

# Differential Subordination and Superordination for Fractional Integral Involving Wanas Operator Defined by Convolution Structure

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## Abstract

In present paper, we use fractional integral and Wanas differential operator to obtain some subordination and superordination results associated with Hadamard product for univalent analytic functions defined in the open unit disk. These results are applied to obtain differential sandwich results. Our results extend corresponding previously known results.

## 1. Introduction and Preliminaries

Denote by  $\mathcal{H}$  the class of analytic functions in the open unit disk  $U = \{z \in \mathbb{C} : |z| < 1\}$ . For a positive integer  $n$  and  $a \in \mathbb{C}$ , assume that  $\mathcal{H}[a, n]$  be the subclass of  $\mathcal{H}$  consisting of functions that have the form:

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots \quad (1.1)$$

Also, let  $\mathcal{A}$  be the subclass of  $\mathcal{H}$  consisting of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \quad (1.2)$$

For the functions  $f \in \mathcal{A}$  given by (1.2) and  $g \in \mathcal{A}$  defined by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

we define the Hadamard product (or convolution) of  $f$  and  $g$  by

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$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z).$$

Let  $f, g \in \mathcal{A}$ . The function  $f$  is said to be subordinate to  $g$ , or  $g$  is said to be superordinate to  $f$ , if there exists a Schwarz function  $w$  analytic in  $U$  with  $w(0) = 0$  and  $|w(z)| < 1$  ( $z \in U$ ) such that  $f(z) = g(w(z))$ . In such a case we write  $f \prec g$  or  $f(z) \prec g(z)$  ( $z \in U$ ). Furthermore, if  $g$  is univalent in  $U$ , then we have the following equivalent (see [9]),  $f \prec g \Leftrightarrow f(0) = g(0)$  and  $f(U) \subset g(U)$ .

Let  $p, h \in \mathcal{H}$  and  $\psi(r, s, t; z): \mathbb{C}^3 \times U \rightarrow \mathbb{C}$ . If  $p$  and  $\psi(p(z), zp'(z), z^2p''(z); z)$  are univalent functions in  $U$  and if  $p$  satisfies the second-order differential superordination:

$$h(z) \prec \psi(p(z), zp'(z), z^2p''(z); z), \quad (1.3)$$

then  $p$  is called a solution of the differential superordination (1.3). (If  $f$  is subordinate to  $g$ , then  $g$  is superordinate to  $f$ ). An analytic function  $q$  is called a subordinated of (1.3), if  $q \prec p$  for all the functions  $p$  satisfying (1.3). A univalent subordinated  $\tilde{q}$  that satisfies  $q \prec \tilde{q}$  for all the subordinateds  $q$  of (1.3) is called the best subordinated.

**Definition 1.1** [17]. For  $f \in \mathcal{A}$ , the Wanas differential operator is defined by

$$W_{\alpha, \beta}^{k, \eta} f(z) = z + \sum_{n=2}^{\infty} \left[ \sum_{m=1}^k \binom{k}{m} (-1)^{m+1} \left( \frac{\alpha^m + n\beta^m}{\alpha^m + \beta^m} \right) \right]^{\eta} a_n z^n, \quad (1.4)$$

where  $\alpha \in \mathbb{R}, \beta \geq 0$  with  $\alpha + \beta > 0, k \in \mathbb{N}, \eta \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ .

**Definition 1.2** [6]. The fractional integral of order  $\lambda$  ( $\lambda > 0$ ) is defined for a function  $f$  by

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(t)}{(z-t)^{\lambda-1}} dt,$$

where  $f$  is an analytic function in a simply-connected region of the  $z$ -plane containing the origin, and the multiplicity of  $(z-t)^{\lambda-1}$  is removed by requiring  $\log(z-t)$  to be real, when  $(z-t) > 0$ .

From Definition 1.1 and Definition 1.2, we conclude that

$$D_z^{-\lambda} W_{\alpha,\beta}^{k,\eta} f(z) = \frac{1}{\Gamma(2 + \lambda)} z^{1+\lambda} + \sum_{n=2}^{\infty} \frac{\Gamma(n + 1)}{\Gamma(n + 1 + \lambda)} \left[ \sum_{m=1}^k \binom{k}{m} (-1)^{m+1} \left( \frac{\alpha^m + n\beta^m}{\alpha^m + \beta^m} \right) \right]^\eta a_n z^{n+\lambda}.$$

From [18] we need this result

$$z \left( D_z^{-\lambda} W_{\alpha,\beta}^{k,\eta} f(z) \right)' = \left[ 1 + \sum_{m=1}^k \binom{k}{m} (-1)^{m+1} \left( \frac{\alpha}{\beta} \right)^m \right] D_z^{-\lambda} W_{\alpha,\beta}^{k,\eta+1} f(z) + \left[ \lambda - \sum_{m=1}^k \binom{k}{m} (-1)^{m+1} \left( \frac{\alpha}{\beta} \right)^m \right] D_z^{-\lambda} W_{\alpha,\beta}^{k,\eta} f(z). \tag{1.5}$$

Special cases of this operator can be found in [1,2,4,7,8,11,13,14,16]. For more details see [19].

Very recently, Xu et al. [20], Tang and Deniz [15], Rahrovi [10], Attiya and Yassen [3] and Seoudy [12] have studied differential subordinations and superordinations for different conditions of analytic functions.

