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On q-Calculus Related Generalization of Close-to-Convexity

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Abstract

We introduce and study q-analogue of certain classes of analytic functions which are related with generalized close-to-convexity. Distortion, inclusion results and growth rate of coefficient problem are investigated for these classes. Some applications of our results are highlighted.

1 Introduction

Let $\mathcal{A}$ denote the class of functions which are analytic in the open unit disc $E = \{z : |z| < 1\}$ and are denoted by power series given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

The subclass of $\mathcal{A}$ of all univalent functions is denoted as S.

Let $f, g \in \mathcal{A}$, $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ and f be given by (1.1). Then the

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convolution(Hadamrd product) of f and g is defined as

$$(f \star g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g \star f)(z), \quad z \in E.$$

The subordination of two analytic functions f_1 and f_2 in E is denoted as $f_1 \prec f_2$, if there exists a Schwarz function $w(z)$, analytic in E with $w(0) = 0$ and $|w(z)| < 1, \ (z \in E)$ such that $f_1(z) = f_2(w(z)), \ z \in E$. If f_2 is univalent in E, then

$$f_1(z) \prec f_2(z),$$

if and only if,

$$f_1(0) = f_2(0) \quad \text{and} \quad f_1(E) \subset f_2(E).$$

Let $C(\delta)$, $S^\star(\delta)$ and $K(\delta)$ denote the classes of S, which consists of, respectively, the convex, starlike and close-to-convex functions of order δ, $0 \leq \delta < 1$. The classes $C(0) = C, \quad S^\star(0) = S^\star$ and $K(0) = K$ are well known [2].

Let p be analytic in E with $p(0) = 1$. Then $p(z)$ is said to belong to the class $P[A, B]$, which is called Janowski class for $-1 \leq B < A \leq 1$, see [6].

The class $P[A, B] \subset P(\gamma), \quad \gamma = \frac{1-A}{1-B}$, where $P(\gamma)$ is the class of Caratheodaory functions satisfying the condition $Re\{p(z)\} > \gamma$ and $P(0) = P$. Also, it is known that $P[A, B]$ is a convex set.

By taking $A = q, B = -q^2, \quad q \in (0, 1)$, we have special subclass

$$P[q, -q^2] = P_q \subset P\Big(\frac{1-q}{1+q^2}\Big), \quad 0 < \alpha = \frac{1-q}{1+q^2} < 1.$$

For $q \to 1^-, \quad P_q \to P$, we note that $p \in P_q$ implies that

$$p(z) \prec L_q(z) = \frac{1+qz}{1-q^2 z}$$

and it maps the unit disc E onto the closed disc centered at $d = \frac{1+q^2}{1-q^4}$ with radius $r = \frac{q(1+q)}{1-q^4}$.

Since $P_q \subset P[A,B]$, therefore $L_q = \frac{1+qz}{1-q^2z}$, $\quad q \in (0,1)$ is convex univalent in E see, [6].

In this paper, we shall use the concept of q-calculus to define and study certain classes of analytic functions which are q-analogue of C, $S^\star$, K and related generalizations. For the applications of q-calculus in geometric functions, see [11–19] and the references therein.

Here we recall some basic definitions and results of q-calculus required in our study as follows.

(i). Let $f \in \mathcal{A}$. Then q-derivative of f is defined as

$$D_q f(z) = \frac{f(qz) - f(z)}{(q-1)z}, \quad z \neq 0,$$

and $D_q f(0) = f'(0)$ and $q \in (0,1)$, see [4,5]. Also

$$D_q(z^n) = [n]_q z^{n-1}, \quad [n]_q = \frac{1-q^n}{1-q}.$$

We note that $D_q f(z) \to f'(z)$ and $[n]_q \to 1^{-1}$.

(ii).

$$D_q\bigg(\log f(z)\bigg) = \frac{\ln(\frac{1}{q})}{1-q}\frac{D_q f(z)}{f(z)}, \quad f \in \mathcal{A}, \quad q \in (0,1).$$

For details, see [22].

