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Fixed Point Results of Rational Type-contraction Mapping in b-Metric Spaces with an Application

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Abstract

In this article, we establish the existence of fixed points of rational type contractions in the setting of b-metric spaces and we verify the T-stability of the P property for some mappings. Also, we present a few examples to illustrate the validity of the results obtained in the paper. Finally, results are applied to find the solution for an integral equation.

1. Introduction

Fixed point theory is one of the most well-known and established theories in mathematics and has a variety of applications. In this theory, contraction is one of the main tools to prove the existence and uniqueness of a fixed point. In [4] Banach proved a very significant result in complete metric spaces which gives unique fixed point on complete metric space. In 1989, Bakhtin [3] and Czerwik [9] presented the notion of b-metric spaces as a generalization of metric spaces. Recently, Kamran et al. [16] gave the concept of extended b-metric space and introduced a counterpart of Banach contraction principle. Some well-known results in this direction are involved (see [1, 6, 8, 10, 14, 19-21, 23]).

Theorem 1.1 (see [14]). *Let T be a continuous self mapping on a complete metric space (X,d). If T is a rational type contraction, there exist $\alpha,\beta \in [0,1)$, where $\alpha+\beta < 1$ such that*

$$d(Tp,Tq) \leq \alpha d(p,q) + \beta \frac{d(p,Tp)d(q,Tq)}{d(p,q)} \tag{1.1}$$

for all $p,q \in X, p \neq q$, then T has a unique fixed point in X.

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Theorem 1.2 (see [10]). *Let T be a continuous self mapping on a complete metric space (X,d). If T is a rational type contraction, there exist $\alpha,\beta\in[0,1)$, where $\alpha+\beta<1$ such that*

$$d(Tp,Tq)\leq \alpha d(p,q)+\beta\cdot\frac{d(q,Tq)[1+d(p,Tp)]}{1+d(p,q)} \tag{1.2}$$

for all $p,q\in X$, then T has a unique fixed point in X.

Fisher [12] refined the result of Khan [19] in the following way.

Theorem 1.3 (see [12]). *Let T be a self mapping on a complete metric space (X,d). If T is a rational type contraction, T satisfies the inequality*

$$d(Tp,Tq)\leq k\begin{cases}\frac{d(p,Tp)d(p,Tq)+d(p,Tq)d(q,Tp)}{d(p,Tq)+d(q,Tp)}, & if\ d(p,Tq)+d(q,Tp)\neq 0\\ 0, & if\ d(p,Tq)+d(q,Tp)=0\end{cases} \tag{1.3}$$

for all $p,q\in X$, where $0\leq k<1$. Then, T has a unique fixed point in X.

In this work we prove fixed points of rational type contractions in the context of b-metric spaces and show that the P property is T-stable for some mappings. Also, we give a few examples to show the applicability of the findings made in the paper. The solution to an integral equation is then determined using some findings.

2. Preliminaries

In this section, we present a few key terms and definitions that will be used in our discussion.

Definition 2.1 (see [3,9]). Let X be a set and let $s\geq 1$ be a given real number. A function $d: X\times X\to R^{+}$ is said to be a *b-metric* if and only if for all $p,q,r\in X$ the following conditions are satisfied:

1. $d(p,q)=0$ if and only if $p=q$;
2. $d(p,q)=d(q,p)$;
3. $d(p,r)\leq s.[d(p,q)+d(q,r)]$.

Then the pair (X,d) is called a *b-metric space*.

Every metric space is b-metric for $s=1$, which is obvious from the concept of b-metric, but the opposite is not true.

The following examples gives us evidence that b-metric space is indeed different from metric space.

Example 2.1 (see [22])**.** Let (X, d) be a metric space and let the mapping $d: X \times X \to [0, \infty)$ be defined by

$$d(p, q) = (d(p, q))^{\eta}, \text{ for all } p, q \in X$$

where $\eta > 1$ is a fixed real number. Then (X, d) is a b-metric space with $s = 2^{\eta - 1}$.

In particular, if $X = \mathbb{R}$, $d(p, q) = |p - q|$ is the usual Euclidean metric and

$$d(p, q) = (p - q)^2, \text{ for all } p, q \in \mathbb{R}$$

then $(\mathbb{R}, d)$ is a b-metric with $s = 2$. However, $(\mathbb{R}, d)$ is not a metric space on $\mathbb{R}$ since the axiom 3 in Defintion 2.1 does not hold. Indeed,

$$d(-2,2) - 16 > 8 - 4 + 4 - d(-2,0) + d(0,2).$$

Example 2.2 (see [18])**.** Let X be the set of Lebesgue measurable functions on $[0,1]$ such that

$$\int_0^1 |p(t)|^2 \, \mathrm{d}t < \infty.$$

Define $d: X \times X \to [0, \infty)$ by

$$d(p, q) = \int_0^1 |p(t) - q(t)|^2 \, \mathrm{d}t.$$

Then d satisfies the following properties

1. $d(p, q) = 0$ if and only if $p = q$,
2. $d(p, q) = d(q, p)$, for any $p, q \in X$,
3. $d(p, q) \leq 2(d(p, r) + d(r, q))$, for any points $p, q, r \in X$.

Clearly, (X, d) is a b-metric space with $s = 2$ but is not a metric space. For example, take $p(t) = 0, q(t) = 1$ and $r(t) = \frac{1}{2}$, for all $t \in [0,1]$. Then

$$d(0,1) = 1 > \frac{1}{2} = \frac{1}{4} + \frac{1}{4} = d\left(0, \frac{1}{2}\right) + d\left(\frac{1}{2}, 1\right).$$

We present now the concepts of convergence, Cauchy sequence and completeness in b-metric spaces.

Definition 2.2 (see [5])**.** Let (X, d) be a b-metric space. Then, a sequence $\{x_n\}$ in X is called:

1. *convergent* if and only if there exists $u \in X$ such that $d(p_n, u) \to 0$, as $n \to +\infty$. In this case we write $\lim_{n\to\infty} p_n = u$.

2. *Cauchy* if and only if $d(p_n, p_m) \to 0$, as $n, m \to +\infty$.

Definition 2.3 (see [2])**.** The b-metric space (X, d) is said *complete* if every Cauchy sequence in X converges in X.

3. Main Results

The following lemma is useful in proving all main results.

Lemma 3.1 (see [13])**.** *Let (X, d) be a b-metric space with coefficient $s \geq 1$ and $T: X \to X$ be a mapping. Suppose that $\{p_n\}$ is a sequence in X induced by $p_{n+1} = Tp_n$ such that*

$$d(p_n, p_{n+1}) \leq \lambda d(p_{n-1}, p_n)$$

for all $n \in \mathbb{N}$, where $\lambda \in [0,1)$ is a constant. Then $\{p_n\}$ is a Cauchy sequence.

Now, we will prove of our main theorems.

Theorem 3.1. *Let (X, d) be a b-complete b-metric space with coefficient $s \geq 1$ and $T: X \to X$ be a mapping such that*

$$d(Tp, Tq) \leq \lambda_1 d(p, q) + \lambda_2 \frac{d(p,Tp)d(q,Tp) + d(q,Tq)d(p,Tq)}{d(p,Tq) + d(q,Tp)} \tag{3.1}$$

for all $p, q \in X$ and $\lambda_1, \lambda_2 \geq 0, d(p, Tq) + d(p, Tq) \neq 0$ with $\lambda_1 + \lambda_2 < 1$. Then T has a unique fixed point in X.