The main object of the present paper is to find sufficient condition for certain normalized analytic functions  $f$  in  $U$  such that  $(f * \Psi)(z) \neq 0$  and  $f$  to satisfy

$$q_1(z) < \left( \frac{D_z^{-\lambda} W_{\alpha,\beta}^{k,\eta+1} (f * \Phi)(z)}{D_z^{-\lambda} W_{\alpha,\beta}^{k,\eta} (f * \Psi)(z)} \right)^\delta < q_2(z),$$

and

$$q_1(z) < \left( \frac{t_1 D_z^{-\lambda} W_{\alpha,\beta}^{k,\eta+1} (f * \Phi)(z) + t_2 D_z^{-\lambda} W_{\alpha,\beta}^{k,\eta} (f * \Psi)(z)}{(t_1 + t_2)z} \right)^\delta < q_2(z),$$

where  $q_1$  and  $q_2$  are given univalent functions in  $U$  and  $\Phi(z) = z + \sum_{n=2}^{\infty} t_n z^n$ ,  $\Psi(z) =$

$z + \sum_{n=2}^{\infty} \varphi_n z^n$  are analytic functions in  $U$  with  $t_n \geq 0, \varphi_n \geq 0$  and  $t_n \geq \varphi_n$ . Also, we

obtain the number of results as their special cases.

To establish our main results, we need the following definition and lemmas:

**Definition 1.3** [9]. Denote by  $Q$  the set of all functions  $f$  that are analytic and injective on  $\overline{U} \setminus E(f)$ , where

$$E(f) = \left\{ \zeta \in \partial U : \lim_{z \rightarrow \zeta} f(z) = \infty \right\}$$

and are such that  $f'(\zeta) \neq 0$  for  $\zeta \in \partial U \setminus E(f)$ .

**Lemma 1.1** [9]. Let  $q$  be univalent in the unit disk  $U$  and let  $\theta$  and  $\phi$  be analytic in a domain  $D$  containing  $q(U)$  with  $\phi(w) \neq 0$  when  $w \in q(U)$ . Set  $Q(z) = zq'(z)\phi(q(z))$  and  $h(z) = \theta(q(z)) + Q(z)$ . Suppose that

(1)  $Q(z)$  is starlike univalent in  $U$ ,

(2)  $\operatorname{Re} \left\{ \frac{zh'(z)}{Q(z)} \right\} > 0$  for  $z \in U$ .

If

$$\theta(p(z)) + zp'(z)\phi(p(z)) < \theta(q(z)) + zq'(z)\phi(q(z)), \quad (1.6)$$

then  $p < q$  and  $q$  is the best dominant of (1.6).

**Lemma 1.2** [5]. Let  $q$  be convex univalent in the unit disk  $U$  and let  $\theta$  and  $\phi$  be analytic in a domain  $D$  containing  $q(U)$ . Suppose that

(1)  $\operatorname{Re} \left\{ \frac{\theta'(q(z))}{\phi(q(z))} \right\} > 0$  for  $z \in U$ ,

(2)  $Q(z) = zq'(z)\phi(q(z))$  is starlike univalent in  $U$ .

If  $p \in \mathcal{H}[q(0), 1] \cap Q$ , with  $p(U) \subset D$ ,  $\theta(p(z)) + zp'(z)\phi(p(z))$  is univalent in  $U$  and

$$\theta(q(z)) + zq'(z)\phi(q(z)) < \theta(p(z)) + zp'(z)\phi(p(z)), \quad (1.7)$$

then  $q < p$  and  $q$  is the best subdominant of (1.7).

## 2. Main Results

**Theorem 2.1.** Let  $\Phi, \Psi \in \mathcal{A}$ ,  $\rho, \gamma, \mu, \varepsilon, \sigma, \tau, \delta \in \mathbb{C}$  such that  $\delta \neq 0$  and  $\sigma, \tau$  are not simultaneously zero,  $q$  be convex univalent in  $U$  with  $q(0) = 1$  and assume that

$$\operatorname{Re} \left\{ \frac{1}{\sigma q(z) + \tau} \left( -\varepsilon + \gamma q^2(z) + 2\mu q^3(z) - \sigma zq'(z) - \frac{2\tau zq'(z)}{q(z)} \right) + \frac{zq''(z)}{q'(z)} + 1 \right\} > 0. \quad (2.1)$$

If  $f \in \mathcal{A}$  satisfies the differential subordination

$$M_1(f, \Phi, \Psi, \rho, \gamma, \mu, \varepsilon, \sigma, \tau, \delta, \alpha, \beta, k, \eta, \lambda; z) < \rho + \gamma q(z) + \mu q^2(z) + \frac{\varepsilon}{q(z)} + \left(\frac{\sigma}{q(z)} + \frac{\tau}{q^2(z)}\right) zq'(z), \tag{2.2}$$

where

$$\begin{aligned} &M_1(f, \Phi, \Psi, \rho, \gamma, \mu, \varepsilon, \sigma, \tau, \delta, \alpha, \beta, k, \eta, \lambda; z) \\ &= \rho + \left(\frac{D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta+1}(f * \Phi)(z)}{D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta}(f * \Psi)(z)}\right)^\delta \left(\gamma + \mu \left(\frac{D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta+1}(f * \Phi)(z)}{D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta}(f * \Psi)(z)}\right)^\delta\right) \\ &+ \varepsilon \left(\frac{D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta}(f * \Psi)(z)}{D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta+1}(f * \Phi)(z)}\right)^\delta + \delta \left(\sigma + \tau \left(\frac{D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta}(f * \Psi)(z)}{D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta+1}(f * \Phi)(z)}\right)^\delta\right) \\ &\times \left(1 + \sum_{m=1}^k \binom{k}{m} (-1)^{m+1} \left(\frac{\alpha}{\beta}\right)^m\right) \left(\frac{D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta+2}(f * \Phi)(z)}{D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta+1}(f * \Phi)(z)} - \frac{D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta+1}(f * \Psi)(z)}{D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta}(f * \Psi)(z)}\right), \end{aligned} \tag{2.3}$$

then

$$\left(\frac{D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta+1}(f * \Phi)(z)}{D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta}(f * \Psi)(z)}\right)^\delta < q(z)$$

and  $q$  is the best dominant of (2.2).