(iii). For $0 < q < 1$, it can easily be checked that the function $h_q(z)$ defined as

$$\begin{aligned} h_q(z) &= \frac{1}{1-q}\log\frac{1-qz}{1-z} = \sum_{n=1}^{\infty}\frac{1-q^n}{1-q}\frac{z^n}{n}, \quad z \in E \\ &= \sum_{n=1}^{\infty}[n]_q\frac{z^n}{n} \end{aligned} \tag{1.2}$$

is convex univalent. Then, by Alexander relation between the classes C and $S^\star$, it follows that the function $zh'_q(z)$ is starlike and is given by

$$zh'_q(z) = \frac{z}{(1-qz)(1-q)} = \sum_{n=1}^{\infty}[n]_q \frac{z^n}{n}, \quad z \in E. \tag{1.3}$$

When $q \to 1^{-1}$, $zh'_q(z)$ reduces to well known Koebe function

$$k(z = \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} nz^z, \quad z \in E.$$

By putting $H_q(z) = zh'_q(z)$, it can be shown that some calculation that

$$\frac{zH'_z(z)}{H_q(z)} \in P(\beta), \quad \beta = \frac{1-q}{2(1+q)}.$$

That is $H_q = zh'_q$ is starlike of order β. It is observed in [20] that

$$\frac{zh'_q(z) \star zh'_q(z)}{zh'_q(z)} = \frac{H_q(z) \star H_q(z)}{H_q(z)} = L_q(z).$$

(iv). From the definition of q-derivative and relation (1.3), we use convolution to have

$$zD_q f(z) = f(z) \star zh'_q(z).$$

If $f \in C$ and $zh'_z \in S^\star(\beta)$, then it is known [21] that

$$zD_f \in S^\star(\beta), \quad \beta = \frac{1-q}{2(1+q)},$$

where $\beta \in (0, \frac{1}{2})$ for $q \in (0,1)$.

Throughout our discussion in this paper, $z \in E$ and $q \in (0,1)$ unless otherwise stated.

We now define the following new classes of analytic functions

Definition 1.1. *Let $f \in \mathcal{A}$. Then f is said to belong to the class $V_{m,q}$, if and only if, there exist $f_1, f_2 \in C$ such that*

$$D_q f(z) = \frac{\Big(D_q f_1(z)\Big)^{(\frac{m}{4}+\frac{1}{2})}}{\Big(D_q f_2(z)\Big)^{(\frac{m}{4}-\frac{1}{2})}}, \quad m \geq 2. \tag{1.4}$$

For $m = 2$, $V_{2,q} = C_q$.

Remark 1.1. *For $f_i \in C$, $\quad S_i^\star(z) = zD_q f_i \in S^\star(\beta), \quad i = 1, 2.$*

It is known [2] that $\{\frac{s_i^\star(z)}{z}\} = \left(\frac{s_i(z)}{z}\right)^{1-\beta}, \quad s_i \in S^\star, \quad i = 1, 2.$

Using these observations, we can write (1.4) as follows.

For $f \in V_{m,q}$, $\quad m \geq 2$, we have

$$D_q f(z) = \frac{\Big(D_q s_1(z)\Big)^{(1-\beta)(\frac{m}{4}+\frac{1}{2})}}{\Big(D_q s_2(z)\Big)^{(1-\beta)(\frac{m}{4}-\frac{1}{2})}}, \quad \beta = \frac{1-q}{2(1+q)}. \tag{1.5}$$

For $q \to 1^{-1}$, $\quad V_{m,q} \to V_m$ is the class of functions with bounded boundary rotation, see [1, 2].