Proof. Let p_0 be arbitrary in X, we define a sequence $\{p_n\}$ in X such that

$$p_{n+1} = Tp_n,$$

for all $n \in \mathbb{N}$, from the condition (3.1) with $p = p_n$ and $q = p_{n-1}$. Therefore

$$\begin{aligned} d(p_n, p_{n+1}) &= d(Tp_{n-1}, Tp_n) \\ &\leq \lambda_1 d(p_{n-1}, p_n) + \lambda_2 \frac{d(p_{n-1},Tp_{n-1})d(p_n,Tp_{n-1}) + d(p_n,Tp_n)d(p_{n-1},Tp_n)}{d(p_{n-1},Tp_n) + d(p_n,Tp_{n-1})} \\ &\leq \lambda_1 d(p_{n-1}, p_n) + \lambda_2 \frac{d(p_{n-1},p_n)d(p_n,p_n) + d(p_n,p_{n+1})d(p_{n-1},p_{n+1})}{d(p_{n-1},p_{n+1}) + d(p_n,p_n)} \\ &\leq \lambda_1 d(p_{n-1}, p_n) + \lambda_2 d(p_n, p_{n+1}). \end{aligned}$$

It follows that

$$(1-\lambda_2)d(p_n,p_{n+1}) \leq \lambda_1 d(p_{n-1},p_n) \tag{3.2}$$

$$d(p_n,p_{n+1}) \leq \left(\frac{\lambda_1}{1-\lambda_2}\right) d(p_{n-1},p_n).$$

Put $\lambda = \frac{\lambda_1}{1-\lambda_2}$. In view of $\lambda_1 + \lambda_2 < 1$, then $0 \leq \lambda < 1$. Thus, by Lemma 3.1, $\{p_n\}$ is a b-Cauchy sequence in X. Since (X,d) is b-complete, there exists some point $u^* \in X$ such that $p_n \to u^*$ as $n \to \infty$.

By (3.1), it is easy to see that

$$d(p_{n+1},Tu^*) = d(Tp_n,Tu^*) \tag{3.3}$$

$$\leq \lambda_1 d(p_n,u^*) + \lambda_2 \frac{d(p_n,Tp_n)d(u^*,Tp_n)+d(u^*,Tp^*)d(p_n,Tu^*)}{d(p_n,Tu^*)+d(u^*,Tp_n)}$$

$$\leq \lambda_1 d(p_n,u^*) + \lambda_3 \frac{d(p_n,p_{n+1})d(u^*,p_{n+1})+d(u^*,Tu^*)d(p_n,Tu^*)}{d(p_n,Tu^*)+d(u^*,p_{n+1})}. \tag{3.4}$$

Taking the limit as $n \to \infty$ by both parties of (3.4), we have $\lim_{n\to\infty} d(p_{n+1}, Tu^*) = 0$. That is, $p_n \to Tu^*$. Hence, $Tu^* = u^*$, u^* is a fixed point of T.

Finally, we prove the uniqueness of the fixed point. Indeed, if there is another fixed point v^*, then by (3.1),

$$d(u^*,v^*) = d(Tu^*,Tv^*)$$

$$\leq \lambda_1 d(u^*,v^*) + \lambda_2 \frac{d(u^*,Tu^*)d(v^*,Tu^*)+d(v^*,Tv^*)d(u^*,Tv^*)}{d(u^*,Tv^*)+d(v^*,Tu^*)}$$

$$\leq \lambda_1 d(u^*,v^*) + \lambda_2 \frac{d(u^*,u^*)d(v^*,u^*)+d(v^*,v^*)d(u^*,v^*)}{d(u^*,v^*)+d(v^*,u^*)}\}$$

$$d(u^*,v^*) \leq \lambda_1 d(u^*,v^*). \tag{3.5}$$

Since $\lambda_1 + \lambda_2 < 1$ implies $\lambda_1 < 1$, we obtain that $d(u^*,v^*) = 0$, i.e., $u^* = v^*$.

Example 3.1. *Let* $X = [0,1]$ *be equipped with the b-complete b-metric given by* $d(p,q) = |p-q|^2$ *with* $s = 2$. *Consider the mapping* $T: X \to X$ *defined by*

$$T(p) = \frac{1}{18}p^2 e^{-p^2}$$

for all $p,q \in X$

$$d(Tp,Tq) = \left|\frac{1}{18}p^2 e^{-p^2} - \frac{1}{18}q^2 e^{-q^2}\right|^2$$

$$\leq \tfrac{1}{9}\left|p^2 e^{-p^2} - q^2 e^{-q^2}\right|^2$$

$$\leq \tfrac{1}{9}|p-q|^2 \leq \tfrac{1}{3}d(p,q)$$

$$\leq \lambda_1 d(p,q) + \lambda_2 \frac{d(p,Tq)d(p,Tp)+d(q,Tq)d(p,Tq)}{d(p,Tq)+d(q,Tp)}.$$

Then, from Theorem 3.1, T has unique fixed point.

Theorem 3.2. *Let* (X,d) *be a b-complete b-metric space with coefficient* $s \geq 1$ *and* $T: X \to X$ *be a mapping such that*

$$d(Tp,Tq) \leq \lambda_1 d(p,q) + \lambda_2 \frac{d(p,Tp)d(p,Tq)+d(q,Tq)d(q,Tp)}{d(p,Tq)+d(q,Tp)}$$

$$+\lambda_3 \frac{d(p,Tp)d(q,Tp)+d(q,Tq)d(p,Tq)}{d(p,Tq)+d(q,Tp)}, \tag{3.6}$$

where $\lambda_1, \lambda_2, \lambda_3$ *are nonnegative constants with* $\lambda_1 + \lambda_2 + \lambda_3 < 1$. *Then* T *has a unique fixed point in* X.

Proof. Choose $p_0 \in X$ and construct a Picard iterative sequence $\{p_n\}$ by $p_{n+1} = Tp_n$. If there exists $n_0 \in \mathbb{N}$ such that $p_{n_0} = p_{n_0+1}$, then $p_{n_0} = p_{n_0+1} = Tp_{n_0}$, i.e., p_{n_0} is a fixed point of T. Next, without loss of generality, let $p_n \neq p_{n+1}$ for all $n \in \mathbb{N}$. By (3.6), we get

$$d(p_n, p_{n+1}) = d(Tp_{n-1}, Tp_n)$$

$$\leq \lambda_1 d(p_{n-1}, p_n) + \lambda_2 \frac{d(p_{n-1},Tp_{n-1})d(p_{n-1},Tp_n)+d(p_n,Tp_n)d(p_n,Tp_{n-1})}{d(p_{n-1},Tp_n)+d(p_n,Tp_{n-1})}$$

$$+\lambda_3 \frac{d(p_{n-1},Tp_{n-1})d(p_n,Tp_{n-1})+d(p_n,Tp_n)d(p_{n-1},Tp_n)}{d(p_{n-1},Tp_n)+d(p_n,Tp_{n-1})}$$

$$\leq \lambda_1 d(p_{n-1}, p_n) + \lambda_2 \frac{d(p_{n-1},p_n)d(p_{n-1},p_{n+1})+d(p_n,p_{n+1})d(p_n,p_n)}{d(p_{n-1},p_{n+1})+d(p_n,p_n)}$$

$$+\lambda_3 \frac{d(p_{n-1},p_n)d(p_n,p_n)+d(p_n,p_{n+1})d(p_{n-1},p_{n+1})}{d(p_{n-1},p_{n+1})+d(p_n,p_n)}$$

$$\leq \lambda_1 d(p_{n-1}, p_n) + \lambda_2 d(p_{n-1}, p_n) + \lambda_3 d(p_n, p_{n+1}).$$

It follows that

$$(1-\lambda_3)d(p_n, p_{n+1}) \leq (\lambda_1 + \lambda_2)d(p_{n-1}, p_n) \tag{3.7}$$

$$d(p_n, p_{n+1}) \leq \left(\frac{\lambda_1+\lambda_2}{1-\lambda_3}\right) d(p_{n-1}, p_n).$$

Put $\lambda = \frac{\lambda_1+\lambda_2+}{1-\lambda_3}$. In view of $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 < 1$, then $0 \leq \lambda < 1$. Thus, by

Lemma 3.1, $\{p_n\}$ is a b-Cauchy sequence in X. Since (X, d) is b-complete, there exists some point $u^* \in X$ such that $p_n \to u^*$ as $n \to \infty$.