**Proof.** Let the function  $p$  be defined by

$$p(z) = \left(\frac{D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta+1}(f * \Phi)(z)}{D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta}(f * \Psi)(z)}\right)^\delta, \quad (z \in U). \tag{2.4}$$

Then the function  $p$  is analytic in  $U$  and  $p(0) = 1$ .

A simple computation using (2.4) gives

$$\frac{zp'(z)}{p(z)} = \delta \left( z \left(\frac{D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta+2}(f * \Phi)(z)}{D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta+1}(f * \Phi)(z)}\right)' - z \left(\frac{D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta+1}(f * \Psi)(z)}{D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta}(f * \Psi)(z)}\right)' \right).$$

In view of (1.5), we obtain

$$\frac{zp'(z)}{p(z)} = \delta \left( \left( 1 + \sum_{m=1}^k \binom{k}{m} (-1)^{m+1} \left(\frac{\alpha}{\beta}\right)^m \right) \left( \frac{D_z^{-\lambda} W_{\alpha,\beta}^{k,\eta+2}(f * \Phi)(z)}{D_z^{-\lambda} W_{\alpha,\beta}^{k,\eta+1}(f * \Phi)(z)} - \frac{D_z^{-\lambda} W_{\alpha,\beta}^{k,\eta+1}(f * \Psi)(z)}{D_z^{-\lambda} W_{\alpha,\beta}^{k,\eta}(f * \Psi)(z)} \right) \right).$$

Also, we find that

$$\begin{aligned} & \rho + \gamma p(z) + \mu p^2(z) + \frac{\varepsilon}{p(z)} + \left( \frac{\sigma}{p(z)} + \frac{\tau}{p^2(z)} \right) zp'(z) \\ &= M_1(f, \Phi, \Psi, \rho, \gamma, \mu, \varepsilon, \sigma, \tau, \delta, \alpha, \beta, k, \eta, \lambda; z), \end{aligned} \quad (2.5)$$

where  $M_1(f, \Phi, \Psi, \rho, \gamma, \mu, \varepsilon, \sigma, \tau, \delta, \alpha, \beta, k, \eta, \lambda; z)$  is given by (2.3).

By using (2.5) in (2.2), we have

$$\begin{aligned} & \rho + \gamma p(z) + \mu p^2(z) + \frac{\varepsilon}{p(z)} + \left( \frac{\sigma}{p(z)} + \frac{\tau}{p^2(z)} \right) zp'(z) \\ &< \rho + \gamma q(z) + \mu q^2(z) + \frac{\varepsilon}{q(z)} + \left( \frac{\sigma}{q(z)} + \frac{\tau}{q^2(z)} \right) zq'(z). \end{aligned}$$

By setting

$$\theta(w) = \rho + \gamma w + \mu w^2 + \frac{\varepsilon}{w} \quad \text{and} \quad \phi(w) = \frac{\sigma}{w} + \frac{\tau}{w^2},$$

it can be easily observed that  $\theta(w)$  and  $\phi(w)$  are analytic in  $\mathbb{C} \setminus \{0\}$  and that  $\phi(w) \neq 0, w \in \mathbb{C} \setminus \{0\}$ . Also, we get

$$Q(z) = zq'(z)\phi(q(z)) = \left( \frac{\sigma}{q(z)} + \frac{\tau}{q^2(z)} \right) zq'(z)$$

and

$$h(z) = \theta(q(z)) + Q(z) = \rho + \gamma q(z) + \mu q^2(z) + \frac{\varepsilon}{q(z)} + \left( \frac{\sigma}{q(z)} + \frac{\tau}{q^2(z)} \right) zq'(z).$$

In light of the hypothesis of Theorem 2.1, we see that  $Q(z)$  is starlike univalent in  $U$  and

$$\begin{aligned} & \operatorname{Re} \left\{ \frac{zh'(z)}{Q(z)} \right\} \\ &= \operatorname{Re} \left\{ \frac{1}{\sigma q(z) + \tau} \left( -\varepsilon + \gamma q^2(z) + 2\mu q^3(z) - \sigma z q'(z) - \frac{2\tau z q'(z)}{q(z)} \right) + \frac{z q''(z)}{q'(z)} + 1 \right\} > 0. \end{aligned}$$

Hence the result now follows by an application of Lemma 1.1.

By fixing  $\Phi(z) = \Psi(z) = \frac{z}{1-z}$  in Theorem 2.1, we obtain the following corollary:

**Corollary 2.1.** *Let  $\rho, \gamma, \mu, \varepsilon, \sigma, \tau, \delta \in \mathbb{C}$  such that  $\delta \neq 0$  and  $\sigma, \tau$  are not simultaneously zero,  $q$  be convex univalent in  $U$  with  $q(0) = 1$  and assume that (2.1) holds true. If  $f \in \mathcal{A}$  satisfies the differential subordination*

$$\begin{aligned} & M_2(f, \rho, \gamma, \mu, \varepsilon, \sigma, \tau, \delta, \alpha, \beta, k, \eta, \lambda; z) \\ & < \rho + \gamma q(z) + \mu q^2(z) + \frac{\varepsilon}{q(z)} + \left( \frac{\sigma}{q(z)} + \frac{\tau}{q^2(z)} \right) z q'(z), \end{aligned} \tag{2.6}$$

where

$$\begin{aligned} & M_2(f, \rho, \gamma, \mu, \varepsilon, \sigma, \tau, \delta, \alpha, \beta, k, \eta, \lambda; z) \\ &= \rho + \left( \frac{D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta+1} f(z)}{D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta} f(z)} \right)^\delta \left( \gamma + \mu \left( \frac{D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta+1} f(z)}{D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta} f(z)} \right)^\delta \right) \\ &+ \varepsilon \left( \frac{D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta} f(z)}{D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta+1} f(z)} \right)^\delta + \delta \left( \sigma + \tau \left( \frac{D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta} f(z)}{D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta+1} f(z)} \right)^\delta \right) \\ &\times \left( 1 + \sum_{m=1}^k \binom{k}{m} (-1)^{m+1} \left( \frac{\alpha}{\beta} \right)^m \right) \left( \frac{D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta+2} f(z)}{D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta+1} f(z)} - \frac{D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta+1} f(z)}{D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta} f(z)} \right), \end{aligned} \tag{2.7}$$

then

$$\left( \frac{D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta+1} f(z)}{D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta} f(z)} \right)^\delta < q(z)$$

and  $q$  is the best dominant of (2.6).