Definition 1.2. *Let $f \in \mathcal{A}$. Then f is said to belong to the class $T^\star_{m,q}$, if and only if, there exists $g \in V_{m,q}$ such that*

$$\frac{D_q f(z)}{D_q g(z)} \in P_q, \quad m \geq 2.$$

When $q \to 1^{-1}$, $\quad T^\star_{m,q} \to T_m$ and T_m is the class of generalized close-to-convex functions introduced in [9].

Also, for $m = 2$, we obtain the class $K_{2,q} = K_q$ of q-close-to-convex functions.

Definition 1.3. *Let* $f \in \mathcal{A}$ *with* $\frac{f(z))(D_q f(z))}{z} \neq 0$, *in* E. *Then* $f \in Q^{\gamma}_{m,q}(a)$, *if there exists* $g \in T^{\star}_{m,q}$ *such that*

$$zD_q f(z) + af(z)) = (a+1)z\big(D_q g(z)\big)^{\gamma}, \quad Re\{a\} \geq 0, \quad 0 \leq \gamma \leq 1. \tag{1.6}$$

We note that $Q^{1}_{m,q}(0) = T^{\star}_{m,q}$ *and* $Q^{\gamma}_{m,q}(0) = Q^{\gamma}_{m,q}$

We shall need the integral representation of hypergeometric function $\mathbb{G}$ given as below:

Let $a, b, c \in \mathbb{C} \quad (c \neq 0, -1, -2, \ldots)$. Then

$$\mathbb{G}(a, b; c : z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{t-1}(1-t)^{c-b-1}(1-tz)^{-a} dt,$$
$$(Re\{(c)\} > Re\{(b)\} > 0).$$

See [8].

2 Preliminary Results

Lemma 2.1. *Let* h_q *be given by (1.2). Then*

$$zh'_q(z) = \frac{z}{(1-qz)(1-z)} \in S^{\star}(\beta), \quad \beta = \frac{1-q}{2(1+q)}.$$

By simple calculations, it can be verified that $L_q(z)$ that

$$L_q(z) = \frac{zh'_q(z) \star zh'_q(z)}{zh'_q(z)} = \frac{1+qz}{1-q^2 z}, \quad q \in (0,1).$$

Also L_q satisfies the following.

(i). $Re\{L_q(z)\} = Re\{\frac{1+qz}{1-q^2 z}\} > \frac{1-q}{1-q^2 z}$, see [6].

(ii). $Re\{L'_q(z)\} = \frac{q(1+q)}{1-q^2} > 0. \quad z \in E$. This shows that $L_q(z)$ is univalent in E.

(iii).

$$Re\left\{\frac{(zL_q'(z))'}{L_q'(z)}\right\} = Re\{\frac{(1-q(q+1)z)+q^3z^2}{(1+qz)(1-q^2z)}\} \geq 0, \quad \text{in} \quad E,$$

since $T(r) = 1 - q(q+1)r + q^3r^2$, with $T(0) = 1$ is decreasing in $(0, 1)$.

Lemma 2.2. *[3] Let $p \in P$. Then, for $z = re^{i\theta}$,*

$$\int_0^{2\pi} |p(re^{i\theta})|^\lambda d\theta < c(r)\frac{1}{(1-r)^{\lambda-1}}, \quad z \in E, \quad \lambda > 1.$$

Lemma 2.3. *[7] Let $f \in \mathcal{A}$. Then, for $z = re^{i\theta}$, $\quad 0 \leq \theta_1 < \theta_2 \leq 2\pi$, a necessary and sufficient condition for $f \in S$ (close-to-convex) is*

$$\int_{\theta_1}^{\theta_2} Re\left\{\frac{(zf'(z))'}{f'(z)}\right\}d\theta > -\pi.$$

3 Main Results

Theorem 3.1. *Let $f \in T^\star_{m,q}$. Then, with $F = f \star h_q$, there exist close-to-convex functions K_1 and K_2 such that*

$$D_qf(z) = F'(z) = \frac{\left(K_1'(z)\right)^{(1-\beta)(\frac{m}{4}+\frac{1}{2})}}{\left(K^\star{}_2(z)\right)^{(1-\beta)(\frac{m}{4}-\frac{1}{2})}}, \quad \beta = \frac{1-q}{2(1+q)}. \tag{3.1}$$