By (3.6), it is easy to see that

$$d(u^*, Tu^*) \leq s\{d(u^*, p_{n+1}) + d(p_{n+1}, Tu^*)\} \tag{3.8}$$

$$= s\{d(u^*, p_{n+1})\} + s\{d(Tp_n, Tu^*)\}$$

$$\leq s\{d(u^*, p_{n+1})\} + s\{\lambda_1 d(p_n, u^*)$$

$$+ \lambda_2 \frac{d(p_n,Tp_n)d(p_n,Tu^*)+d(u^*,Tu^*)d(u^*,Tp_n)}{d(p_n,Tu^*)+d(u^*,Tp_n)}$$

$$+\lambda_3 \frac{d(p_n,Tp_n)d(u^*,Tp_n)+d(u^*,Tu^*)d(p_n,Tu^*)}{d(p_n,Tu^*)+d(u^*,Tp_n)}\}$$

$$\leq s\{d(u^*, p_{n+1})\} + s\{\lambda_1 d(p_n, u^*)$$

$$+\lambda_2 \frac{d(p_n,p_{n+1})d(p_n,Tu^*)+d(u^*,Tu^*)d(u^*,p_{n+1})}{d(p_n,Tu^*)+d(u^*,p_{n+1})} \tag{3.9}$$

$$+\lambda_3 \frac{d(p_n,p_{n+1})d(u^*,p_{n+1})+d(u^*,Tu^*)d(p_n,Tu^*)}{d(p_n,Tu^*)+d(u^*,p_{n+1})}\}.$$

Taking the limit as $n \to \infty$ by both parties of (3.9), we have $\lim_{n\to\infty} d(u^*, Tu^*) = 0$. Hence, $Tu^* = u^*$ and u^* is a fixed point of T.

Finally, we prove the uniqueness of the fixed point. Indeed, if there is another fixed point v^*, then by (3.6),

$$d(u^*, v^*) = d(Tu^*, Tv^*)$$

$$\leq \lambda_1 d(u^*, v^*) + \lambda_2 \frac{d(u^*,Tu^*)d(u^*,Tv^*)+d(v^*,Tv^*)d(v^*,Tu^*)}{d(u^*,Tv^*)+d(v^*,Tu^*)}$$

$$+\lambda_3 \frac{d(u^*,Tu^*)d(v^*,Tu^*)+d(v^*,Tv^*)d(u^*,Tv^*)}{d(u^*,Tv^*)+d(v^*,Tu^*)}$$

$$\leq \lambda_1 d(u^*, v^*) + \lambda_2 \frac{d(u^*,u^*)d(u^*,v^*)+d(v^*,v^*)d(v^*,u^*)}{d(u^*,v^*)+d(v^*,u^*)}$$

$$+\lambda_3 \frac{d(u^*,u^*)d(v^*,u^*)+d(v^*,v^*)d(u^*,v^*)}{d(u^*,v^*)+d(v^*,u^*)}$$

$$d(u^*, v^*) \leq \lambda_1 d(u^*, v^*). \tag{3.10}$$

Since $\lambda_1 + \lambda_2 + \lambda_3 < 1$ implies $\lambda_1 + \lambda_3 < 1$, we have $d(u^*, v^*) = 0$, i.e., $u^* = v^*$.

Example 3.2. Let $X = [0,1]$ and define a mapping $d: X \times X \to \mathbb{R}^+$ by $d(p,q) = |p - q|^\eta (\eta \geq 1)$. We claim that (X,d) is a b-complete b-metric space with coefficient $s = 2^{\eta-1}$. Define a mapping $T: X \to X$ by $Tp = e^{p-2\lambda}$, where $\lambda > 1 + ln2$ is a constant. Then by mean value theorem of differentials, for any $p, q \in X$ and $p \neq q$, there exists some real number ξ belonging to between p and q such that

$$d(Tp,Tq) = \left|\mathrm{e}^{p-2\lambda} - \mathrm{e}^{q-2\lambda}\right|^\eta = \left(\mathrm{e}^{\xi-2\lambda}\right)^\eta |p-q|^\eta$$

$$\leq \left(\mathrm{e}^{2-2\lambda}\right)^\eta d(p,q)$$

$$\leq \lambda_1 d(p,q) + \lambda_2 \frac{d(p,Tp)d(p,Tq) + d(q,Tq)d(q,Tp)}{d(p,Tq) + d(q,Tp)}$$

$$+\lambda_3 \frac{d(p,Tp)d(q,Tp) + d(q,Tq)d(p,Tq)}{d(p,Tq) + d(q,Tp)},$$

where $\lambda_1 = \left(\mathrm{e}^{2-2\lambda}\right)^\eta, \lambda_2 = \lambda_3 = 0$. Obviously, $\lambda_1 + \lambda_2 + \lambda_3 < 1$. Hence, all the conditions of Theorem 3.2 are satisfied and T has a unique fixed point in X, i.e., p_0 is the fixed point.

Theorem 3.3. *Let (X,d) be a b-complete b-metric space with coefficient $s \geq 1$ and $T: X \to X$ be a mapping such that*

$$d(Tp,Tq) \leq \lambda_1 d(p,q) + \lambda_2 \frac{d(p,Tp)d(q,Tq)}{d(p,q)} + \lambda_3 \frac{d(p,Tq)d(q,Tp)}{d(,q)}$$

$$+\lambda_4[d(p,Tp) + d(q,Tq)] + \lambda_5[d(q,Tp) + d(p,Tq)] \tag{3.11}$$

for all $p, q \in X$ and $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ and λ_5 are nonnegative constants with $\lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_4 + 2s\lambda_5 < 1$. Then T has a unique fixed point in X.

Proof. Let p_0 be arbitrary in X, we define a sequence $\{p_n\}$ in X such that

$$p_{n+1} = Tp_n,$$

for all $n \in \mathbb{N}$, from the condition (3.11) with $p = p_n$ and $q = p_{n-1}$. Therefore

$$d(p_n, p_{n+1}) = d(Tp_{n-1}, Tp_n)$$

$$\leq \lambda_1 d(p_{n-1}, p_n) + \lambda_2 \frac{d(p_{n-1},Tp_{n-1})d(p_n,Tp_n)}{d(p_{n-1},p_n)} + \lambda_3 \frac{d(p_{n-1},Tp_n)d(p_n,Tp_{n-1})}{d(p_{n-1},p_n)}$$

$$+\lambda_4[d(p_{n-1},Tp_{n-1}) + d(p_n,Tp_n)] + \lambda_5[d(p_n,Tp_{n-1}) + d(p_{n-1},Tp_n)]$$

$$\leq \lambda_1 d(p_{n-1}, p_n) + \lambda_2 \frac{d(p_{n-1},p_n)d(p_n,p_{n+1})}{d(p_{n-1},p_n)} + \lambda_3 \frac{d(p_{n-1},p_{n+1})d(p_n,p_n)}{d(p_{n-1},p_n)}$$

$$+\lambda_4[d(p_{n-1},p_n)+d(p_n,p_{n+1})]+\lambda_5[d(p_n,p_n)+d(p_{n-1},p_{n+1})]$$

$$\leq \lambda_1 d(p_{n-1},p_n)+\lambda_2 d(p_n,p_{n+1})+\lambda_4[d(p_{n-1},p_n)+d(p_n,p_{n+1})]$$

$$+\lambda_5[d(p_{n-1},p_{n+1})]$$

$$\leq \lambda_1 d(p_{n-1},p_n)+\lambda_2 d(p_n,p_{n+1})+\lambda_4[d(p_{n-1},p_n)+d(p_n,p_{n+1})]$$

$$+s\lambda_5[d(p_{n-1},p_n)+d(p_n,p_{n+1})].$$

It follows that

$$(1-\lambda_2-\lambda_4-s\lambda_5)d(p_n,p_{n+1})\leq(\lambda_1+\lambda_4+s\lambda_5)d(p_{n-1},p_n) \tag{3.12}$$

$$d(p_n,p_{n+1})\leq\left(\frac{\lambda_1+\lambda_4+s_5}{1-\lambda_2-\lambda_4-s\lambda_5}\right)d(p_{n-1},p_n).$$

Put $\frac{\lambda_1+\lambda_4+s_5}{1-\lambda_2-\lambda_4-s\lambda_5}$. In view of $\lambda_1+\lambda_2+\lambda_3+2\lambda_4+2s\lambda_5<1$, then $0\leq\lambda<1$. Thus, by Lemma 3.1, $\{p_n\}$ is a b-Cauchy sequence in X. Since (X,d) is b-complete, there exists some point $u^*\in X$ such that $p_n\to u^*$ as $n\to\infty$.

