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Almost $\eta-$Ricci Solitons on the Pseudosymmetric Lorentzian Para-Kenmotsu Manifolds

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Abstract

In this paper, we consider Lorentzian para-Kenmotsu manifold admitting almost $\eta-$Ricci solitons by virtue of some curvature tensors. Ricci pseudosymmetry concepts of Lorentzian para-Kenmotsu manifolds admitting $\eta-$Ricci soliton have introduced according to the choice of some curvature tensors such as Riemann, concircular, projective, $\mathcal{M}-$projective, W_1 and W_2. After then, according to the choice of the curvature tensors, necessary conditions are given for Lorentzian para-Kenmotsu manifold admitting $\eta-$Ricci soliton to be Ricci semisymmetric. Then some characterizations are given and classifications have made under the some conditions.

1 Introduction

Para-Kenmotsu and special para-Kenmotsu manifolds, also known as almost paracontact metric manifolds, were defined in 1989 by Sinha and Sai Prasad [1].

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Sinha and Sai Prasad obtained important characterizations of para-Kenmotsu manifolds. In the following years, para-Kenmotsu manifolds attracted a lot of attention and many authors revealed the important properties of these manifolds. In 2018, Lorentzian para-Kenmotssu manifolds, known as Lorentzian almost paracontact metric manifolds, were introduced [2]. Then, the concept of $q-$semisymmetry for Lorentzian para-Kenmotsu manifolds is studied [3]. M. Atçeken studied invariant submanifolds of Lorentzian para-Kenmotsu manifolds in 2022 and in this study he gave the necessary and sufficient conditions for the an invarıant submanifold of Lorentzian para-Kenmotsu manifold to be total geodesic [4].

The notion of Ricci flow was introduced by Hamilton in 1982. With the help of this concept, Hamilton found the canonical metric on a smooth manifold. Then Ricci flow has become a powerful tool for the study of Riemannian manifolds, especially for those manifolds with positive curvature. Perelman used Ricci flow and it surgery to prove Poincare conjecture in [5], [6]. The Ricci flow is an flow is an evolution equation for metrics on a Riemannian manifold defined as follows:

$$\frac{\partial}{\partial t}g\left(t\right)=-2S\left(g\left(t\right)\right).$$

A Ricci soliton emerges as the limit of the solitons of the Ricci flow. A solution to the Ricci flow is called Ricci soliton if it moves only by a one parameter group of diffeomorphism and scaling.

During the last two decades, the geometry of Ricci solitons has been the focus of attention of many mathematicians. In particular, it has become more important after Perelman applied Ricci solitons to solve the long standing Poincare conjecture posed in 1904. In [7], Sharma studied the Ricci solitons in contact geometry. Thereafter Ricci solitons in contact metric manifolds have been studied by various authors such as Bagewadi et al. in [8–11], Bejan and Crasmareanu in [12], Blaga in [13], Chandra et al. in [14], Chen and Deshmukh in [15], Deshmukh et al. in [16], He and Zhu in [17], Atçeken et al. in [18], Nagaraja and Premalatta in [19], Tripathi in [20] and many others.

Motivated by all these studies, we consider Lorentzian para-Kenmotsu manifold admitting almost $\eta-$Ricci solitons in some curvature tensors. Ricci pseudosymmetry concepts of Lorentzian para-Kenmotsu manifolds admitting $\eta-$Ricci soliton have introduced according to the choice of some curvature tensors such as Riemann, concircular, projective, $\mathcal{M}-$projective, W_1 and W_2. After then, according to the choice of the curvature tensors, necessary conditions are given for Lorentzian para-Kenmotsu manifold admitting $\eta-$Ricci soliton to be Ricci semisymmetric. Then some characterizations are obtained and classifications have made under the some conditions.

2 Preliminary

Let $\hat{M}^n$ be an $n-$dimensional Lorentzian metric manifold. This means that it is endowed with a structure (ϕ, ξ, η, g), where ϕ is a $(1,1)-$type tensor field, ξ is a vector field, η is a $1-$form on $\hat{M}^n$ and g is a Lorentzian metric tensor satisfying;

$$\begin{cases} \phi^2\omega_1 = \omega_1 + \eta(\omega_1)\xi, \\ \\ g(\phi\omega_1, \phi\omega_2) = g(\omega_1, \omega_2) + \eta(\omega_1)\eta(\omega_2), \end{cases} \tag{1}$$

$$\eta(\xi) = -1, \eta(\omega_1) = g(\omega_1, \xi), \tag{2}$$

for all vector fields ω_1, ω_2 on $\hat{M}^n$. Then $\hat{M}^n(\phi, \xi, \eta, g)$ is said to be Lorentzian almost paracontact manifold.

A Lorentzian almost paracontact manifold $\hat{M}^n(\phi, \xi, \eta, g)$ is called Lorentzian para-Kenmotsu manifold if

$$(\nabla_{\omega_1}\phi)\omega_2 = -g(\phi\omega_1, \omega_2)\xi - \eta(\omega_2)\phi\omega_1, \tag{3}$$

for all $\omega_1, \omega_2 \in \Gamma(T\hat{M})$, where ∇ and $\Gamma\left(T\hat{M}\right)$ denote the Levi-Civita connection and differentiable vector fields set on $\hat{M}^n$, respectively.

Lemma 1. *Let $\hat{M}^n(\phi,\xi,\eta,g)$ be the $n-$dimensional Lorentzian para-Kenmotsu manifold. The following relations are provided for $\hat{M}^n(\phi,\xi,\eta,g)$.*

$$\nabla_{\omega_1}\xi = -\phi^2\omega_1 = -\omega_1 - \eta(\omega_1)\xi, \tag{4}$$

$$(\nabla_{\omega_1}\eta)\omega_2 = -g(\omega_1,\omega_2) - \eta(\omega_1)\eta(\omega_2), \tag{5}$$

$$R(\omega_1,\omega_2)\xi = \eta(\omega_2)\omega_1 - \eta(\omega_1)\omega_2, \tag{6}$$

$$\eta(R(\omega_1,\omega_2)\omega_3) = g(\eta(\omega_1)\omega_2 - \eta(\omega_2)\omega_1,\omega_3), \tag{7}$$

$$S(\omega_1,\xi) = (n-1)\eta(\omega_1), \tag{8}$$

where R and S are the Riemann curvature tensor and Ricci curvature tensor of $\hat{M}^n(\phi,\xi,\eta,g)$, respectively.