Proof. Using Definition 1.1, Remark 1.1 and the inclusions result $P_q \subset P$, we obtain the required (3.1). □

Remark 3.1. *(i). The function $h_q \in C$, given by (1.2) and*

$$H_q(z) = zh_q'(z) \in S^\star(\beta), \quad \beta = \frac{1-q}{2(1+q)}.$$

These functions will play the same role throughout in our present study.

(ii). For $m = 2$, $F' = \left(K_1'\right)^{1-\beta}$.

Taking logarithmic differentiation and integrating from θ_1 *to* θ_2, $0 \leq \theta_1 < \theta_2 \leq 2\pi$, $z = re^{i\theta}$, *we have*

$$\int_{\theta_1}^{\theta_2} Re\{\frac{(zF'(z))'}{F'(z)}\}d\theta = (1-\beta)\int_{\theta_1}^{\theta_2} Re\{\frac{(zK_1'(z))'}{K_1'(z)}\}d\theta \geq -(1-\beta)\pi.$$

Since $(1-\beta) < 1$, *it follows that* F *is close-to-convex in* E, *by using Lemma 2.2.*

(iii). *Let* $F(z) = (f \star h_q)(z)$, $m = 2$. *Then the growth rate of Hankel determinant* H_l *defined by the coefficient in its series expansion is* $H_l = O(1)\cdot n^{2-l}$, $l \geq 1, n \neq 1$, *see [10]. The exponent is shown is best possible.*

Theorem 3.2. *Let* $f \in V_{m,q}$, $m > 2$. *Then* $f \in T^{\star}_{2,q}$ *for* $|z| < r^{\star}$, *where*

$$r^{\star} = Sin\{\frac{\pi(1+q)}{(m-2)(1+3q)}\}. \tag{3.2}$$

Proof. From Definition 1.1 and Definition 1.2 together Remark 1.1, we can write

$$\begin{aligned} \frac{D_q f(z)}{D_q g_1(z)} &= \left(\frac{D_q g_1(z)}{D_q g_2(z)}\right)^{(\frac{m}{4}-\frac{1}{2})}, \quad g_1, g_2 \in C \\ &= \left(\frac{s_1(z)}{s_2(z)}\right)^{(1-\beta)(\frac{m}{4}-\frac{1}{2})}, \quad s_1, s_2 \in S^{\star}. \end{aligned} \tag{3.3}$$

For $s_1, s_2 \in S^{\star}$, it is known [2] that

$$|\arg \frac{s_1(z)}{z}| \leq 2Sin^{-1}r.$$

So, from (3.3), we have

$$|\arg \frac{D_q f(z)}{D_q g(z)}| \leq (\frac{m}{4} - \frac{1}{2})(1-\beta)\{4Sin^{-1}r\}.$$

This gives us $|\arg \frac{D_q f(z)}{D_q g(z)}| \leq \frac{\pi}{2}$, which implies $(m-2)(1\beta)Sin^{-1}r < \frac{\pi}{2}$. Therefore

$$r < r^{\star} = Sin\{\frac{\pi(1+q)}{(m-2)(1+3q)}\}, \quad m > 2.$$

□

Theorem 3.3. *Let $f \in \mathcal{A}$ and $g \in V_{m,q}$. Define the function h as*

$$D_q h(z) = \frac{D_q(zD_q f(z))}{1 + zqD_q(D_q g(z))}. \tag{3.4}$$

If

$$\frac{D_q(zD_q f(z))}{D_q(zD_q g(z))} \in P_q,$$

then $h \in T^{\star}_{m,q}$ in E.

