + 1 - \delta.$$

Thus, $f \in G(\eta,m,n;\delta h + 1 - \delta)$.

Also, for $\delta > \delta_0$, we have

$$Re\left(\frac{f(mz)-f(nz)}{(m-n)z}\right) \longrightarrow \frac{\delta}{\eta}\int_0^1 \frac{u^{\frac{1}{\eta}-1}}{1+u} du + 1 - \delta < \frac{1}{2} (z \to 1),$$

which implies that $f \notin G(0,m,n;h)$. Therefore, the bound δ_0 cannot be increased when $h(z)=\frac{1}{1-z}$ and this completes the proof of the theorem.

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