Example 1. *Let us consider the $5-$dimensional manifold*

$$\hat{M}^5 = \{(x_1,x_2,x_3,x_4,z)|\, z>0\},$$

where (x_1,x_2,x_3,x_4,z) denote the standard coordinates of $\mathbb{R}^5$. Then let e_1,e_2,e_3,e_4,e_5 be vector fields on $\hat{M}^5$ given by

$$e_1 = z\frac{\partial}{\partial x_1}, e_2 = z\frac{\partial}{\partial x_2}, e_3 = \frac{\partial}{\partial x_3}, e_4 = \frac{\partial}{\partial x_4}, e_5 = \frac{\partial}{\partial z}$$

which are linearly independent at each point of $\hat{M}^5$ and we define a Lorentzian metric tensor g on $\hat{M}^5$ as

$$g(e_i,e_i) = 1, 1 \le i \le 4$$

$$g(e_i,e_j) = 0, 1 \le i \ne j \le 5$$

$$g(e_5,e_5) = -1.$$

Let η be the $1-$form defined by $\eta(\omega_1) = g(\omega_1,e_5)$ for all $\omega_1 \in \Gamma\left(T\hat{M}\right)$. Now, we define the tensor field $(1,1)-$type φ such that

$$\varphi e_1 = -e_2, \varphi e_3 = -e_4, \varphi e_5 = 0.$$

Then for $\omega_1 = x_i e_i, \omega_2 = y_j e_j \in \Gamma\left(T\hat{M}\right), 1 \leq i, j \leq 5$, *we can easily see that*

$$\varphi^2 \omega_1 = \omega_1 + \eta(\omega_1)\xi, \xi = e_5, \eta(\omega_1) = g(\omega_1, \xi)$$

and

$$g(\varphi\omega_1, \varphi\omega_2) = g(\omega_1, \omega_2) + \eta(\omega_1)\eta(\omega_2).$$

By direct calculations, only non-vanishing components are

$$[e_i, e_5] = -e_i, 1 \leq i \leq 4.$$

From Kozsul's formula, we can compute

$$\tilde{\nabla}_{e_i} e_5 = -e_i, 1 \leq i \leq 4.$$

Thus for $\omega_1 = x_i e_i, \omega_2 = y_j e_j \in \Gamma\left(T\hat{M}\right)$, *we have*

$$\tilde{\nabla}_{\omega_1}\xi = -\omega_1 - \eta(\omega_1)\xi,$$

and

$$\left(\tilde{\nabla}_{\omega_1}\varphi\right)\omega_2 = -g(\varphi\omega_1, \omega_2)\xi - \eta(\omega_2)\varphi\omega_1,$$

that is, $\hat{M}^5(\varphi, \xi, \eta, g)$ *is a Lorentzian para-Kenmotsu manifold [4].*

Precisely, a Ricci soliton on a Riemannian manifold (M, g) is defined as a triple (g, ξ, λ) on M satisfying

$$L_\xi g + 2S + 2\lambda g = 0, \tag{9}$$

where L_ξ is the Lie derivative operator along the vector field ξ and λ is a real constant. We note that if ξ is a Killing vector field, then Ricci soliton reduces to an Einstein metric (g, λ). Futhermore, generalization is the notion of $\eta-$Ricci soliton defined by J.T. Cho and M. Kimura as a quadruple (g, ξ, λ, μ) satisfying

$$L_\xi g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0, \tag{10}$$

where λ and μ are real constants and η is the dual of ξ and S denotes the Ricci tensor of M. Furthermore if λ and μ are smooth functions on M, then it called almost $\eta-$Ricci soliton on M.

Suppose the quartet (g, ξ, λ, μ) is almost $\eta-$Ricci soliton on manifold M. Then,

· If $\lambda < 0$, then M is shriking.

· If $\lambda = 0$, then M is steady.

· If $\lambda > 0$, then M is expanding.

Let M be a Riemannian manifold, T is $(0, k)-$type tensor field and A is $(0, 2)-$type tensor field. In this case, Tachibana tensor field $Q(A, T)$ is defined as

$$\begin{aligned} Q(A,T)(X,...,X_k;\omega_1,\omega_2) = -T((\omega_1 \wedge_A \omega_2)X_1,...,X_k) - \\ ... - T(X_1,...,X_{k-1},(\omega_1 \wedge_A \omega_2)X_k), \end{aligned} \tag{11}$$

where,

$$(\omega_1 \wedge_A \omega_2)\omega_3 = A(\omega_2,\omega_3)\omega_1 - A(\omega_1,\omega_3)\omega_2, \tag{12}$$

$k \geq 1, X_1, X_2, ..., X_k, \omega_1, \omega_2 \in \Gamma(TM)$.

3 Almost $\eta-$Ricci Solitons on Ricci Pseudosymmetric and Ricci Semisymmetric of Lorentzian para-Kenmotsu Manifolds

Now let (g, ξ, λ, μ) be almost $\eta-$Ricci soliton on Lorentzian para-Kenmotsu manifold. Then we have

$$(L_\xi g)(\omega_1,\omega_2) = L_\xi g(\omega_1,\omega_2) - g(L_\xi\omega_1,\omega_2) - g(\omega_1, L_\xi\omega_2)$$

$$= \xi g(\omega_1,\omega_2) - g([\xi,\omega_1],\omega_2) - g(\omega_1,[\xi,\omega_2])$$

$$= g(\nabla_\xi\omega_1,\omega_2) + g(\omega_1,\nabla_\xi\omega_2) - g(\nabla_\xi\omega_1,\omega_2)$$

$$+g(\nabla_{\omega_1}\xi,\omega_2) - g(\nabla_\xi\omega_2,\omega_1) + g(\omega_1,\nabla_{\omega_2}\xi),$$

for all $\omega_1,\omega_2 \in \Gamma(TM)$. If we use (4) in the last equation, then we have

$$(L_\xi g)(\omega_1,\omega_2) = -2g(\omega_1,\omega_2) - 2\eta(\omega_1)\eta(\omega_2). \tag{13}$$

Thus, in a Lorentzian para-Kenmotsu manifold, from (10) and (13), we have

$$S(\omega_1,\omega_2) = (1-\lambda)g(\omega_1,\omega_2) + (1-\mu)\eta(\omega_1)\eta(\omega_2). \tag{14}$$

Thus, we can easily give the following result.