By (3.11), it is easy to see that

$$d(u^*,Tu^*)\leq s\{d(u^*,p_{n+1})+d(p_{n+1},Tu^*)\} \tag{3.13}$$

$$=s\{d(u^*,p_{n+1})\}+s\{d(Tp_n,Tu^*)\}$$

$$\leq s\{d(u^*,p_{n+1})\}+s\lambda_1 d(p_n,u^*)+s\lambda_2\frac{d(p_n,Tp_n)d(u^*,Tu^*)}{d(p_n,u^*)}+s\lambda_3\frac{d(p_n,Tu^*)d(u^*,Tp_n)}{d(p_n,u^*)}$$

$$+s\lambda_4[d(p_n,Tp_n)+d(u^*,Tu^*)]+s\lambda_5[d(u^*,Tp_n)+d(p_n,Tu^*)]$$

$$\leq s\{d(u^*,p_{n+1})\}+s\lambda_1 d(p_n,u^*)+\lambda_2\frac{d(p_n,p_{n+1})d(u^*,Tu^*)}{d(p_n,u^*)}+\lambda_3\frac{d(p_n,Tu^*)d(u^*,p_{n+1})}{d(p_n,u^*)}$$

$$+s\lambda_4[d(p_n,p_{n+1})+d(u^*,Tu^*)]+s\lambda_5[d(u^*,p_{n+1})+d(p_n,Tu^*)]$$

$$\leq s\{d(u^*,p_{n+1})\}+s\lambda_1 d(p_n,u^*)+\lambda_2\frac{d(p_n,p_{n+1})d(u^*,Tu^*)}{d(p_n,u^*)}+\lambda_3\frac{d(p_n,Tu^*)d(u^*,p_{n+1})}{d(p_n,u^*)}$$

$$+s\lambda_4 d(p_n,p_{n+1})+s^2\lambda_4[d(u^*,p_n)+d(p_n,Tu^*)]+s\lambda_5[d(u^*,p_{n+1})+d(p_n,Tu^*)].$$

Taking the limit as $n\to\infty$ by both parties of (3.14), we have $\lim_{n\to\infty}d(u^*,\,Tu^*)=0$. Hence, $Tu^*=u^*$and x^* is a fixed point of T.

Finally, we prove the uniqueness of the fixed point. Indeed, if there is another fixed point v^*, then by (3.11),

$$
\begin{aligned}
d(u^*,v^*) &= d(Tu^*,Tv^*) \\
&\leq \lambda_1 d(u^*,v^*) + \lambda_2 \frac{d(u^*,Tu^*)d(v^*,Tv^*)}{d(u^*,v^*)} + \lambda_3 \frac{d(u^*,Tv^*)d(v^*,Tu^*)}{d(u^*,v^*)} \\
&\quad +\lambda_4[d(u^*,Tu^*) + d(v^*,Tu^*)] + \lambda_5[d(v^*,Tu^*) + d(u^*,Tv^*)] \\
&\leq \lambda_1 d(u^*,v^*) + \lambda_2 \frac{d(u^*,u^*)d(v^*,v^*)}{d(u^*,v^*)} + \lambda_3 \frac{d(u^*,v^*)d(v^*,u^*)}{d(u^*,v^*)}\} \\
&\quad +\lambda_4[d(u^*,u^*) + d(v^*,v^*)] + \lambda_5[d(v^*,u^*) + d(u^*,v^*)] \\
d(u^*,v^*) &\leq \lambda_1 d(u^*,v^*) + \lambda_3 d(u^*,v^*) + 2\lambda_5 d(u^*,v^*) \\
&\leq (\lambda_1 + \lambda_3 + 2\lambda_5) d(u^*,v^*) \qquad (3.14)
\end{aligned}
$$

since $0 < \lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_4 + 2s\lambda_5 < 1$ implies $\lambda_1 + \lambda_3 + 2\lambda_5 < 1$, then we have $d(x^*,y^*) = 0$. Thus, we proved that T have a unique fixed point in X.

Example 3.3. Let $X = [0,1]$ with the usual metric. We define an operator $T: X \to X$ as follows:

$$
Tp = \begin{cases} \frac{p}{12} & \text{if } p \in \left[0, \frac{1}{4}\right] \\ \frac{p}{6} - \frac{1}{48} & \text{if } p \in \left(\frac{1}{4}, 1\right]. \end{cases}
$$

Then T is continuous and non-decreasing. Take $\lambda_1 = \frac{1}{5}$. Then, for any $\lambda_2, \lambda_3 \in [0,1)$ with $\lambda_1 + \lambda_2 + \lambda_3 < 1$.

Case 01. If $p, q \in \left[0, \frac{1}{4}\right]$, then

$$
\begin{aligned}
d(Tp,Tq) &= \frac{1}{12}|p-q| \\
&\leq \frac{1}{5}|p-q| = \frac{1}{5} d(p,q) \\
&\leq \frac{1}{5} d(p,q) + \lambda_2 \frac{d(p,Tp)d(q,Tq)}{d(p,q)} + \lambda_3 \frac{d(p,Tq)d(q,T)}{d(p,q)} \\
&\quad +\lambda_4[d(p,Tp) + d(q,Tq)] + \lambda_5[d(q,Tp) + d(p,Tq)].
\end{aligned}
$$

Thus, all the conditions of Theorem 3.3 are satisfied.

Case 02. If $p, q \in \left(\frac{1}{4}, 1\right]$, then

$$
d(Tp,Tq) = \frac{1}{12}|p-q|
$$

$$\leq \frac{1}{5}|p-q| = \frac{1}{5}d(p,q)$$

$$\leq \frac{1}{5}d(p,q) + \lambda_2 \frac{d(p,Tp)d(q,Tq)}{d(p,q)} + \lambda_3 \frac{d(p,Tq)d(q,Tp)}{d(p,q)}$$

$$+\lambda_4[d(p,Tq)+d(q,Tq)] + \lambda_5[d(q,Tp)+d(p,Tq)].$$

Thus, all the conditions of Theorem 3.3 are satisfied.

Case 03. If $p \in \left(\frac{1}{4},1\right]$ and $q \in \left[0,\frac{1}{4}\right]$, then we can easily evaluate that $\frac{1}{48}|4p-1| \leq \frac{1}{48}$. Further, we have

$$d(Tp,Tq) = \left|\frac{p}{6} - \frac{1}{48} - \frac{q}{12}\right|$$

$$\leq \frac{1}{12}|p-q| + \frac{1}{48}|2p-1|$$

$$\leq \frac{1}{5}d(p,q) + \lambda_2 \frac{d(p,Tp)d(q,Tq)}{d(p,q)} + \lambda_3 \frac{d(p,Tq)d(q,Tp)}{d(p,q)}$$

$$+\lambda_4[d(p,Tp)+d(q,Tq)] + \lambda_5[d(q,Tp)+d(p,Tq)].$$

Thus all conditions of Theorem 3.3 are satisfied in X.Therefore, $0 \in X$ is the unque fixed point of T.

Theorem 3.4. *Let (X,d) be a b-complete b-metric space with coefficient $s \geq 1$ and $T: X \to X$ be a mapping such that*

$$d(Tp,Tq) \leq \lambda_1 d(p,q) + \lambda_2 \frac{d(p,Tp)d(q,Tq)}{d(p,q)} + \lambda_3 \frac{d(q,Tq)[1+d(p,Tp)]}{1+d(p,q)} \tag{3.15}$$

for all $p,q \in X$ and $\lambda_1, \lambda_2, \lambda_3$ are nonnegative constants with $\lambda_1 + \lambda_2 + \lambda_3 < 1$. Then T has a unique fixed point in X.