Proof. We have

$$D_q(zD_q g(z)) = D_q g(z) + zqD_q(D_q g(z)) = D_q g(z)\{1 + zqD_q(D_q g(z))\}.$$

Therefore, for $g \in V_{m,q}$,

$$\begin{aligned} \frac{D_q(zD_q f(z))}{D_q(zD_q g(z))} &= \frac{D_q(zD_q f(z))}{D_q g(z)\{1 + zqD_q(D_q g(z))} \\ &= \frac{D_q(zD_q f(z))}{\{1 + zqD_q(D_q g(z))} \frac{1}{D_q g(z)} \\ &= \frac{D_q h(z)}{D_q g(z)} \in P_q, \quad \text{using (3.4)} \end{aligned}$$

and this implies $h \in T^{\star}_{m,q}$ in E. □

Theorem 3.4. *Let $0 < \gamma_1 < \gamma_2 \leq 1$. Then $Q^{\gamma_1}_{m,q}(a) \subset Q^{\gamma_2}_{m,q}(a)$.*

Proof. Let $f \in Q^{\gamma_1}_{m,q}(a)$. We can write

$$\begin{aligned} zD_qf(z) + af(z) &= (a+1)z\big(D_qg(z)\big)^{\gamma_1}, \quad g \in V_{m,q} \\ &= (a+1)z(D_qG(z))^{\gamma_2}, \end{aligned} \tag{3.5}$$

where

$$D_qG(z) = \big(D_qg(z)\big)^{\frac{\gamma_1}{\gamma_2}}. \tag{3.6}$$

We now show that $G \in V_{m,q}$.

From (3.6), we have

$$\begin{aligned} D_qG(z) &= \left\{\frac{\big(D_qg_1(z)\big)^{(\frac{m}{4}+\frac{1}{2})}}{\big(D_qg_2(z)\big)^{(\frac{m}{4}-\frac{1}{2})}}\right\}^{\frac{\gamma_1}{\gamma_2}} \\ &= \left\{\frac{\big(D_qg_1(z)\big)^{\frac{\gamma_1}{\gamma_2}(\frac{m}{4}+\frac{1}{2})}}{\big(D_qg_2(z)\big)^{\frac{\gamma_1}{\gamma_2}(\frac{m}{4}-\frac{1}{2})}}\right\} \\ &= \left\{\frac{\big(D_qg^{\star}{}_1(z)\big)^{(\frac{m}{4}+\frac{1}{2})}}{\big(D_qg^{\star}{}_2(z)\big)^{(\frac{m}{4}-\frac{1}{2})}}\right\}, \end{aligned} \tag{3.7}$$

where $g_i^{\star} = g_i^{\frac{\gamma 1}{\gamma_2}}$, $(\gamma_1 < \gamma_2)$ and $g_i^{\star} \in C$. Thus, from (3.5) and (3.7), it follows that $f \in Q^{\gamma 2}_{m,q}(a)$ in E. This completes the proof.

□

Remark 3.2. *The class $C_q(\gamma)$ is defined as:*

Let $f \in \mathcal{A}$. Then $f \in C_q(\gamma)$, if and only if, $\frac{D_q(zD_qf(z))}{D_qf(z)} \in P(\gamma)$, $\gamma \in [0, 1)$.

Using Theorem 3.1 and a result due to Kaplan [7] for close-to-convex functions, we easily obtain the following.

Theorem 3.5. *Let $f \in Q^{\gamma}_{m,q}$ and $F = f \star h$. Then, for $0 \le \theta_1 < \theta_2 \le 2\pi$, $z = re^{i\theta}$, we have*

$$\int_{\theta_1}^{\theta_2} Re\left\{\frac{(zF'(z))'}{F'(z)}\right\}d\theta > -\frac{\gamma m(1-\beta)}{2}\pi, \quad \beta = \frac{1-q}{2(1+q)} \in (0, \frac{1}{2}). \tag{3.8}$$

We note that F is close-to-convex (univalent), if $m \leq \frac{2}{\gamma(1-\beta)} = \frac{1+3q}{\gamma(1+q)}$, by Lemma 2.2.