Corollary 1. *The $n-$dimensional Lorentz para-Kenmotsu manifold admitting almost $\eta-$Ricci soliton $\left(\hat{M}^n, g, \xi, \lambda, \mu\right)$ is an $\eta-$Einstein manifold.*

For $\omega_2 = \xi$ in (14), this implies that

$$S(\xi,\omega_1) = (\mu-\lambda)\eta(\omega_1). \tag{15}$$

Taking into account of (8) and (15), we conclude that

$$\mu - \lambda = n - 1. \tag{16}$$

Definition 1. *Let $\hat{M}^n$ be an $n-$dimensional Lorentzian para-Kenmotsu manifold. If $R\cdot S$ and $Q(g,S)$ are linearly dependent, then the $\hat{M}^n$ is said to be* ***Ricci pseudosymmetric****.*

In this case, there exists a function h_1 on $\hat{M}^n$ such that

$$R\cdot S = h_1 Q(g,S).$$

In particular, if $h_1 = 0$, the manifold $\hat{M}^n$ is said to be **Ricci semisymmetric**.

Let us now investigate the Ricci pseudosymmetric case of the $n-$dimensional Lorentzian para-Kenmotsu manifold.

Theorem 1. *Let $\hat{M}^n$ be Lorentzian para-Kenmotsu manifold and (g, ξ, λ, μ) be almost $\eta-$Ricci soliton on $\hat{M}^n$. If $\hat{M}^n$ is a Ricci pseudosymmetric, then $\hat{M}^n$ is either an $\eta-$Einstein manifold provided $\lambda = 2 - n$ and $\mu = 1$ or $h_1 = 1$.*

Proof. Let us assume that Lorentzian para-Kenmotsu manifold $\hat{M}^n$ be Ricci pseudosymmetric and (g, ξ, λ, μ) be almost $\eta-$Ricci soliton on Lorentzian para-Kenmotsu manifold $\hat{M}^n$. That means

$$(R(\omega_1, \omega_2) \cdot S)(\omega_4, \omega_5) = h_1 Q(g, S)(\omega_4, \omega_5; \omega_1, \omega_2),$$

for all $\omega_1, \omega_2, \omega_4, \omega_5 \in \Gamma\left(T\hat{M}^n\right)$. From the last equation, we can easily write

$$\begin{aligned} &S(R(\omega_1, \omega_2)\omega_4, \omega_5) + S(\omega_4, R(\omega_1, \omega_2)\omega_5) \\ &= h_1\{S((\omega_1 \wedge_g \omega_2)\omega_4, \omega_5) + S(\omega_4, (\omega_1 \wedge_g \omega_2)\omega_5)\}. \end{aligned} \tag{17}$$

If we putting $\omega_5 = \xi$ in (17), we get

$$\begin{aligned} &S(R(\omega_1, \omega_2)\omega_4, \xi) + S(\omega_4, R(\omega_1, \omega_2)\xi) \\ &= h_1\{S(g(\omega_2, \omega_4)\omega_1 - g(\omega_1, \omega_4)\omega_2, \xi) \\ &+ S(\omega_4, \eta(\omega_2)\omega_1 - \eta(\omega_1)\omega_2)\}. \end{aligned} \tag{18}$$

If we make use of (6) and (8) in (18), we have

$$\begin{aligned} &S(\omega_4, \eta(\omega_2)\omega_1 - \eta(\omega_1)\omega_2) \\ &+ (n-1)\eta(R(\omega_1, \omega_2)\omega_4) \\ &= h_1\{(n-1)g(\eta(\omega_1)\omega_2 - \eta(\omega_2)\omega_1, \omega_4) \\ &+ S(\omega_4, \eta(\omega_2)\omega_1 - \eta(\omega_1)\omega_2)\}. \end{aligned} \tag{19}$$

If we use (7) in the (19), we get

$$\begin{aligned}
&(n-1)\, g\left(\eta\left(\omega_1\right) \omega_2-\eta\left(\omega_2\right) \omega_1, \omega_4\right) \\
&+S\left(\eta\left(\omega_2\right) \omega_1-\eta\left(\omega_1\right) \omega_2, \omega_4\right) \\
&=h_1\left\{(n-1)\, g\left(\eta\left(\omega_1\right) \omega_2-\eta\left(\omega_2\right) \omega_1, \omega_4\right)\right. \\
&\left.+S\left(\omega_4, \eta\left(\omega_2\right) \omega_1-\eta\left(\omega_1\right) \omega_2\right)\right\} .
\end{aligned} \tag{20}$$

In same way, we use (14) in the (20), we can write

$$\begin{aligned}
&[(n-1)+(\lambda-1)]\left[1-h_1\right] \times \\
&g\left(\eta\left(\omega_1\right) \omega_2-\eta\left(\omega_2\right) \omega_1, \omega_4\right)=0.
\end{aligned} \tag{21}$$

It is clear from (21),

$$h_1=1 \text{ or } \lambda=2-n.$$

This completes the proof. □

We can give the results obtained from this theorem as follows.

Corollary 2. *Let $\hat{M}^n$ be Lorentz para-Kenmotsu manifold and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on $\hat{M}^n$. If $\hat{M}^n$ is a Ricci semisymmetric, then $\hat{M}^n$ is an $\eta-$Einstein manifold provided $\lambda=2-n$ and $\mu=1$.*

Corollary 3. *Let $\hat{M}^n$ be Lorentz para-Kenmotsu manifold and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on $\hat{M}^n$. If $\hat{M}^n$ is a Ricci semisymmetric, then $\hat{M}^n$ is always shriking.*

For an $n-$dimensional semi-Riemann manifold M, the concircular curvature tensor is defined as

$$C\left(\omega_1, \omega_2\right) \omega_3=R\left(\omega_1, \omega_2\right) \omega_3-\frac{r}{n\,(n-1)}\left[g\left(\omega_2, \omega_3\right) \omega_1-g\left(\omega_1, \omega_3\right) \omega_2\right]. \tag{22}$$

For an $n-$dimensional Lorentzian para-Kenmotsu manifold, if we choose $\omega_3 = \xi$ in (22), we can write

$$C(\omega_1, \omega_2)\xi = \left[1 - \frac{r}{n(n-1)}\right][\eta(\omega_2)\omega_1 - \eta(\omega_1)\omega_2], \tag{23}$$

and similarly if we take the inner product of both sides of (22) by ξ, we get

$$\eta(C(\omega_1, \omega_2)\omega_3) = \left[1 - \frac{r}{n(n-1)}\right] g(\eta(\omega_1)\omega_2 - \eta(\omega_2)\omega_1, \omega_3). \tag{24}$$

Definition 2. *Let $\hat{M}^n$ be an $n-$dimensional Lorentz para-Kenmotsu manifold. If $C \cdot S$ and $Q(g, S)$ are linearly dependent, then the manifold is said to be* ***concircular Ricci pseudosymmetric.***

In this case, there exists a function h_2 on $\hat{M}^n$ such that

$$C \cdot S = h_2 Q(g, S).$$

In particular, if $h_2 = 0$, the manifold $\hat{M}^n$ is said to be **concircular Ricci semisymmetric.**

Thus we have the following theorem.