**Theorem 2.2.** *Let  $\Phi, \Psi \in \mathcal{A}$ ,  $\rho, \gamma, \mu, \varepsilon, \sigma, \tau, \delta \in \mathbb{C}$  such that  $\delta \neq 0$  and  $\sigma, \tau$  are not simultaneously zero,  $q$  be convex univalent in  $U$  with  $q(0) = 1$  and assume that*

$$\operatorname{Re} \left\{ \frac{q'(z)}{\sigma q(z) + \tau} (\gamma q^2(z) + 2\mu q^3(z) - \varepsilon) \right\} > 0. \quad (2.8)$$

Let  $f \in \mathcal{A}$ ,

$$\left( \frac{D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta+1}(f * \Phi)(z)}{D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta}(f * \Psi)(z)} \right)^\delta \in \mathcal{H}[q(0), 1] \cap \mathcal{Q}$$

and  $M_1(f, \Phi, \Psi, \rho, \gamma, \mu, \varepsilon, \sigma, \tau, \delta, \alpha, \beta, k, \eta, \lambda; z)$  as defined by (2.3) be univalent in  $U$ . If

$$\begin{aligned} & \rho + \gamma q(z) + \mu q^2(z) + \frac{\varepsilon}{q(z)} + \left( \frac{\sigma}{q(z)} + \frac{\tau}{q^2(z)} \right) zq'(z) \\ & < M_1(f, \Phi, \Psi, \rho, \gamma, \mu, \varepsilon, \sigma, \tau, \delta, \alpha, \beta, k, \eta, \lambda; z), \end{aligned} \quad (2.9)$$

then

$$q(z) < \left( \frac{D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta+1}(f * \Phi)(z)}{D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta}(f * \Psi)(z)} \right)^\delta$$

and  $q$  is the best subordinator of (2.9).

**Proof.** Let the function  $p$  be defined by (2.4).

In view of (1.5), the superordination (2.9) becomes

$$\begin{aligned} & \rho + \gamma q(z) + \mu q^2(z) + \frac{\varepsilon}{q(z)} + \left( \frac{\sigma}{q(z)} + \frac{\tau}{q^2(z)} \right) zq'(z) \\ & < \rho + \gamma p(z) + \mu p^2(z) + \frac{\varepsilon}{p(z)} + \left( \frac{\sigma}{p(z)} + \frac{\tau}{p^2(z)} \right) zp'(z). \end{aligned}$$

By setting  $\theta(w) = \rho + \gamma w + \mu w^2 + \frac{\varepsilon}{w}$  and  $\phi(w) = \frac{\sigma}{w} + \frac{\tau}{w^2}$ , it is easily observed that  $\theta(w)$  and  $\phi(w)$  are analytic in  $\mathbb{C} \setminus \{0\}$  and that  $\phi(w) \neq 0, w \in \mathbb{C} \setminus \{0\}$ . Also, we get

$$\operatorname{Re} \left\{ \frac{\theta'(q(z))}{\phi(q(z))} \right\} = \operatorname{Re} \left\{ \frac{q'(z)}{\sigma q(z) + \tau} (\gamma q^2(z) + 2\mu q^3(z) - \varepsilon) \right\} > 0.$$

Now Theorem 2.2 follows by applying Lemma 1.2.

By fixing  $\Phi(z) = \Psi(z) = \frac{z}{1-z}$  in Theorem 2.2, we obtain the following corollary:



**Corollary 2.2.** Let  $\rho, \gamma, \mu, \varepsilon, \sigma, \tau, \delta \in \mathbb{C}$  such that  $\delta \neq 0$  and  $\sigma, \tau$  are not simultaneously zero,  $q$  be convex univalent in  $U$  with  $q(0) = 1$  and assume that (2.2) holds true. Let  $f \in \mathcal{A}$ ,

$$\left( \frac{D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta+1} f(z)}{D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta} f(z)} \right)^\delta \in \mathcal{H}[q(0), 1] \cap Q$$

and  $M_2(f, \rho, \gamma, \mu, \varepsilon, \sigma, \tau, \delta, \alpha, \beta, k, \eta, \lambda; z)$  as defined by (2.7) be univalent in  $U$ . If

$$\begin{aligned} & \rho + \gamma q(z) + \mu q^2(z) + \frac{\varepsilon}{q(z)} + \left( \frac{\sigma}{q(z)} + \frac{\tau}{q^2(z)} \right) z q'(z) \\ & < M_2(f, \rho, \gamma, \mu, \varepsilon, \sigma, \tau, \delta, \alpha, \beta, k, \eta, \lambda; z), \end{aligned} \tag{2.10}$$

then

$$q(z) < \left( \frac{D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta+1} f(z)}{D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta} f(z)} \right)^\delta$$

and  $q$  is the best subordinant of (2.10).

Concluding the results of differential subordination and superordination, we state at the following sandwich result.