Theorem 3.6. *Let $f \in Q^{\gamma}_{m,q}$ and let $F = (f \star h_q)$. Then*

(i).

$$\frac{(1-r)^{\{\gamma_1(\frac{m}{4}-1)+1\}}}{(1+r)^{\{\gamma_1(\frac{m}{4}+1)+1\}}} \leq |F'(z)| = |D_q f(z)| \leq \frac{(1+r)^{\{\gamma_1(\frac{m}{4}-1)+1\}}}{(1-r)^{\{\gamma_1(\frac{m}{4}+1)+1\}}}$$

where $\gamma_1 = \gamma(1-\beta), \quad \beta = \frac{1-q}{2(1+q)}$.

Also, for

$$r_1 = r_2^{-1} = \frac{1-r}{1+r}; \quad a = \gamma_1(\frac{m}{2}-1)+2; \quad b = 2(1-\gamma_1); \quad c = a+1; \tag{3.9}$$

we have (ii).

$$\begin{aligned} &2^{(1-\gamma_1)}\frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)}\left\{\mathbb{G}(a,b;c;-1) - r_1^a\mathbb{G}(a,b;c;-r_1)\right\} \\ &\leq |F(z)| \\ &\leq 2^{(1-\gamma_1)}\frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)}\left\{\mathbb{G}(a,b;c;-1) - \mathbb{G}(a,b;c;-r_2)\right\}, \end{aligned}$$

where $\mathbb{G}$ is hypergeometric function and Γ represents Gamma function.

Proof. (i). From Definition 1.3, $f \in Q^{\gamma}_{m,q}$ implies that

$$F'(z) = D_q f(z) = \left(D_q f(z)\right)^{\gamma}, \quad G \in T^{\star}_{m,q}.$$

From Definition 1.2 and Remark 1.1. we can write

$$zF'(z) = zD_q f(z) = \left\{\frac{(s_1(z))^{(1-\beta)(\frac{m}{4}+\frac{1}{2})}}{(s_2(z))^{(1-\beta)(\frac{m}{4}-\frac{1}{2})}} \cdot p(z)\right\}^{\gamma},$$
$$s_i \in S^{\star}, \quad i = 1,2, \quad p \in P_q \subset P.$$

Using distortion result for $s_i \in S^{\star}$ and $p \in P$, the proof of part(i) at once.

We proceed to prove part (ii).

Let d_r denote the radius of the largest schlicht disc centered at the origin contained in the image of $|z| < r$ under $F(z) = (f \star h_q)(z)$. Then there is point $z_0, \quad |z_0| = r$ such that $F(z_0) = d_r$. The ray from (0) to $F(z_0)$ lies entirely in the image and inverse image of the ray is a curve in $|z| < 1$. Using part (i), we have

$$\begin{aligned} d_r = |F(z_0)| &= \int_{\mathcal{C}} |F'(z)||dz| \\ &\geq \int_0^{|z|} \frac{(1-s))^{(\gamma_1(\frac{m}{2}-1)+1)}}{(1+s)^{(\gamma_1(\frac{m}{2}+1)+1)}} ds, \quad \gamma_1 = \gamma(1-\beta) \\ &= \int_0^{|z|} (\frac{1-s}{1+s})^{(\gamma_1(\frac{m}{2}-1)+1)} (1+s)^{-2\gamma_1} ds. \end{aligned} \tag{3.10}$$

Let $\frac{1-s}{1+s} = t$. Then, by simple calculations, we have

$$|F(z_0)| \geq \int_1^{r_1} t^{(\gamma_1(\frac{m}{2}-1)+1)} (-2^{(1-2\gamma 1)})(1+t)^{-2(1-\gamma_1)} dt = -2^{1-2\gamma_1)}[I_1 - I_2]. \tag{3.11}$$

Now put $t = r_1 u$, with $r_1 = \frac{1-r}{1+r}$. Then $dt = r_1 du$, and

$$\begin{aligned} I_1 &= \int_0^{r_1} (r_1 u)^{a-1} (1 + r1u)^{-b} (r_1 du) \\ &= r_1^a \int_0^{r_1} u^{a-1} (1 + r_1 u)^{-b} du \\ &= r_1^a \frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)} \mathbb{G}(a, b; c; -r_1), \end{aligned} \tag{3.12}$$

where a, b, c and r_1 are given by (3.9).