Theorem 2. *Let $\hat{M}^n$ be Lorentzian para-Kenmotsu manifold and (g, ξ, λ, μ) be almost $\eta-$Ricci soliton on $\hat{M}^n$. If $\hat{M}^n$ is a concircular Ricci pseudosymmetric, then we have*

$$h_2 = \frac{(n + \lambda - 2)[n(n-1) - r]}{n(n-1)^2 + [n(n-1) - r](\lambda - 1)}.$$

Proof. Let us assume that Lorentzian para-Kenmotsu manifold $\hat{M}^n$ be concircular Ricci pseudosymmetric and (g, ξ, λ, μ) be almost $\eta-$Ricci soliton on Lorentzian para-Kenmotsu manifold $\hat{M}^n$. That means

$$(C(\omega_1, \omega_2) \cdot S)(\omega_4, \omega_5) = h_2 Q(g, S)(\omega_4, \omega_5; \omega_1, \omega_2),$$

for all $\omega_1, \omega_2, \omega_4, \omega_5 \in \Gamma\left(T\hat{M}^n\right)$. From the last equation, we can easily write

$$\begin{aligned} &S\left(C\left(\omega_1, \omega_2\right)\omega_4, \omega_5\right) + S\left(\omega_4, C\left(\omega_1, \omega_2\right)\omega_5\right) \\ &= h_2\left\{S\left(\left(\omega_1 \wedge_g \omega_2\right)\omega_4, \omega_5\right) + S\left(\omega_4, \left(\omega_1 \wedge_g \omega_2\right)\omega_5\right)\right\}. \end{aligned} \tag{25}$$

If we choose $\omega_5 = \xi$ in (25), we get

$$\begin{aligned} &S\left(C\left(\omega_1, \omega_2\right)\omega_4, \xi\right) + S\left(\omega_4, C\left(\omega_1, \omega_2\right)\xi\right) \\ &= h_2\left\{S\left(g\left(\omega_2, \omega_4\right)\omega_1 - g\left(\omega_1, \omega_4\right)\omega_2, \xi\right)\right. \\ &\left.+S\left(\omega_4, \eta\left(\omega_2\right)\omega_1 - \eta\left(\omega_1\right)\omega_2\right)\right\}. \end{aligned} \tag{26}$$

By using of (8) and (23) in (26), we have

$$\begin{aligned} &S\left(\omega_4, A\left[\eta\left(\omega_2\right)\omega_1 - \eta\left(\omega_1\right)\omega_2\right]\right) \\ &+\left(n-1\right)\eta\left(C\left(\omega_1, \omega_2\right)\omega_4\right) \\ &= h_2\left\{\left(n-1\right)g\left(\eta\left(\omega_1\right)\omega_2 - \eta\left(\omega_2\right)\omega_1, \omega_4\right)\right. \\ &\left.+S\left(\omega_4, \eta\left(\omega_2\right)\omega_1 - \eta\left(\omega_1\right)\omega_2\right)\right\}, \end{aligned} \tag{27}$$

where $A = 1 - \frac{r}{n(n-1)}$. Substituting (24) into (27), we have

$$\begin{aligned} &A\left(n-1\right)g\left(\eta\left(\omega_1\right)\omega_2 - \eta\left(\omega_2\right)\omega_1, \omega_4\right) \\ &+AS\left(\eta\left(\omega_2\right)\omega_1 - \eta\left(\omega_1\right)\omega_2, \omega_4\right) \\ &= h_2\left\{\left(n-1\right)g\left(\eta\left(\omega_1\right)\omega_2 - \eta\left(\omega_2\right)\omega_1, \omega_4\right)\right. \\ &\left.+S\left(\eta\left(\omega_2\right)\omega_1 - \eta\left(\omega_1\right)\omega_2, \omega_4\right)\right\}. \end{aligned} \tag{28}$$

If we use (14) in the (28), we can write

$$\left\{A\left(n+\lambda-2\right) - h_2\left[\left(n-1\right) + A\left(\lambda-1\right)\right]\right\}g\left(\eta\left(\omega_1\right)\omega_2 - \eta\left(\omega_2\right)\omega_1, \omega_4\right) = 0. \tag{29}$$

It is clear from (29),

$$h_2 = \frac{(n+\lambda-2)\left[n(n-1)-r\right]}{n(n-1)^2 + \left[n(n-1)-r\right](\lambda-1)}.$$

This completes the proof. □

We can give the results obtained from this theorem as follows.

Corollary 4. *Let $\hat{M}^n$ be Lorentz para-Kenmotsu manifold and (g, ξ, λ, μ) be almost $\eta-$Ricci soliton on $\hat{M}^n$. If $\hat{M}^n$ is a concircular Ricci semisymmetric, then $\hat{M}^n$ is either manifold with scalar curvature $r = n(n-1)$ or $\eta-$Einstein manifold provided $\lambda = 2-n$ and $\mu = 1$.*

For an $n-$dimensional semi-Riemann manifold M, the projective curvature tensor is defined as

$$P(\omega_1, \omega_2)\omega_3 = R(\omega_1, \omega_2)\omega_3 - \frac{1}{n-1}\left[S(\omega_2, \omega_3)\omega_1 - S(\omega_1, \omega_3)\omega_2\right]. \tag{30}$$

For an $n-$dimensional Lorentzian para-Kenmotsu manifold, if we choose $\omega_3 = \xi$ in (30), we can write

$$P(\omega_1, \omega_2)\xi = 0, \tag{31}$$

and similarly if we take the inner product of both sides of (30) by ξ, we get

$$\eta(P(\omega_1, \omega_2)\omega_3) = 0. \tag{32}$$

Definition 3. *Let $\hat{M}^n$ be an $n-$dimensional Lorentzian para-Kenmotsu manifold. If $P \cdot S$ and $Q(g, S)$ are linearly dependent, then the manifold is said to be* ***projective Ricci pseudosymmetric.***

In this case, there exists a function h_3 on $\hat{M}^n$ such that

$$P \cdot S = h_3 Q(g, S).$$

In particular, if $h_3 = 0$, the manifold $\hat{M}^n$ is said to be **projective Ricci semisymmetric.**

Let us now investigate the projective Ricci pseudosymmetric case of the Lorentzian para-Kenmotsu manifold.