**Theorem 2.3.** Let  $q_1$  and  $q_2$  be convex univalent in  $U$  with  $q_1(0) = q_2(0) = 1$ ,  $\rho, \gamma, \mu, \varepsilon, \sigma, \tau, \delta \in \mathbb{C}$  such that  $\delta \neq 0$  and  $\sigma, \tau$  are not simultaneously zero. Suppose  $q_2$  satisfies (2.1) and  $q_1$  satisfies (2.8). For  $f, \Phi, \Psi \in \mathcal{A}$ , let

$$\left( \frac{D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta+1} (f * \Phi)(z)}{D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta} (f * \Psi)(z)} \right)^\delta \in \mathcal{H}[1, 1] \cap Q$$

and  $M_1(f, \Phi, \Psi, \rho, \gamma, \mu, \varepsilon, \sigma, \tau, \delta, \alpha, \beta, k, \eta, \lambda; z)$  as defined by (2.3) be univalent in  $U$ . If

$$\begin{aligned} & \rho + \gamma q_1(z) + \mu q_1^2(z) + \frac{\varepsilon}{q_1(z)} + \left( \frac{\sigma}{q_1(z)} + \frac{\tau}{q_1^2(z)} \right) z q_1'(z) \\ & < M_1(f, \Phi, \Psi, \rho, \gamma, \mu, \varepsilon, \sigma, \tau, \delta, \alpha, \beta, k, \eta, \lambda; z) \\ & < \rho + \gamma q_2(z) + \mu q_2^2(z) + \frac{\varepsilon}{q_2(z)} + \left( \frac{\sigma}{q_2(z)} + \frac{\tau}{q_2^2(z)} \right) z q_2'(z), \end{aligned}$$

then

$$q_1(z) < \left( \frac{D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta+1} (f * \Phi)(z)}{D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta} (f * \Psi)(z)} \right)^\delta < q_2(z)$$

and  $q_1, q_2$  are respectively the best subordinant and the best dominant.

By making use of Corollaries 2.1 and 2.2, we obtain the following corollary:

**Corollary 2.3.** Let  $q_1$  and  $q_2$  be convex univalent in  $U$  with  $q_1(0) = q_2(0) = 1$ ,  $\rho, \gamma, \mu, \varepsilon, \sigma, \tau, \delta \in \mathbb{C}$  such that  $\delta \neq 0$  and  $\sigma, \tau$  are not simultaneously zero. Suppose  $q_2$  satisfies (2.1) and  $q_1$  satisfies (2.8). For  $f \in \mathcal{A}$ , let

$$\left( \frac{D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta+1} f(z)}{D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta} f(z)} \right)^\delta \in \mathcal{H}[1, 1] \cap Q$$

and  $M_2(f, \rho, \gamma, \mu, \varepsilon, \sigma, \tau, \delta, \alpha, \beta, k, \eta, \lambda; z)$  as defined by (2.7) be univalent in  $U$ . If

$$\begin{aligned} & \rho + \gamma q_1(z) + \mu q_1^2(z) + \frac{\varepsilon}{q_1(z)} + \left( \frac{\sigma}{q_1(z)} + \frac{\tau}{q_1^2(z)} \right) z q_1'(z) \\ & < M_2(f, \rho, \gamma, \mu, \varepsilon, \sigma, \tau, \delta, \alpha, \beta, k, \eta, \lambda; z) \\ & < \rho + \gamma q_2(z) + \mu q_2^2(z) + \frac{\varepsilon}{q_2(z)} + \left( \frac{\sigma}{q_2(z)} + \frac{\tau}{q_2^2(z)} \right) z q_2'(z), \end{aligned}$$

then

$$q_1(z) < \left( \frac{D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta+1} f(z)}{D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta} f(z)} \right)^\delta < q_2(z)$$

and  $q_1, q_2$  are respectively the best subordinant and the best dominant.

**Theorem 2.4.** Let  $\Phi, \Psi \in \mathcal{A}$ ,  $\rho, \gamma, \mu, \varepsilon, \sigma, \tau, \delta, t_1, t_2 \in \mathbb{C}$  such that  $\delta \neq 0, t_1 + t_2 \neq 0$ , and  $\sigma, \tau$  are not simultaneously zero,  $q$  be convex univalent in  $U$  with  $q(0) = 1$  and assume that (2.1) holds true. If  $f \in \mathcal{A}$  satisfies the differential subordination

$$\begin{aligned} & M_3(f, \Phi, \Psi, \rho, \gamma, \mu, \varepsilon, \sigma, \tau, \delta, \alpha, \beta, k, \eta, \lambda, t_1, t_2; z) \\ & < \rho + \gamma q(z) + \mu q^2(z) + \frac{\varepsilon}{q(z)} + \left( \frac{\sigma}{q(z)} + \frac{\tau}{q^2(z)} \right) z q'(z), \end{aligned} \quad (2.11)$$

where

$$\begin{aligned}
 & M_3(f, \Phi, \Psi, \rho, \gamma, \mu, \varepsilon, \sigma, \tau, \delta, \alpha, \beta, k, \eta, \lambda, t_1, t_2; z) \\
 &= \rho + \left( \frac{t_1 D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta+1}(f * \Phi)(z) + t_2 D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta}(f * \Psi)(z)}{(t_1 + t_2)z} \right)^\delta \\
 &\quad \times \left( \gamma + \mu \left( \frac{t_1 D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta+1}(f * \Phi)(z) + t_2 D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta}(f * \Psi)(z)}{(t_1 + t_2)z} \right)^\delta \right) \\
 &+ \varepsilon \left( \frac{(t_1 + t_2)z}{t_1 D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta+1}(f * \Phi)(z) + t_2 D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta}(f * \Psi)(z)} \right)^\delta \\
 &\quad + \delta \left( \sigma + \tau \left( \frac{(t_1 + t_2)z}{t_1 D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta+1}(f * \Phi)(z) + t_2 D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta}(f * \Psi)(z)} \right)^\delta \right) \\
 &\quad \times \left( \frac{t_1 \left[ \begin{aligned} & \left( 1 + \sum_{m=1}^k \binom{k}{m} (-1)^{m+1} \left( \frac{\alpha}{\beta} \right)^m \right) D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta+2}(f * \Phi)(z) \\ & + \left( \lambda - 1 - \sum_{m=1}^k \binom{k}{m} (-1)^{m+1} \left( \frac{\alpha}{\beta} \right)^m \right) D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta+1}(f * \Phi)(z) \end{aligned} \right]}{t_1 D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta+1}(f * \Phi)(z) + t_2 D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta}(f * \Psi)(z)} \right. \\
 &\quad \left. + \frac{t_2 \left[ \begin{aligned} & \left( 1 + \sum_{m=1}^k \binom{k}{m} (-1)^{m+1} \left( \frac{\alpha}{\beta} \right)^m \right) D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta+1}(f * \Psi)(z) \\ & + \left( \lambda - 1 - \sum_{m=1}^k \binom{k}{m} (-1)^{m+1} \left( \frac{\alpha}{\beta} \right)^m \right) D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta}(f * \Psi)(z) \end{aligned} \right]}{t_1 D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta+1}(f * \Phi)(z) + t_2 D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta}(f * \Psi)(z)} \right] \right), \tag{2.12}
 \end{aligned}$$