Following the similar procedure and calculations, we obtain

$$I_2 = \int_0^1 t^{a-1}(1+t)^{-b} dt = \frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)} \mathbb{G}(q, b, c; -1). \tag{3.13}$$

Then, form (3.11), (3.12) and (3.14), we obtain the lower bound for $|F(z)|$.

For the upper bound, we calculate

$$|F(z)| \leq \int_0^{|z|} (\frac{1+s}{1-s})^{a-1} (\frac{1}{1-s})^{-2\gamma_1} ds$$

and obtain

$$|F(z)| \leq 2^{1-2\gamma} \frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)} \left\{ \mathbb{G}(a,b;c;-1) - r_2^a \mathbb{G}(a,b,c;-r_2) \right\}, \quad (r_1 = r_2^{-1}).$$

This completes the proof of part (ii).

□

Corollary 3.1. *Let $r \to 1$ in the lower bound of $|F(z)|$. Then, from Theorem 3.6, it can easily be deduced that the image of E under $F = f \star h_q, \quad f \in Q_{m,q}^{\gamma}$ contains the Schlicht disc $|z| < r_\star$, where $r_\star = \frac{2^{(1+\gamma_1)}}{\gamma_1(\frac{m}{2}-1)+2}, \quad \gamma_1 = \gamma(1-\beta)$.*

We note that following special cases of Corollary 3.1.

(i). $r_\star = \frac{2^{(1-\gamma)}}{\gamma(\frac{m}{2}-1)+2}$.
When $q \to 1^{-1}$ and for $\gamma = 1$, we have the radius $r_\star = \frac{2}{m+2}$.

(ii). For $\gamma = 1, \quad m = 2$, we obtain $r_\star = \frac{1}{2^{(1-\beta)}}$.

We now estimate the growth rate of coefficients of $F = f \star h_q, \quad f \in Q_{m,q}^{\gamma}$.

Theorem 3.7. *Let*

$$f: \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in Q_{m,q}^{\gamma}$$

and let $F = f \star h_q$ with

$$F(z) = z + \sum_{n=2}^{\infty} A_z^n.$$

Then

$$|A_n| \leq \eta(\gamma, m, q) n^\sigma, \quad \sigma = \{\gamma(1-\beta)(\frac{m}{2}+1) + \gamma - 2\},$$

where η is a constant depending on q and γ.

Proof. By Cauchy theorem, for $z = re^{i\theta}$,

$$\begin{aligned} |nA_n| &= \frac{1}{2\pi r^n}\Big|\int_0^{2\pi}\{zF'(z)e^{-in\theta}\}d\theta\Big| \\ &\leq \frac{1}{2\pi r^{n+\gamma}}\int_0^{2\pi}\big|zD_q g(z)\big|^{\gamma}d\theta, \quad g\in T^{\star}_{m,q} \\ &= \frac{1}{2\pi r^{n+\gamma}}I_\gamma(r). \end{aligned} \tag{3.14}$$

Since $g \in T^{\star}_{m,q}$, there exists $g_1 \in V_{m,q}$, such that

$$D_q g(z) = D_q g_1(z)p(z), \quad p \in P_q \subset P. \tag{3.15}$$

We calculate $I_\gamma(r)$ as follows by using (3.15).