Theorem 3. *Let $\hat{M}^n$ be Lorentzian para-Kenmotsu manifold and (g, ξ, λ, μ) be almost $\eta-$Ricci soliton on $\hat{M}^n$. If $\hat{M}^n$ is a projective Ricci pseudosymmetric, then $\hat{M}^n$ is either projective Ricci semisymmetric or $\eta-$Einstein manifold such that $\lambda = 2 - n$ and $\mu = 1$.*

Proof. Let us assume that Lorentzian para-Kenmotsu manifold $\hat{M}^n$ be projective Ricci pseudosymmetric and (g, ξ, λ, μ) be almost $\eta-$Ricci soliton on Lorentzian para-Kenmotsu manifold $\hat{M}^n$. Then we have

$$(P(\omega_1, \omega_2) \cdot S)(\omega_4, \omega_5) = h_3 Q(g, S)(\omega_4, \omega_5; \omega_1, \omega_2),$$

for all $\omega_1, \omega_2, \omega_4, \omega_5 \in \Gamma\left(T\hat{M}^n\right)$. From the last equation, we can easily write

$$\begin{aligned} & S(P(\omega_1, \omega_2)\omega_4, \omega_5) + S(\omega_4, P(\omega_1, \omega_2)\omega_5) \\ & = h_3\{S((\omega_1 \wedge_g \omega_2)\omega_4, \omega_5) + S(\omega_4, (\omega_1 \wedge_g \omega_2)\omega_5)\}. \end{aligned} \tag{33}$$

If we choose $\omega_5 = \xi$ in (33), we get

$$\begin{aligned} & S(P(\omega_1, \omega_2)\omega_4, \xi) + S(\omega_4, P(\omega_1, \omega_2)\xi) \\ & = h_3\{S(g(\omega_2, \omega_4)\omega_1 - g(\omega_1, \omega_4)\omega_2, \xi) \\ & + S(\omega_4, \eta(\omega_2)\omega_1 - \eta(\omega_1)\omega_2)\}. \end{aligned} \tag{34}$$

If we make use of (8) and (31) in (34), we have

$$\begin{aligned} & (n-1)\eta(P(\omega_1, \omega_2)\omega_4) \\ & = h_3\{(n-1)g(\eta(\omega_1)\omega_2 - \eta(\omega_2)\omega_1, \omega_4) \\ & + S(\omega_4, \eta(\omega_2)\omega_1 - \eta(\omega_1)\omega_2)\}. \end{aligned} \tag{35}$$

If we use (32) in the (35), we get

$$\begin{aligned} &h_3 \{(n-1) g (\eta (\omega_1) \omega_2 - \eta (\omega_2) \omega_1, \omega_4) \\ &+S (\eta (\omega_2) \omega_1 - \eta (\omega_1) \omega_2, \omega_4)\} = 0. \end{aligned} \tag{36}$$

If we use (14) in the (36), we can write

$$h_3 (\lambda + n - 2) g (\eta (\omega_1) \omega_2 - \eta (\omega_2) \omega_1, \omega_4) = 0. \tag{37}$$

It is clear from (37) ,

$$h_3 = 0 \text{ or } \lambda = 2 - n.$$

This completes the proof. □

Corollary 5. *Let $\hat{M}^n$ be Lorentzian para-Kenmotsu manifold and (g, ξ, λ, μ) be almost $\eta-$Ricci soliton on $\hat{M}^n$. If $\hat{M}^n$ is a projective Ricci pseudosymmetric, then $\hat{M}^n$ is always shriking provided $h_3 \neq 0$.*

For an $n-$dimensional semi-Riemann manifold M, the $\mathcal{M}-$projective curvature tensor is defined as

$$\begin{aligned} \mathcal{M} (\omega_1, \omega_2) \omega_3 = {} & R (\omega_1, \omega_2) \omega_3 - \tfrac{1}{2(n-1)} [S (\omega_2, \omega_3) \omega_1 - S (\omega_1, \omega_3) \omega_2 \\ & +g (\omega_2, \omega_3) Q\omega_1 - g (\omega_1, \omega_3) Q\omega_2] \end{aligned} \tag{38}$$

For an $n-$dimensional Lorentzian para-Kenmotsu manifold, if we choose $\omega_3 = \xi$ in (38) , we obtain

$$\mathcal{M} (\omega_1, \omega_2) \xi = \frac{1}{2} [\eta (\omega_2) \omega_1 - \eta (\omega_1) \omega_2] - \frac{1}{2 (n-1)} [\eta (\omega_2) Q\omega_1 - \eta (\omega_1) Q\omega_2] , \tag{39}$$

and similarly if we take the inner product of both of sides of (38) by ξ, we get

$$\begin{aligned} \eta (\mathcal{M} (\omega_1, \omega_2) \omega_3) = & \frac{1}{2} g (\eta (\omega_1) \omega_2 - \eta (\omega_2) \omega_1, \omega_3) - \frac{1}{2 (n-1)} S (\eta (\omega_1) \omega_2 \\ & - \eta (\omega_2) \omega_1, \omega_3. \end{aligned} \tag{40}$$

Definition 4. *Let $\hat{M}^n$ be an $n-$dimensional Lorentzian para-Kenmotsu manifold. If $\mathcal{M} \cdot S$ and $Q(g, S)$ are linearly dependent, then it is said to be $\mathcal{M}-$**projective Ricci pseudosymmetric.***

In this case, there exists a function h_4 on $\hat{M}^n$ such that

$$\mathcal{M} \cdot S = h_4 Q(g, S).$$

In particular, if $h_4 = 0$, the manifold $\hat{M}^n$ is said to be $\mathcal{M}-$**projective Ricci semisymmetric.**

Let us now investigate the $\mathcal{M}-$projective Ricci pseudosymmetric case of the Lorentzian para-Kenmotsu manifold.