Proof. Let p_0 be arbitrary in X, we define a sequence $\{p_n\}$ in X such that

$$p_{n+1} = Tp_n,$$

for all $n \in \mathbb{N}$, from the condition (3.16) with $p = p_n$ and $q = p_{n-1}$. Therefore

$$d(p_n, p_{n+1}) = d(Tp_{n-1}, Tp_n)$$

$$\leq \lambda_1 d(p_{n-1}, p_n) + \lambda_2 \frac{d(p_{n-1},Tp_{n-1})d(p_n,Tp_n)}{d(p_{n-1},\mathrm{p})} + \lambda_3 \frac{d(p_n,Tp_n)[1+d(p_{n-1},Tp_{n-1}])}{1+d(p_{n-1},p_n)}$$

$$\leq \lambda_1 d(p_{n-1}, p_n) + \lambda_2 \frac{d(p_{n-1},p_n)d(p_n,p_{n+1})}{d(p_{n-1},p_n)} + \lambda_3 \frac{d(p_n,p_{n+1})[1+d(p_{n-1},p_n)}{1+d(p_{n-1},p_n)}$$

$$\leq \lambda_1 d(p_{n-1}, p_n) + \lambda_2 d(p_n, p_{n+1}) + \lambda_3 d(p_n, p_{n+1}).$$

It follows that

$$(1-\lambda_2-\lambda_3)d(p_n,p_{n+1}) \leq \lambda_1 d(p_{n-1},p_n) \tag{3.16}$$

$$d(p_n,p_{n+1}) \leq \left(\frac{\lambda_1}{1-\lambda_2-\lambda_3}\right) d(p_{n-1},p_n).$$

Put $\lambda = \frac{\lambda_1}{1-\lambda_2-\lambda_3}$. In view of $\lambda_1+\lambda_2+\lambda_3<1$, then $0 \leq \lambda < 1$. Thus, by Lemma 3.1, $\{p_n\}$ is a b-Cauchy sequence in X. Since (X,d) is b-complete, then there exists some point $u^* \in X$ such that $p_n \to u^*$ as $n \to \infty$.

By (3.16), it is easy to see that

$$d(u^*,Tu^*) \leq s\{d(u^*,p_{n+1}) + d(p_{n+1},Tu^*)\} \tag{3.17}$$

$$= s\{d(u^*,p_{n+1})\} + s\{d(Tp_n,Tu^*)\}$$

$$\leq s\{d(u^*,p_{n+1})\} + s\{\lambda_1 d(p_n,u^*) + \lambda_2 \frac{d(p_n,Tp_n)d(u^*,Tu^*)}{d(p_n,u^*)}$$

$$+\lambda_3 \frac{d(u^*,Tu^*)[1+d(p_n,Tp_n)]}{1+d(p_n,u^*)}\}$$

$$\leq s\{d(u^*,p_{n+1})\} + s\{\lambda_1 d(p_n,u^*) + \lambda_2 \frac{d(p_n,p_{n+1})d(u^*,Tu^*)}{d(p_n,u^*)}$$

$$+\lambda_3 \frac{d(u^*,Tu^*)[1+d(u^*,p_{n+1})]}{d(p_n,u^*)}\}. \tag{3.18}$$

Taking the limit as $n \to \infty$ by both parties of (3.19), we have $\lim_{n\to\infty} d(u^*,\, Tu^*) = 0$. Hence, $Tu^* = u^*$ and u^* is a fixed point of T.

Finally, we prove the uniqueness of the fixed point. Indeed, if there is another fixed point v^*, then by (3.16),

$$d(u^*,v^*) = d(Tu^*,Tv^*)$$

$$\leq \lambda_1 d(u^*,v^*) + \lambda_2 \frac{d(u^*,Tu^*)d(v^*,Tv^*)}{d(u^*,v^*)} + \lambda_3 \frac{d(v^*,Tv^*)[1+d(u^*,Tu^*)]}{1+d(u^*,v^*)}$$

$$\leq \lambda_1 d(u^*,v^*) + \lambda_2 \frac{d(u^*,u^*)d(v^*,v^*)}{d(u^*,v^*)} + \lambda_3 \frac{d(v^*,v^*)[1+d(u^*,u^*)]}{1+d(u^*,v^*)}\}$$

$$d(u^*,v^*) \leq \lambda_1 d(u^*,v^*) \tag{3.19}$$

since $0 < \lambda_1+\lambda_2+\lambda_3 < 1$ implies $\lambda_1 < 1$, then we get $d(u^*,v^*) = 0$. Thus, we proved that T have a unique fixed point in X.

Example 3.4. Let (X,d) be a complete b-metric space, where $X = [0,\infty)$ and $d: X \times$

$X \to [0,\infty)$, $d(p,q) = (p-q)^2$. Let $T: X \to X$ be defined by $Tp = \frac{p}{4}$. Obviously,

$$d(Tp,Tq) = \frac{(p-q)^2}{16}, \quad d(p,Tp) = \frac{9p^2}{16}, \quad d(q,Tq) = \frac{9q^2}{16}$$

and, choosing $\lambda_1 = \frac{1}{16}$, $\lambda_2 = \frac{1}{8}$ and $\lambda_3 = \frac{1}{4}$ we get

$$\begin{aligned} d(Tp,Tq) &= \frac{1}{16}(p-q)^2 \\ &\leq \frac{1}{16}(p-q)^2 + \frac{1}{8}\frac{d(p,Tp)d(q,Tq)}{d(p,q)} + \frac{1}{4}\frac{d(q,Tq)[1+d(p,Tp)]}{1+d(p,q)}. \end{aligned}$$

Clearly, $\lambda_1 + \lambda_2 + \lambda_3 = \frac{1}{16} + \frac{1}{8} + \frac{1}{4} = \frac{7}{16} < 1$. We conclude that inequality (3.16) remains valid by an application of Theorem 3.4, T has a unique fixed point. It is seen that 0 is the unique fixed point of T.

Theorem 3.5. *Let (X,d) be a b-complete b-metric space with coefficient $s \geq 1$ and $T: X \to X$ be a mapping such that*

$$\begin{aligned} d(Tp,Tq) \leq \lambda_1 d(p,q) + \lambda_2 \frac{d(p,Tp)d(q,Tq)}{d(p,q)} + \lambda_3 \frac{d(p,Tq)d(q,Tp)}{d(p,q)} \\ + \lambda_4 \frac{d(q,Tq)[1+d(p,Tp)]}{1+d(p,q)} \end{aligned} \tag{3.20}$$

for all $p, q \in X$ and $\lambda_1, \lambda_2, \lambda_3$ are nonnegative constants with $\lambda_1 + \lambda_2 + \lambda_3 < 1$. Then T has a unique fixed point in X.

Proof. Let p be arbitrary in X, we define a sequence $\{p_n\}$ in X such that

$$p_{n+1} = Tp_n,$$

for all $n \in \mathbb{N}$, from the condition (3.20) with $p = p_n$ and $q = p_{n-1}$. Therefore

$$\begin{aligned} d(p_n, p_{n+1}) &= d(Tp_{n-1}, Tp_n) \\ &\leq \lambda_1 d(p_{n-1}, p_n) + \lambda_2 \frac{d(p_{n-1},Tp_{n-1})d(p_n,Tp_n)}{d(p_{n-1},p_n)} \\ &\quad + \lambda_3 \frac{d(p_{n-1},Tp_n)d(p_n,Tp_{n-1})}{d(p_{n-1},p_n)} + \lambda_4 \frac{d(p_n,Tp_n)[1+d(p_{n-1},Tp_{n-1}])}{1+d(p_{n-1},p_n)} \\ &\leq \lambda_1 d(p_{n-1}, p_n) + \lambda_2 \frac{d(p_{n-1},p_n)d(p_n,p_{n+1})}{d(p_{n-1},p_n)} \\ &\quad + \lambda_3 \frac{d(p_{n-1},p_{n+1})d(p_n,p_n)}{d(p_{n-1},p_n)} + \lambda_4 \frac{d(p_n,p_{n+1})[1+d(p_{n-1},p_n)}{1+d(p_{n-1},p_n)} \\ &\leq \lambda_1 d(p_{n-1}, p_n) + \lambda_2 d(p_n, p_{n+1}) + \lambda_4 d(p_n, p_{n+1}). \end{aligned}$$

It follows that

$$(1-\lambda_2-\lambda_4)d(p_n,p_{n+1}) \le \lambda_1 d(p_{n-1},p_n) \tag{3.21}$$

$$d(p_n,p_{n+1}) \le \left(\frac{\lambda_1}{1-\lambda_2-\lambda_4}\right) d(p_{n-1},p_n).$$

Put $\lambda = \frac{\lambda_1}{1-\lambda_2-\lambda_4}$. In view of $\lambda_1+\lambda_2+\lambda_3<1$, then $0 \le \lambda < 1$. Thus, by Lemma 3.1, $\{p_n\}$ is a b-Cauchy sequence in X. Since (X,d) is b-complete, there exists some point $u^* \in X$ such that $p_n \to u^*$ as $n \to \infty$.