Using Remark 1.1 and Definition 1.2, we have

$$I_\gamma(r) = \int_0^{2\pi}\frac{|s_1(z)|^{\gamma(1-\beta)(\frac{m}{4}+\frac{1}{2})}}{|s_2(z)|^{\gamma(1-\beta)(\frac{m}{4}-\frac{1}{2})}}|p(z)|^{\gamma}d\theta, \quad s_1, s_2 \in S^{\star}. \tag{3.16}$$

For $s_2 \in S^{\star}$, the well known distortion result in (3.16), gives us

$$I_\gamma(r) \leq r^{\gamma_1}\big(\frac{1}{4}\big)^{\gamma_1(\frac{m-2}{4})}\int_0^{2\pi}|s_1(z)|^{\gamma_1(\frac{m+2}{4})}|p(z)|^{\gamma}d\theta, \tag{3.17}$$

where $\gamma_1 = \gamma(1-\beta)$.

Now, applying Holder's inequality, subordination for $s_1 \in S^{\star}$ and Lemma 2.3 for $p \in P$, we obtain from (3.17

$$\begin{aligned} I_\gamma(r) &\leq \big(\frac{1}{4}\big)^{\gamma_1(\frac{m-2}{4})}\left(\frac{1}{2\pi}\int_0^{2\pi}|s_1(z)|^{\gamma_1(\frac{m+2}{4})\frac{2}{2-\gamma}}d\theta\right)^{\frac{2-\gamma}{2}}\left(\frac{1}{2\pi}\int_0^{2\pi}|p(z)|^2 d\theta\right)^{\frac{\gamma}{2}} \\ &\leq \eta(\gamma, m, q)\left(\frac{1}{1-r}\right)^{\gamma_1(\frac{m}{2}+1)+\gamma-1}, \quad (r\to 1) \end{aligned} \tag{3.18}$$

where $\gamma_1 = \gamma(1-\beta)$, $\beta = \frac{1-q}{2(1+q)}$, $m \geq 2$ and η is a constant depending on γ, m and q. Thus, with $r = (1-\frac{1}{n})$, $n \to \infty$, it follows from (3.14) and (3.18) that

$$|A_n| \leq \eta(\gamma, m, q)\cdot n^{\sigma}, \quad \sigma = \gamma_1(\frac{m}{2}+1)+\gamma-2.$$

This completes the proof. □

Remark 3.3.
(i). *As a special case, we note that* $A_n = O(1) \cdot n^{\frac{m}{2}}$ *for* $q \to 1^{-1}$ *and* $\gamma = 1$, $0(1)$ *is a constant. This result has been proved in [9].*

(ii). *Also* $F(z) = (f(z) \star h_q(z))$, $f \in Q^{\gamma}_{m,q}$ *and*

$$h_q(z) = z + \sum_{n=2}^{\infty} [n]_q \cdot \frac{z^n}{n}.$$

Therefore $A_n = \frac{[n]_a}{n} a_n$ *and this yields the growth rate of the coefficients for* $f \in Q^{\gamma}_{m,q}$ *as*

$$a_n = 0(1) \cdot n^{\sigma_1}, \quad n\sigma_1 = \sigma[n]_q.$$

When $q \to 1^{-1}$, $\sigma_1 = \sigma$.

(iii). *For* $\gamma = 1$, $m = 2$, *we have* $A_n = 0(1) n^{1-2\beta}$, $1 - 2\beta = \frac{2q}{1+q}$.

Conclusion

The concept of q-calculus has been used to define and study the classes of $Q^{\gamma}_{m,q}(a)$ containing the q-analogue of analytic functions with bounded boundary rotation. We have shown that the q-derivative of certain convex functions has played an important role to investigate some interesting properties such as distortion, inclusion and rate of growth of coefficients of these new classes. Some inclusion properties with references to parameters involved , Hankel determinant problem and study of certain q-analogue of linear operators, which may include Bernadi and Carlson-Shaeffer operators, can be explored as open problems.

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