Theorem 4. *Let $\hat{M}^n$ be Lorentzian para-Kenmotsu manifold and (g, ξ, λ, μ) be almost $\eta-$Ricci soliton on $\hat{M}^n$. If $\hat{M}^n$ is a $\mathcal{M}-$projective Ricci pseudosymmetric provided $n \neq 1$ and $\lambda \neq 2 - n$, then we have*

$$h_4 = \frac{\lambda + n - 2}{2(n-1)}.$$

Proof. Let us assume that Lorentzian para-Kenmotsu manifold $\hat{M}^n$ be projective $\mathcal{M}-$projective Ricci pseudosymmetric and (g, ξ, λ, μ) be almost $\eta-$Ricci soliton on Lorentzian para-Kenmotsu manifold $\hat{M}^n$. That means

$$(\mathcal{M}(\omega_1, \omega_2) \cdot S)(\omega_4, \omega_5) = h_4 Q(g, S)(\omega_4, \omega_5; \omega_1, \omega_2),$$

for all $\omega_1, \omega_2, \omega_4, \omega_5 \in \Gamma\left(T\hat{M}^n\right)$. From the last equation, we can easily write

$$\begin{aligned} &S(\mathcal{M}(\omega_1, \omega_2)\omega_4, \omega_5) + S(\omega_4, \mathcal{M}(\omega_1, \omega_2)\omega_5) \\ &= h_4 \{S((\omega_1 \wedge_g \omega_2)\omega_4, \omega_5) + S(\omega_4, (\omega_1 \wedge_g \omega_2)\omega_5)\}. \end{aligned} \tag{41}$$

If we choose $\omega_5 = \xi$ in (41), we get

$$\begin{aligned} & S\left(\mathcal{M}\left(\omega_1, \omega_2\right) \omega_4, \xi\right)+S\left(\omega_4, \mathcal{M}\left(\omega_1, \omega_2\right) \xi\right) \\ & =h_4\left\{S\left(g\left(\omega_2, \omega_4\right) \omega_1-g\left(\omega_1, \omega_4\right) \omega_2, \xi\right)\right. \\ & \left.+S\left(\omega_4, \eta\left(\omega_2\right) \omega_1-\eta\left(\omega_1\right) \omega_2\right)\right\} . \end{aligned} \tag{42}$$

If we make use of (8) and (39) in (42), we have

$$\begin{aligned} & (n-1) \eta\left(\mathcal{M}\left(\omega_1, \omega_2\right) \omega_4\right) \\ & +S\left(\omega_4, \tfrac{1}{2}\left[\eta\left(\omega_2\right) \omega_1-\eta\left(\omega_1\right) \omega_2\right]-\tfrac{1}{2(n-1)}\left[\eta\left(\omega_2\right) Q \omega_1-\eta\left(\omega_1\right) Q \omega_2\right]\right) \\ & =h_4\left\{(n-1) g\left(\eta\left(\omega_1\right) \omega_2-\eta\left(\omega_2\right) \omega_1, \omega_4\right)\right. \\ & \left.+S\left(\omega_4, \eta\left(\omega_2\right) \omega_1-\eta\left(\omega_1\right) \omega_2\right)\right\} . \end{aligned} \tag{43}$$

By using (40) in the (43), we get

$$\begin{aligned} & \tfrac{(n-1)}{2} g\left(\eta\left(\omega_1\right) \omega_2-\eta\left(\omega_2\right) \omega_1, \omega_4\right)-S\left(\eta\left(\omega_1\right) \omega_2-\eta\left(\omega_2\right) \omega_1, \omega_4\right) \\ & -\tfrac{1}{2(n-1)} S\left(\eta\left(\omega_2\right) Q \omega_1-\eta\left(\omega_1\right) Q \omega_2, \omega_4\right) \\ & =h_4\left\{(n-1) g\left(\eta\left(\omega_1\right) \omega_2-\eta\left(\omega_2\right) \omega_1, \omega_4\right)\right. \\ & \left.+S\left(\eta\left(\omega_2\right) \omega_1-\eta\left(\omega_1\right) \omega_2, \omega_4\right)\right\} . \end{aligned} \tag{44}$$

If we put (14) in (44), we can write

$$\begin{aligned} & \tfrac{1}{2}(n-1) g\left(\eta\left(\omega_1\right) \omega_2-\eta\left(\omega_2\right) \omega_1, \omega_4\right) \\ & +(\lambda-1) g\left(\eta\left(\omega_1\right) \omega_2-\eta\left(\omega_2\right) \omega_1, \omega_4\right) \\ & +\tfrac{(\lambda-1)}{2(n-1)} S\left(\eta\left(\omega_2\right) \omega_1-\eta\left(\omega_1\right) \omega_2, \omega_4\right) \\ & =h_4[\lambda+n-2] g\left(\eta\left(\omega_1\right) \omega_2-\eta\left(\omega_2\right) \omega_1, \omega_4\right) \end{aligned} \tag{45}$$

Again, if we use (14) in the (45), we obtain

$$\left\{\frac{1}{2(n-1)}(\lambda-1)^2+(\lambda-1)+\frac{1}{2}(n-1)-h_4[\lambda+n-2]\right\}\times \\ g(\eta(\omega_1)\omega_2-\eta(\omega_2)\omega_1,\omega_4)=0. \tag{46}$$

It is clear from (46),

$$h_4=\frac{\lambda+n-2}{2(n-1)}.$$

This completes the proof. □

We can give the following corollary.