By (3.21), it is easy to see that

$$d(u^*,Tu^*) \le s\{d(u^*,p_{n+1}) + d(p_{n+1},Tu^*)\} \tag{3.22}$$

$$= s\{d(u^*,p_{n+1})\} + s\{d(Tp_n,Tu^*)\}$$

$$\le s\{d(u^*,p_{n+1})\} + s\{\lambda_1 d(p_n,u^*) + \lambda_2 \frac{d(p_n,Tp_n)d(u^*,Tu^*)}{d(p_n,u^*)}$$

$$+\lambda_3 \frac{d(p_n,Tu^*)d(u^*,Tp_n)}{d(p_n,u^*)} + \lambda_4 \frac{d(u^*,Tu^*)[1+d(p_n,Tp_n)]}{1+d(p_n,u^*)}\}$$

$$\le s\{d(u^*,p_{n+1})\} + s\{\lambda_1 d(p_n,u^*) + \lambda_2 \frac{d(p_n,p_{n+1})d(u^*,Tu^*)}{d(p_n,u^*)}$$

$$+\lambda_3 \frac{d(p_n,Tu^*)d(u^*,Tp_{n+1})}{d(p_n,u^*)} + \lambda_4 \frac{d(u^*,Tu^*)[1+d(p_n,p_{n+1})]}{d(p_n,u^*)}\}. \tag{3.23}$$

Taking the limit as $n \to \infty$ by both parties of (3.24), we have $\lim_{n\to\infty} d(u^*, Tu^*) = 0$. Hence, $Tu^* = u^*$ and u^* is a fixed point of T.

Finally, we prove the uniqueness of the fixed point. Indeed, if there is another fixed point v^*, then by (3.21),

$$d(u^*,v^*) = d(Tu^*,Tv^*)$$

$$\le \lambda_1 d(u^*,v^*) + \lambda_2 \frac{d(u^*,Tu^*)d(v^*,Tv^*)}{d(u^*,v^*)} + \lambda_3 \frac{d(u^*,Tv^*)d(v^*,Tu^*)}{d(u^*,v^*)}$$

$$+\lambda_4 \frac{d(v^*,Tv^*)[1+d(u^*,Tu^*)]}{1+d(u^*,v^*)}$$

$$\le \lambda_1 d(u^*,v^*) + \lambda_2 \frac{d(u^*,u^*)d(v^*,v^*)}{d(u^*,v^*)} + \lambda_3 \frac{d(u^*,v^*)d(v^*,u^*)}{d(u^*,v^*)} + \lambda_4 \frac{d(v^*,v^*)[1+d(u^*,u^*)]}{1+d(u^*,v^*)}\}$$

$$d(u^*,v^*) \le \lambda_1 d(u^*,v^*) + \lambda_3 d(u^*,v^*)$$

$$\le (\lambda_1+\lambda_3) d(u^*,v^*) \tag{3.24}$$

since $0 < \lambda_1 + \lambda_2 + \lambda_3 < 1$ implies $\lambda_1 + \lambda_3 < 1$, then we get $d(u^*, v^*) = 0$. Thus, we proved that T have a unique fixed point in X.

Example 3.5. Let $X = [0,1]$ be equipped with the b-metric $d(p,q) = |p-q|^2$ for all $p, q \in X$. Then (X, d) is a b-metric space with parameter $s = 2$ and it is complete. Let $T: X \longrightarrow X$ be defined as

$$T(p) = \frac{p}{5}, \quad p \in [0,1].$$

Then for $p, q \in X$,

$$\begin{aligned} 2d(Tp, Tq) &= 2d\left(\frac{p}{5}, \frac{q}{5}\right) \\ &= \frac{2}{25}|p-q|^2 \\ &\leq \frac{2}{25} d(p,q) + \frac{4}{25} \frac{d(p,Tp)d(p,Tq) + d(q,Tq)d(q,Tp)}{d(p,Tq) + d(q,Tp)} + \frac{2}{5} d(Tp, Tq). \end{aligned}$$

Clearly, $\lambda_1 + \lambda_2 + \lambda_3 = \frac{2}{25} + \frac{4}{25} + \frac{2}{5} = \frac{16}{25} < 1$. We conclude that inequality (3.26) remains valid by an application of Theorem 3.6, T has a unique fixed point. It is seen that 0 is the unique fixed point of T.

Theorem 3.6. *Let (X, d) be a b-complete b-metric space with coefficient $s \geq 1$ and $T: X \to X$ be a mapping such that*

$$sd(Tp, Tq) \leq \lambda_1 d(p,q) + \lambda_2 \frac{d(p,Tp)d(p,Tq) + d(q,Tq)d(q,Tp)}{d(p,Tq) + d(q,Tp)} + \lambda_3 d(Tp, Tq) \quad (3.25)$$

for all $p, q \in X$ and $\lambda_1, \lambda_2, \lambda_3 \geq 0$, $d(p,Tq) + d(q,Tp) \neq 0$ with $\lambda_1 + \lambda_2 + \lambda_3 < 1$. Then T has a unique fixed point in X.

Proof. Let p be arbitrary in X, we define a sequence $\{p_n\}$ in X such that

$$p_{n+1} = Tp_n,$$

for all $n \in \mathbb{N}$, from the condition (3.26) with $p = p_n$ and $q = p_{n-1}$. Therefore

$$\begin{aligned} d(p_n, p_{n+1}) &= sd(Tp_{n-1}, Tp_n) \\ &\leq \lambda_1 d(p_{n-1}, p_n) + \lambda_2 \frac{d(p_{n-1},Tp_{n-1})d(p_{n-1},Tp_n) + d(p_n,Tp_n)d(p_n,Tp_{n-1})}{d(p_{n-1},Tp_n) + d(p_n,Tp_{n-1})} \\ &\quad + \lambda_3 d(Tp_{n-1}, Tp_n) \\ &\leq \lambda_1 d(p_{n-1}, p_n) + \lambda_2 \frac{d(p_{n-1},p_n)d(p_{n-1},p_{n+1}) + d(p_n,p_{n+1})d(p_n,p_n)}{d(p_{n-1},p_{n+1}) + d(p_n,p_n)} \end{aligned}$$

$$+\lambda_3 d(p_n, p_{n+1})$$

$$\leq \lambda_1 d(p_{n-1}, p_n) + \lambda_2 d(p_{n-1}, p_n) + \lambda_3 d(p_n, p_{n+1}).$$

It follows that

$$(s-\lambda_3) d(p_n, p_{n+1}) \leq (\lambda_1 + \lambda_2) d(p_{n-1}, p_n) \tag{3.26}$$

$$d(p_n, p_{n+1}) \leq \left(\frac{\lambda_1+\lambda_2}{s-\lambda_3}\right) d(p_{n-1}, p_n).$$

Put $\lambda = \frac{\lambda_1+\lambda_2}{s-\lambda_3}$. In view of $\lambda_1 + \lambda_2 + \lambda_3 < 1$, then $0 \leq \lambda < 1$. Thus, by Lemma 3.1, $\{p_n\}$ is a b-Cauchy sequence in X. Since (X, d) is b-complete, there exists some point $u^* \in X$ such that $p_n \to u^*$ as $n \to \infty$.