then

$$\left( \frac{t_1 D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta+1}(f * \Phi)(z) + t_2 D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta}(f * \Psi)(z)}{(t_1 + t_2)z} \right)^\delta < q(z)$$

and  $q$  is the best dominant of (2.11).

**Proof.** Let the function  $p$  be defined by

$$p(z) = \left( \frac{t_1 D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta+1}(f * \Phi)(z) + t_2 D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta}(f * \Psi)(z)}{(t_1 + t_2)z} \right)^\delta, \quad (z \in U). \quad (2.13)$$

Then the function  $p$  is analytic in  $U$  and  $p(0) = 1$ .

A simple computation using (2.13) gives

$$\frac{zp'(z)}{p(z)} = \delta \left( \frac{t_1 z \left( D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta+1}(f * \Phi)(z) \right)' + t_2 z \left( D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta}(f * \Psi)(z) \right)'}{t_1 D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta+1}(f * \Phi)(z) + t_2 D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta}(f * \Psi)(z)} - 1 \right).$$

In view of (1.5), we obtain

$$\frac{zp'(z)}{p(z)} = \delta \left( \frac{t_1 \left[ \begin{aligned} & \left( 1 + \sum_{m=1}^k \binom{k}{m} (-1)^{m+1} \left( \frac{\alpha}{\beta} \right)^m \right) D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta+2}(f * \Phi)(z) \\ & + \left( \lambda - 1 - \sum_{m=1}^k \binom{k}{m} (-1)^{m+1} \left( \frac{\alpha}{\beta} \right)^m \right) D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta+1}(f * \Phi)(z) \end{aligned} \right]}{t_1 D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta+1}(f * \Phi)(z) + t_2 D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta}(f * \Psi)(z)} + \frac{t_2 \left[ \begin{aligned} & \left( 1 + \sum_{m=1}^k \binom{k}{m} (-1)^{m+1} \left( \frac{\alpha}{\beta} \right)^m \right) D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta+1}(f * \Psi)(z) \\ & + \left( \lambda - 1 - \sum_{m=1}^k \binom{k}{m} (-1)^{m+1} \left( \frac{\alpha}{\beta} \right)^m \right) D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta}(f * \Psi)(z) \end{aligned} \right]}{t_1 D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta+1}(f * \Phi)(z) + t_2 D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta}(f * \Psi)(z)} \right), \quad (2.14)$$

Also, we find that

$$\begin{aligned} & \rho + \gamma p(z) + \mu p^2(z) + \frac{\varepsilon}{p(z)} + \left( \frac{\sigma}{p(z)} + \frac{\tau}{p^2(z)} \right) zp'(z) \\ & = M_3(f, \Phi, \Psi, \rho, \gamma, \mu, \varepsilon, \sigma, \tau, \delta, \alpha, \beta, k, \eta, \lambda, t_1, t_2; z), \end{aligned} \quad (2.15)$$

where  $M_3(f, \Phi, \Psi, \rho, \gamma, \mu, \varepsilon, \sigma, \tau, \delta, \alpha, \beta, k, \eta, \lambda, t_1, t_2; z)$  is given by (2.12).

By using (2.15) in (2.11), we have

$$\rho + \gamma p(z) + \mu p^2(z) + \frac{\varepsilon}{p(z)} + \left( \frac{\sigma}{p(z)} + \frac{\tau}{p^2(z)} \right) zp'(z)$$

$$< \rho + \gamma q(z) + \mu q^2(z) + \frac{\varepsilon}{q(z)} + \left( \frac{\sigma}{q(z)} + \frac{\tau}{q^2(z)} \right) zq'(z).$$

By setting

$$\theta(w) = \rho + \gamma w + \mu w^2 + \frac{\varepsilon}{w} \quad \text{and} \quad \phi(w) = \frac{\sigma}{w} + \frac{\tau}{w^2},$$

it can be easily observed that  $\theta(w)$  and  $\phi(w)$  are analytic in  $\mathbb{C} \setminus \{0\}$  and that  $\phi(w) \neq 0, w \in \mathbb{C} \setminus \{0\}$ . Also, we get

$$Q(z) = zq'(z)\phi(q(z)) = \left( \frac{\sigma}{q(z)} + \frac{\tau}{q^2(z)} \right) zq'(z)$$

and

$$h(z) = \theta(q(z)) + Q(z) = \rho + \gamma q(z) + \mu q^2(z) + \frac{\varepsilon}{q(z)} + \left( \frac{\sigma}{q(z)} + \frac{\tau}{q^2(z)} \right) zq'(z).$$

In light of the hypothesis of Theorem 2.4, we see that  $Q(z)$  is starlike univalent in  $U$  and

$$\begin{aligned} & \operatorname{Re} \left\{ \frac{zh'(z)}{Q(z)} \right\} \\ &= \operatorname{Re} \left\{ \frac{1}{\sigma q(z) + \tau} \left( -\varepsilon + \gamma q^2(z) + 2\mu q^3(z) - \sigma zq'(z) - \frac{2\tau zq'(z)}{q(z)} \right) + \frac{zq''(z)}{q'(z)} + 1 \right\} > 0. \end{aligned}$$

Hence the result now follows by an application of Lemma 1.1.