Corollary 6. *Let $\hat{M}^n$ be Lorentzian para-Kenmotsu manifold and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on $\hat{M}^n$. If $\hat{M}^n$ is a $\mathcal{M}-$projective Ricci semisymmetric, then $\lambda=2-n$ that is $\hat{M}^n$ is always shriking.*

For an $n-$dimensional semi-Riemann manifold M, the W_1-curvature tensor is defined as

$$W_1(\omega_1,\omega_2)\omega_3=R(\omega_1,\omega_2)\omega_3+\frac{1}{n-1}[S(\omega_2,\omega_3)\omega_1-S(\omega_1,\omega_3)\omega_2]. \tag{47}$$

For an $n-$dimensional Lorenrzian para-Kenmotsu manifold $\hat{M}^n$, if we choose $\omega_3=\xi$ in (47), we can write

$$W_1(\omega_1,\omega_2)\xi=2[\eta(\omega_2)\omega_1-\eta(\omega_1)\omega_2], \tag{48}$$

and similarly if we take the inner product of both of sides of (47) by ξ, we get

$$\eta(W_1(\omega_1,\omega_2)\omega_3)=2g(\eta(\omega_1)\omega_2-\eta(\omega_2)\omega_1,\omega_3). \tag{49}$$

Definition 5. *Let $\hat{M}^n$ be an $n-$dimensional Lorentzian para-Kenmotsu. If $W_1\cdot S$ and $Q(g,S)$ are linearly dependent, then the manifold is said to be W_1-**Ricci pseudosymmetric.***

In this case, there exists a function h_5 on $\hat{M}^n$ such that

$$W_1 \cdot S = h_5 Q(g, S).$$

In particular, if $h_5 = 0$, the manifold $\hat{M}^n$ is said to be W_1-**Ricci semisymmetric.**

Let us now investigate the W_1-Ricci pseudosymmetric case of the Lorentzian para-Kenmotsu manifold.

Theorem 5. *Let $\hat{M}^n$ be Lorentzian para-Kenmotsu manifold and (g, ξ, λ, μ) be almost $\eta-$Ricci soliton on $\hat{M}^n$. If $\hat{M}^n$ is W_1-Ricci pseudosymmetric, then $\hat{M}^n$ is either an $\eta-$Einstein manifold provided $\lambda = 2 - n$ and $\mu = 1$ or $h_5 = 2$.*

Proof. Let us assume that Lorentzian para-Kenmotsu manifold $\hat{M}^n$ be W_1-Ricci pseudosymmetric and (g, ξ, λ, μ) be almost $\eta-$Ricci soliton on Lorentzian para-Kenmotsu manifold $\hat{M}^n$. That means

$$(W_1(\omega_1, \omega_2) \cdot S)(\omega_4, \omega_5) = h_5 Q(g, S)(\omega_4, \omega_5; \omega_1, \omega_2),$$

for all $\omega_1, \omega_2, \omega_4, \omega_5 \in \Gamma\left(T\hat{M}^n\right)$. From the last equation, we can easily write

$$\begin{aligned} & S(W_1(\omega_1, \omega_2)\omega_4, \omega_5) + S(\omega_4, W_1(\omega_1, \omega_2)\omega_5) \\ & = H_5\{S((\omega_1 \wedge_g \omega_2)\omega_4, \omega_5) + S(\omega_4, (\omega_1 \wedge_g \omega_2)\omega_5)\}. \end{aligned} \tag{50}$$

If we choose $\omega_5 = \xi$ in (50), we get

$$\begin{aligned} & S(W_1(\omega_1, \omega_2)\omega_4, \xi) + S(\omega_4, W_1(\omega_1, \omega_2)\xi) \\ & = h_5\{S(g(\omega_2, \omega_4)\omega_1 - g(\omega_1, \omega_4)\omega_2, \xi) \\ & +S(\omega_4, \eta(\omega_1)\omega_2 - \eta(\omega_2)\omega_1)\}. \end{aligned} \tag{51}$$

If we make use of (8) and (48) in (51), we have

$$\begin{aligned} &2S\left(\omega_4,\eta\left(\omega_2\right)\omega_1-\eta\left(\omega_1\right)\omega_2\right)\\ &+(n-1)\,\eta\left(W_1\left(\omega_1,\omega_2\right)\omega_4\right)\\ &=h_5\left\{(n-1)\,g\left(\eta\left(\omega_1\right)\omega_2-\eta\left(\omega_2\right)\omega_1,\omega_4\right)\right.\\ &\left.+S\left(\omega_4,\eta\left(\omega_2\right)\omega_1-\eta\left(\omega_1\right)\omega_2\right)\right\}. \end{aligned} \tag{52}$$

If we use (49) in the (52), we get

$$\begin{aligned} &2\left(n-1\right)g\left(\eta\left(\omega_1\right)\omega_2-\eta\left(\omega_2\right)\omega_1,\omega_4\right)\\ &+2S\left(\eta\left(\omega_2\right)\omega_1-\eta\left(\omega_1\right)\omega_2,\omega_4\right)\\ &=h_5\left\{(n-1)\,g\left(\eta\left(\omega_1\right)\omega_2-\eta\left(\omega_2\right)\omega_1,\omega_4\right)\right.\\ &\left.+S\left(\eta\left(\omega_2\right)\omega_1-\eta\left(\omega_1\right)\omega_2,\omega_4\right)\right\}. \end{aligned} \tag{53}$$

If we use (14) in the (53), we can write

$$[n+\lambda-2]\,[2-h_5]\,g\left(\eta\left(\omega_1\right)\omega_2-\eta\left(\omega_2\right)\omega_1,\omega_4\right)=0. \tag{54}$$

It is clear from (54),

$$h_5=2 \text{ or } \lambda=2-n.$$

This completes the proof. □

We can give the following corollaries.

Corollary 7. *Let $\hat{M}^n$ be Lorentzian para-Kenmotsu manifold and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on $\hat{M}^n$. If $\hat{M}^n$ is a W_1-Ricci semisymmetric, then $\hat{M}^n$ is an $\eta-$Einstein manifold provided $\lambda=2-n$ and $\mu=1$.*

Corollary 8. *Let $\hat{M}^n$ be Lorentzian para-Kenmotsu manifold and (g, ξ, λ, μ) be almost $\eta-$Ricci soliton on $\hat{M}^n$. If $\hat{M}^n$ is a W_1-Ricci semisymmetric, then $\hat{M}^n$ is always shriking.*

For an $n-$dimensional semi-Riemann manifold M, the W_2-curvature tensor is defined as

$$W_2(\omega_1, \omega_2)\omega_3 = R(\omega_1, \omega_2)\omega_3 - \frac{1}{n}\left[g(\omega_2, \omega_3) Q\omega_1 - g(\omega_1, \omega_3) Q\omega_2\right]. \tag{55}$$