By (3.26), it is easy to see that

$$d(u^*, Tp^*) \leq s\{d(u^*, p_{n+1}) + d(p_{n+1}, Tu^*)\} \tag{3.27}$$

$$= s\{d(u^*, p_{n+1})\} + s\{d(Tp_n, Tu^*)\}$$

$$\leq s\{d(u^*, p_{n+1})\} + \{\lambda_1 d(p_n, u^*) + \lambda_2 \frac{d(p_n,Tp_n)d(p_n,Tu^*)+d(u^*,Tu^*)d(u^*,Tp_n)}{d(p_n,Tu^*)+d(u^*,Tp_n)}$$

$$+\lambda_3 d(Tp_n, Tu^*)$$

$$\leq s\{d(u^*, p_{n+1})\}$$

$$+\left\{\lambda_1 d(p_n, u^*) + \lambda_2 \frac{d(p_n,p_{n+1})d(p_n,Tu^*)+d(u^*,Tu^*)d(u^*,p_{n+1})}{d(p_n,Tu^*)+d(u^*,p_{n+1})} + \lambda_3 d(x_{n+1}, Tx^*)\right\}. \tag{3.28}$$

Taking the limit as $n \to \infty$ by both parties of (3.29), we have $\lim_{n\to\infty} d(u^*, Tu^*) = 0$. Hence, $Tu^* = u^*$ and u^* is a fixed point of T.

Finally, we prove the uniqueness of the fixed point. Indeed, if there is another fixed point v^*, then by (3.26),

$$sd(u^*, v^*) = sd(Tu^*, Tv^*)$$

$$\leq \lambda_1 d(u^*, v^*) + \lambda_2 \frac{d(u^*,Tv^*)d(v^*,Tu^*)+d(v^*,Tv^*)d(u^*,Tv^*)}{d(u^*,Tv^*)+d(v^*,Tu^*)} + \lambda_3 d(Tu^*, Tv^*)$$

$$\leq \lambda_1 d(u^*, v^*) + \lambda_2 \frac{d(u^*,u^*)d(v^*,u^*)+d(v^*,v^*)d(u^*,v^*)}{d(u^*,v^*)+d(v^*,u^*)}\} + \lambda_3 d(x^*, y^*) d(u^*, v^*)$$

$$\leq \lambda_1 d(u^*, v^*) + \lambda_3 d(u^*, v^*)$$

$$\leq (\lambda_1 + \lambda_3) d(u^*, v^*) \tag{3.29}$$

since $0 < \lambda_1 + \lambda_2 + \lambda_3 < 1$ implies $\lambda_1 + \lambda_3 < 1$, then we get $d(u^*, v^*) = 0$. Thus, we proved that T have a unique fixed point in X.

Example 3.6. Let $X = \{1,2,3\}$, and let $d: X \times X \to [0, +\infty)$ be a mapping satisfies the following condition for all $p, q \in X$:

1. $d(p, q) = 0$, where $p = q$;
2. $d(1,2) = 1, \quad d(1,3) = 4, \quad d(2,3) = 2.$

It is easy to check that d is a b-metric with $s = \frac{4}{3}$. Consider mapping $T: X \to X$, by

$$T(1) = T(2) = 1, \quad T(3) = 2.$$

Let $\lambda_1 = \frac{1}{2}, \lambda_2 = \frac{1}{5}$ and $\lambda_3 = \frac{1}{10}$. Clearly, $\lambda_1 + \lambda_2 + \lambda_3 = \frac{4}{5} < 1$. Next, we will verify the condition (3.26). It have the following cases to be considered.

Case 1. $d(Tp, Tq) = 0$. Clearly, the inequality (3.26) holds.

Case 2. $d(Tp, Tq) = 1$, that is, $Tp = 1, Tq = 2$ or $Tp = 2$, $Tq = 1$. When $Tp = 1$, $Tq = 2$, we get

Case 2.1. $p = 1, q = 3$, we can get $d(p, q) = 4$, then

$$\begin{aligned}\frac{4}{3} \times 1 &< 2\\ &= \frac{1}{2} \times 4 = \lambda_1 d(p, q)\\ &\le \lambda_1 d(p, q) + \lambda_2 \frac{d(p,Tp)d(p,Tq)+d(q,Tq)d(q,Tp)}{d(p,Tq)+d(q,Tp)} + \lambda_3 d(Tp, Tq).\end{aligned}$$

Thus, the inequality (3.26) holds.

Case 2.2. $p = 2, q = 3$, we can get $d(p, q) = 2$, then

$$\begin{aligned}\frac{4}{3} \times 1 &< \frac{7}{5}\\ &= \frac{1}{2} \times 4 + \frac{2}{5} = \lambda_1 d(p, q) + \lambda_2 \frac{d(p,Tp)d(p,Tq)+d(q,Tq)d(q,Tp)}{d(p,Tq)+d(q,Tp)}\\ &\le \lambda_1 d(p, q) + \lambda_2 \frac{d(p,Tp)d(p,Tq)+d(q,Tq)d(q,Tp)}{d(p,Tq)+d(q,Tp)} + \lambda_3 d(Tp, Tq).\end{aligned}$$

Thus, the inequality (3.26) holds.

When $Tp = 2, Tq = 1$, we get

Case 2.3. $p = 3, q = 1$, we can get $d(p, q) = 4$, then

$$\frac{4}{3} \times 1 < 2$$

$$= \frac{1}{2} \times 4 = \lambda_1 d(p, q)$$

$$\leq \lambda_1 d(p, q) + \lambda_2 \frac{d(p,Tp)d(p,Tq)+d(q,Tq)d(q,Tp)}{d(p,Tq)+d(q,Tp)} + \lambda_3 d(Tp, Tq).$$

Thus, the inequality (3.26) holds.

Case 2.4. $p = 2, q = 3$, we can get $d(p, q) = 2$, then

$$\frac{4}{3} \times 1 < \frac{7}{5}$$

$$= \frac{1}{2} \times 4 + \frac{2}{5} = \lambda_1 d(x, y) + \lambda_2 \frac{d(p,Tp)d(p,Tq)+d(q,Tq)d(q,Tp)}{d(p,Tq)+d(q,Tp)}$$

$$\leq \lambda_1 d(p, q) + \lambda_2 \frac{d(p,Tp)d(p,Tq)+d(q,Tq)d(q,Tp)}{d(p,Tq)+d(q,Tp)} + \lambda_3 d(Tp, Tq).$$

Thus, the inequality (3.26) holds.

Remark 3.1.

1) If $s = 1$ and $\lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0$ in Theorem 3.4, we get the Banach Theorem [4].

2) If $s = 1$ and $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_5 = 0$ in Theorem 3.4, we get the Kanan Theorem [17].

3) If $s = 1$ and $\lambda_2 = \lambda_3 = \lambda_5 = 0$ in Theorem 3.4, we get the Fisher Theorem [11].

4) If $s = 1$ and $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$ in Theorem 3.4, we get the Chaterjee Theorem [7].

5) If $s = 1$ and $\lambda_3 = \lambda_4 = \lambda_5 = 0$ in Theorem 3.4, we get the result of Jaggi [2].

6) If $s = 1$ and $\lambda_2 = \lambda_3 = 0$ in Theorem 3.6, we get the result of Dass and Gupta [6].

Theorem 3.7. *Let (X, d) be a b-metric space with coefficient $s \geq 1$. Let $T: X \to X$ be a mapping such that $F(T) \neq \emptyset$ and that*

$$d(Tp, T^2p) \leq \lambda d(p, Tp) \tag{3.30}$$

for all $p \in X$, where $0 \leq \lambda < 1$ is a constant. Then T has the P property.

Proof. We always assume that $n > 1$, since the statement for $n = 1$ is trivial. Let $z \in F(T^n)$. By the hypotheses, we get

$$\begin{aligned} d(z, Tz) &= d(TT^{n-1}z, T^2T^{n-1}z) \\ &\leq \lambda d(T^{n-1}z, T^n z) \\ &= \lambda d(TT^{n-2}z, T^2T^{n-2}z) \\ &\leq \lambda^2 d(T^{n-2}z, T^{n-1}z) \\ &\leq \cdots \leq \lambda^n d(z, Tz) \to 0 \ (n \to \infty). \end{aligned}$$

Hence, $d(z, Tz) = 0$, that is, $Tz = z$.