By fixing  $\Phi(z) = \Psi(z) = \frac{z}{1-z}$  in Theorem 2.4, we obtain the following corollary:

**Corollary 2.4.** *Let  $\rho, \gamma, \mu, \varepsilon, \sigma, \tau, \delta, t_1, t_2 \in \mathbb{C}$  such that  $\delta \neq 0, t_1 + t_2 \neq 0$  and  $\sigma, \tau$  are not simultaneously zero,  $q$  be convex univalent in  $U$  with  $q(0) = 1$  and assume that (2.1) holds true. If  $f \in \mathcal{A}$  satisfies the differential subordination*

$$\begin{aligned} & M_4(f, \rho, \gamma, \mu, \varepsilon, \sigma, \tau, \delta, \alpha, \beta, k, \eta, \lambda, t_1, t_2; z) \\ & < \rho + \gamma q(z) + \mu q^2(z) + \frac{\varepsilon}{q(z)} + \left( \frac{\sigma}{q(z)} + \frac{\tau}{q^2(z)} \right) zq'(z), \end{aligned} \tag{2.16}$$

where

$$M_4(f, \rho, \gamma, \mu, \varepsilon, \sigma, \tau, \delta, \alpha, \beta, k, \eta, \lambda, t_1, t_2; z)$$

$$\begin{aligned}
&= \rho + \left( \frac{t_1 D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta+1} f(z) + t_2 D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta} f(z)}{(t_1 + t_2)z} \right)^\delta \\
&\quad \times \left( \gamma + \mu \left( \frac{t_1 D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta+1} f(z) + t_2 D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta} f(z)}{(t_1 + t_2)z} \right)^\delta \right) \\
&+ \varepsilon \left( \frac{(t_1 + t_2)z}{t_1 D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta+1} f(z) + t_2 D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta} f(z)} \right)^\delta \\
&\quad + \delta \left( \sigma + \tau \left( \frac{(t_1 + t_2)z}{t_1 D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta+1} f(z) + t_2 D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta} f(z)} \right)^\delta \right) \\
&\times \left( \frac{t_1 \left[ \begin{aligned} &\left( 1 + \sum_{m=1}^k \binom{k}{m} (-1)^{m+1} \left( \frac{\alpha}{\beta} \right)^m \right) D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta+2} f(z) \\ &+ \left( \lambda - 1 - \sum_{m=1}^k \binom{k}{m} (-1)^{m+1} \left( \frac{\alpha}{\beta} \right)^m \right) D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta+1} f(z) \end{aligned} \right]}{t_1 D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta+1} f(z) + t_2 D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta} f(z)} \right. \\
&\quad \left. + \frac{t_2 \left[ \begin{aligned} &\left( 1 + \sum_{m=1}^k \binom{k}{m} (-1)^{m+1} \left( \frac{\alpha}{\beta} \right)^m \right) D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta+1} f(z) \\ &+ \left( \lambda - 1 - \sum_{m=1}^k \binom{k}{m} (-1)^{m+1} \left( \frac{\alpha}{\beta} \right)^m \right) D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta} f(z) \end{aligned} \right]}{t_1 D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta+1} f(z) + t_2 D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta} f(z)} \right), \quad (2.17)
\end{aligned}$$

then

$$\left( \frac{t_1 D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta+1} f(z) + t_2 D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta} f(z)}{(t_1 + t_2)z} \right)^\delta < q(z)$$

and  $q$  is the best dominant of (2.16).

**Theorem 2.5.** Let  $\Phi, \Psi \in \mathcal{A}$ ,  $\rho, \gamma, \mu, \varepsilon, \sigma, \tau, \delta, t_1, t_2 \in \mathbb{C}$  such that  $\delta \neq 0, t_1 + t_2 \neq 0$  and  $\sigma, \tau$  are not simultaneously zero,  $q$  be convex univalent in  $U$  with  $q(0) = 1$  and assume

that (2.8) holds true. Let  $f \in \mathcal{A}$ ,

$$\left( \frac{t_1 D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta+1}(f * \Phi)(z) + t_2 D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta}(f * \Psi)(z)}{(t_1 + t_2)z} \right)^\delta \in \mathcal{H}[q(0), 1] \cap Q$$

and  $M_3(f, \Phi, \Psi, \rho, \gamma, \mu, \varepsilon, \sigma, \tau, \delta, \alpha, \beta, k, \eta, \lambda, t_1, t_2; z)$  as defined by (2.12) be univalent in  $U$ . If

$$\begin{aligned} & \rho + \gamma q(z) + \mu q^2(z) + \frac{\varepsilon}{q(z)} + \left( \frac{\sigma}{q(z)} + \frac{\tau}{q^2(z)} \right) zq'(z) \\ & < M_3(f, \Phi, \Psi, \rho, \gamma, \mu, \varepsilon, \sigma, \tau, \delta, \alpha, \beta, k, \eta, \lambda, t_1, t_2; z), \end{aligned} \tag{2.18}$$

then

$$q(z) < \left( \frac{t_1 D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta+1}(f * \Phi)(z) + t_2 D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta}(f * \Psi)(z)}{(t_1 + t_2)z} \right)^\delta$$

and  $q$  is the best subordinant of (2.18).