For an $n-$dimensional Lorentzian para-Kenmotsu manifold $\hat{M}^n$, if we choose $\omega_3 = \xi$ in (55), we can write

$$\begin{aligned} W_2(\omega_1, \omega_2)\xi &= [\eta(\omega_2)\omega_1 - \eta(\omega_1)\omega_2] \\ &-\tfrac{1}{(n-1)}\left[\eta(\omega_1) Q\omega_2 - \eta(\omega_2) Q\omega_1\right], \end{aligned} \tag{56}$$

and similarly if we take the inner product of both sides of (56) by ξ, we get

$$\begin{aligned} \eta(W_2(\omega_1, \omega_2)\omega_3) &= g(\eta(\omega_1)\omega_2 - \eta(\omega_2)\omega_1, \omega_3) \\ &+\tfrac{1}{(n-1)} S(\eta(\omega_1)\omega_2 - \eta(\omega_2)\omega_1, \omega_3). \end{aligned} \tag{57}$$

Definition 6. *Let $\hat{M}^n$ be an $n-$dimensional Lorentzian para-Kenmotsu manifold. If $W_2 \cdot S$ and $Q(g, S)$ are linearly dependent, then the manifold is said to be W_2-**Ricci pseudosymmetric.***

In this case, there exists a function h_6 on $\hat{M}^n$ such that

$$W_2 \cdot S = h_6 Q(g, S).$$

In particular, if $h_6 = 0$, the manifold $\hat{M}^n$ is said to be W_2-**Ricci semisymmetric.**

Let us now investigate the W_2-Ricci pseudosymmetric case of the Lorentzian para-Kenmotsu manifold.

Theorem 6. *Let $\hat{M}^n$ be Lorentzian para-Kenmotsu manifold and (g, ξ, λ, μ) be almost $\eta-$Ricci soliton on $\hat{M}^n$. If $\hat{M}^n$ is a W_2-Ricci pseudosymmetric, then*

$$h_6 = \frac{n + \lambda - 2}{n - 1},$$

provided $n \neq 1$.

Proof. Let us assume that Lorentzian para-Kenmotsu manifold be W_2-Ricci pseudosymmetric and (g, ξ, λ, μ) be almost $\eta-$Ricci soliton on Lorentzian para-Kenmotsu manifold. That means

$$(W_2(\omega_1, \omega_2) \cdot S)(\omega_4, \omega_5) = h_6 Q(g, S)(\omega_4, \omega_5; \omega_1, \omega_2),$$

for all $\omega_1, \omega_2, \omega_4, \omega_5 \in \Gamma\left(T\hat{M}^n\right)$. From the last equation, we can easily write

$$\begin{aligned} & S(W_2(\omega_1, \omega_2)\omega_4, \omega_5) + S(\omega_4, W_2(\omega_1, \omega_2)\omega_5) \\ & = h_6 \{S((\omega_1 \wedge_g \omega_2)\omega_4, \omega_5) + S(\omega_4, (\omega_1 \wedge_g \omega_2)\omega_5)\}. \end{aligned} \tag{58}$$

If we choose $\omega_5 = \xi$ in (58), we get

$$\begin{aligned} & S(W_2(\omega_1, \omega_2)\omega_4, \xi) + S(\omega_4, W_2(\omega_1, \omega_2)\xi) \\ & = h_6 \{S(g(\omega_2, \omega_4)\omega_1 - g(\omega_1, \omega_4)\omega_2, \xi) \\ & + S(\omega_4, \eta(\omega_2)\omega_1 - \eta(\omega_1)\omega_2)\}. \end{aligned} \tag{59}$$

If we make use of (8) and (56) in (59), we have

$$\begin{aligned} & (n-1)\eta(W_2(\omega_1, \omega_2)\omega_4) \\ & + S(\omega_4, [\eta(\omega_2)\omega_1 - \eta(\omega_1)\omega_2] \\ & - \tfrac{1}{(n-1)}[\eta(\omega_1)Q\omega_2 - \eta(\omega_2)Q\omega_1]\Big) \\ & = h_6\{(n-1)g(\eta(\omega_1)\omega_2 - \eta(\omega_2)\omega_1, \omega_4) \\ & + S(\omega_4, \eta(\omega_1)\omega_2 - \eta(\omega_2)\omega_1)\}. \end{aligned} \tag{60}$$

If we use (57) in the (60), we get

$$\begin{aligned}&(n-1)\,g\left(\eta\left(\omega_1\right)\omega_2-\eta\left(\omega_2\right)\omega_1,\omega_4\right)\\&-\tfrac{1}{(n-1)}S\left(\omega_4,\eta\left(\omega_1\right)Q\omega_2-\eta\left(\omega_2\right)Q\omega_1\right)\\&=H_6\left\{S\left(\omega_4,\eta\left(\omega_2\right)\omega_1-\eta\left(\omega_1\right)\omega_2\right)\right.\\&\left.+2n\left(f_1-f_3\right)g\left(\eta\left(\omega_1\right)\omega_2-\eta\left(\omega_2\right)\omega_1,\omega_4\right)\right\}.\end{aligned} \tag{61}$$

If we use (14) in the (61), we have

$$\begin{aligned}&(n-1)\,g\left(\eta\left(\omega_1\right)\omega_2-\eta\left(\omega_2\right)\omega_1,\omega_4\right)\\&-\tfrac{1-\lambda}{n-1}S\left(\eta\left(\omega_1\right)\omega_2-\eta\left(\omega_2\right)\omega_1,\omega_4\right)\\&=h_6\left[n-\lambda\right]g\left(\eta\left(\omega_1\right)\omega_2-\eta\left(\omega_2\right)\omega_1,\omega_4\right)\end{aligned} \tag{62}$$

Again, if we use (14) in (62) , we obtain

$$\begin{aligned}&\left\{(n-1)-\tfrac{(1-\lambda)^2}{n-1}-h_6\left(n-\lambda\right)\right\}\times\\&g\left(\eta\left(\omega_1\right)\omega_2-\eta\left(\omega_2\right)\omega_1,\omega_4\right)=0.\end{aligned} \tag{63}$$

It is clear from (63) ,

$$h_6=\frac{n+\lambda-2}{n-1}.$$

This completes the proof. □

We can give the results obtained from this theorem as follows.

Corollary 9. *Let $\hat{M}^n$ be Lorentzian para-Kenmotsu manifold and (g,ξ,λ,μ) be almost $\eta-$Ricci soliton on $\hat{M}^n$. If $\hat{M}^n$ is a W_2-Ricci semisymmetric, then $\hat{M}^n$ is an $\eta-$Einstein manifold provided $\lambda=2-n$ and $\mu=1$ or $\lambda=n$, $\mu=2n-1$ and it is always shriking.*

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