Theorem 3.8. *Under the conditions of Theorem* 3.2, T *has the* P *property.*

Proof. We have to prove that the mapping T satisfies (3.31) In fact, for any $p \in X$, for one thing, we have

$$\begin{aligned} d(Tp, T^2p) &= d(Tp, TTp) \\ &\leq \lambda_1 d(p, Tp) + \lambda_2 \frac{d(p,Tp)d(p,TTp) + d(Tp,TTp)d(Tp,Tp)}{d(p,TTp) + d(Tp,Tp)} \\ &\quad + \lambda_3 \frac{d(p,Tp)d(Tp,Tp) + d(Tp,TTp)d(p,TTp)}{d(p,TTp) + d(Tp,Tp)} \\ &\leq \lambda_1 d(p, Tp) + \lambda_2 d(p, Tp) + \lambda_3 d(Tp, T^2p) \end{aligned}$$

$$(1 - \lambda_3) d(Tp, T^2p) \leq (\lambda_1 + \lambda_2) d(p, Tp) \tag{3.31}$$

$$d(Tx, T^2x) \leq \frac{\lambda_1 + \lambda_2}{1 - \lambda_3} d(p, Tp).$$

Denote that $\lambda = \frac{\lambda_1 + \lambda_2}{1 - \lambda_3}$. Note that $\lambda_1 + \lambda_2 + \lambda_3 < 1$, then $\lambda < 1$. Accordingly, (3.31) is satisfied. Consequently, by Theorem 3.2, T has the P property.

4. Application to Non-Linear Integral Equations

Let $X = C[a, b]$ be the set of all real valued continuous functions on $[a, b]$, where $[a, b]$ is a closed and bounded interval in $\mathbb{R}$. For $\eta > 1$ a real number, define $d: X \times X \to \mathbb{R}_+$ by:

$$d(p, q) = \sup_{t \in [a,b]} |p(t) - q(t)|^\eta$$

for all $p, q \in X$. Therefore, (X, d) is a complete b-metric space with $s = 2^{\eta-1}$. In this section, we apply Theorem 3.6 to establish the existence of solution of nonlinear integral equation of Fredholm type defined by:

$$p(t) = g(t) + \lambda \int_a^b K(t, s, p(s))ds, \tag{4.1}$$

where $p \in C[a, b]$ is the unknown function, $\lambda \in \mathbb{R}, t, s \in [a, b], K: [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ and $g: [a, b] \to \mathbb{R}$ are given continuous functions.

Theorem 4.1. *Assume that the following conditions are fulfilled.*

1. *There exists a continuous function* $\psi: [a, b] \times [a, b] \to \mathbb{R}_+$ *such that for all* $p, q \in X, \lambda \in \mathbb{R}$ *and* $t, s \in [a, b]$, *we have*

$$|K(t, s, p(s)) - K(t, s, y(s))|^\eta \leq \psi(t, s)M(p, q),$$

where

$$M(p, q) \leq \lambda_1 d(p, q) + \lambda_2 \frac{d(p,Tq)d(p,Tq)+d(q,Tq)d(q,Tp)}{d(p,Tq)+d(q,Tp)} + \lambda_3 d(Tp, Tq).$$

2. $|\lambda| \leq 1$,

3. $\sup\limits_{t\in[a,b]} \int_a^b \psi(t, s)ds \leq \frac{1}{2^{\eta-1}(b-a)^{\eta-1}}$.

Then, the nonlinear integral equation (4.1) *has a solution* $z \in C[a, b]$.

Proof. Define a mapping $T: X \to X$ by:

$$Tp(t) = g(t) + \lambda \int_a^b K(t, s, p(s))ds,$$

for all $t \in [a, b]$. So, the existence of a solution of (4.1) is equivalent to the existence and uniqueness of fixed point of T. Let $\beta \in \mathbb{R}$ such that $\frac{1}{\eta} + \frac{1}{\beta} = 1$. Using the Holder inequality, (1), (2) and (3), we have

$$T(Tp, Tq) = \sup_{t\in[a,b]} |Tp(t) - Tq(t)|^\eta$$

$$\leq |\lambda|^\eta \sup_{t\in[a,b]} \left(\int_a^b |(K(t, s, p(s)) - K(t, s, q(s)))|ds\right)^\eta$$

$$\leq \sup_{t\in[a,b]} \left[\left(\int_a^b 1^\beta ds\right)^{\frac{1}{\beta}} \left(\int_a^b |(K(t, s, p(s)) - K(t, s, q(s)))|^\eta ds\right)^{\frac{1}{\eta}}\right]^\eta$$

$$\leq (b-a)^{\frac{\eta}{\beta}} \sup_{t\in[a,b]} \left(\int_a^b |(K(t,s,p(s)) - K(t,s,q(s)|^\eta ds\right)$$

$$\leq (b-a)^{\eta-1} \sup_{t\in[a,b]} \left(\int_a^b \psi(t,s)dsM(p,q)\right)$$

$$\leq (b-a)^{\eta-1} \sup_{t\in[a,b]} \left(\int_a^b \psi(t,s)ds\right) M(p,q)$$

$$\leq \frac{1}{2^{\eta-1}} M(p,q).$$

Thus

$$sd(Tp,Tq) \leq \lambda_1 d(p,q) + \lambda_2 \frac{d(p,Tp)d(p,Tq)+d(q,Tq)d(q,Tp)}{d(p,Tq)+d(q,Tp)} + \lambda_3 d(Tp,Tq).$$

Hence, all the conditions of Theorem 3.6 hold. Consequently, the integral equation (4.1) has a solution $z \in C[a,b]$.

Example 4.1. Let $X = C[0,1]$ be a set of all continuous functions on $[0,1]$. Define $d: X \times X \to \mathbb{R}_+$ by:

$$d(p,q) = \sup_{t\in[0,1]} |p(t) - q(t)|^2,$$

for all $p, q \in X$. Therefore, (X,d) is a complete b-metric space with $s = 2$. Consider the following problem:

$$p(t) = 3t + \frac{\pi}{2}\int_0^1 \frac{t}{2} spds \tag{4.2}$$

is the exact solution of (4.2).

Customize $K(t,s,p(s)) = \frac{t}{2}sx, g(t) = 3t$ and $\lambda = \frac{\pi}{2}$ in Theorem 4.1. Note that:

1. K and g are continuous functions.
2. $|\lambda| = \left|\frac{\pi}{2}\right| < 1.$
3. $\psi(t,s) = (ts)^2$, then

$$\sup_{t\in[0,1]} \int_0^1 \psi(t,s)ds = \sup_{t\in[0,1]} \int_0^1 (ts)^2 ds$$

$$= \sup_{t\in[0,1]} t\left[\frac{s^3}{3}\right]_0^1$$

$$= \frac{1}{3} \sup_{t \in [0,1]} t$$

$$\leq \frac{1}{3} < \frac{1}{2} = \frac{1}{s}.$$

4. For $s \in [0,1]$, we have

$$\begin{aligned} |K(t,s,p(s)) - K(t,s,q(s))|^2 &= \frac{1}{4}(ts)^2|p-q|^2 \\ &\leq \frac{1}{4}(ts)^2 \sup_{t \in [0,1]} |p-q|^2 \\ &= \frac{1}{4}\psi(t,s)d(p,q), \end{aligned}$$

with $\psi(t,s) = (ts)^2$ and

$$M(p,q) \leq \lambda_1 d(p,q) + \lambda_2 \frac{d(p,Tp)d(p,Tq) + d(q,Tq)d(q,Tp)}{d(p,Tq) + d(q,Tp)} + \lambda_3 d(Tp,Tq),$$

where $\lambda_1 = \frac{1}{4}, \lambda_2 = \lambda_3 = 0$ it means that $\lambda_1 + \lambda_2 + \lambda_3 < 1$.

Therefore, the conditions of Theorem 4.1 are justified, hence the mapping T has a unique fixed point in $C[0,1]$, with is the unique solution of problem (4.2).

Data Availability Statement

The results data used to support the fndings of this study are included within the article.

Disclosure statement

No potential conflict of interest was reported by the authors.

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