**Proof.** Let the function  $p$  be defined by (2.13).

In view of (1.5), the superordination (2.18) becomes

$$\begin{aligned} & \rho + \gamma q(z) + \mu q^2(z) + \frac{\varepsilon}{q(z)} + \left( \frac{\sigma}{q(z)} + \frac{\tau}{q^2(z)} \right) zq'(z) \\ & < \rho + \gamma p(z) + \mu p^2(z) + \frac{\varepsilon}{p(z)} + \left( \frac{\sigma}{p(z)} + \frac{\tau}{p^2(z)} \right) zp'(z). \end{aligned}$$

By setting  $\theta(w) = \rho + \gamma w + \mu w^2 + \frac{\varepsilon}{w}$  and  $\phi(w) = \frac{\sigma}{w} + \frac{\tau}{w^2}$ , it is easily observed that  $\theta(w)$  and  $\phi(w)$  are analytic in  $\mathbb{C} \setminus \{0\}$  and that  $\phi(w) \neq 0, w \in \mathbb{C} \setminus \{0\}$ . Also, we get

$$Re \left\{ \frac{\theta'(q(z))}{\phi(q(z))} \right\} = Re \left\{ \frac{q'(z)}{\sigma q(z) + \tau} (\gamma q^2(z) + 2\mu q^3(z) - \varepsilon) \right\} > 0.$$

Now Theorem 2.5 follows by applying Lemma 1.2.

By fixing  $\Phi(z) = \Psi(z) = \frac{z}{1-z}$  in Theorem 2.2, we obtain the following corollary:

**Corollary 2.5.** Let  $\rho, \gamma, \mu, \varepsilon, \sigma, \tau, \delta, t_1, t_2 \in \mathbb{C}$  such that  $\delta \neq 0, t_1 + t_2 \neq 0$  and  $\sigma, \tau$  are not simultaneously zero,  $q$  be convex univalent in  $U$  with  $q(0) = 1$  and assume that (2.8) holds true. Let  $f \in \mathcal{A}$ ,

$$\left( \frac{t_1 D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta+1} f(z) + t_2 D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta} f(z)}{(t_1 + t_2)z} \right)^\delta \in \mathcal{H}[q(0), 1] \cap Q$$

and  $M_4(f, \rho, \gamma, \mu, \varepsilon, \sigma, \tau, \delta, \alpha, \beta, k, \eta, \lambda, t_1, t_2; z)$  as defined by (2.17) be univalent in  $U$ . If

$$\begin{aligned} & \rho + \gamma q(z) + \mu q^2(z) + \frac{\varepsilon}{q(z)} + \left( \frac{\sigma}{q(z)} + \frac{\tau}{q^2(z)} \right) z q'(z) \\ & < M_4(f, \rho, \gamma, \mu, \varepsilon, \sigma, \tau, \delta, \alpha, \beta, k, \eta, \lambda, t_1, t_2; z), \end{aligned} \quad (2.19)$$

then

$$q(z) < \left( \frac{t_1 D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta+1} f(z) + t_2 D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta} f(z)}{(t_1 + t_2)z} \right)^\delta$$

and  $q$  is the best subordinator of (2.19).

Concluding the results of differential subordination and superordination, we state at the following sandwich result.

**Theorem 2.6.** Let  $q_1$  and  $q_2$  be convex univalent in  $U$  with  $q_1(0) = q_2(0) = 1$ ,  $\rho, \gamma, \mu, \varepsilon, \sigma, \tau, \delta, t_1, t_2 \in \mathbb{C}$  such that  $\delta \neq 0, t_1 + t_2 \neq 0$ , and  $\sigma, \tau$  are not simultaneously zero. Suppose  $q_2$  satisfies (2.1) and  $q_1$  satisfies (2.8). For  $f, \Phi, \Psi \in \mathcal{A}$ , let

$$\left( \frac{t_1 D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta+1} (f * \Phi)(z) + t_2 D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta} (f * \Psi)(z)}{(t_1 + t_2)z} \right)^\delta \in \mathcal{H}[1, 1] \cap Q$$

and  $M_3(f, \Phi, \Psi, \rho, \gamma, \mu, \varepsilon, \sigma, \tau, \delta, \alpha, \beta, k, \eta, \lambda, t_1, t_2; z)$  as defined by (2.12) be univalent in  $U$ . If

$$\begin{aligned} & \rho + \gamma q_1(z) + \mu q_1^2(z) + \frac{\varepsilon}{q_1(z)} + \left( \frac{\sigma}{q_1(z)} + \frac{\tau}{q_1^2(z)} \right) z q_1'(z) \\ & < M_3(f, \Phi, \Psi, \rho, \gamma, \mu, \varepsilon, \sigma, \tau, \delta, \alpha, \beta, k, \eta, \lambda, t_1, t_2; z) \\ & < \rho + \gamma q_2(z) + \mu q_2^2(z) + \frac{\varepsilon}{q_2(z)} + \left( \frac{\sigma}{q_2(z)} + \frac{\tau}{q_2^2(z)} \right) z q_2'(z), \end{aligned}$$

then

$$q_1(z) < \left( \frac{t_1 D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta+1} (f * \Phi)(z) + t_2 D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta} (f * \Psi)(z)}{(t_1 + t_2)z} \right)^\delta < q_2(z)$$

and  $q_1, q_2$  are respectively the best subordinator and the best dominant.



By making use of Corollaries 2.4 and 2.5, we obtain the following corollary:

**Corollary 2.6.** Let  $q_1$  and  $q_2$  be convex univalent in  $U$  with  $q_1(0) = q_2(0) = 1$ ,  $\rho, \gamma, \mu, \varepsilon, \sigma, \tau, \delta, t_1, t_2 \in \mathbb{C}$  such that  $\delta \neq 0, t_1 + t_2 \neq 0$ , and  $\sigma, \tau$  are not simultaneously zero. Suppose  $q_2$  satisfies (2.8) and  $q_1$  satisfies (2.1). For  $f \in \mathcal{A}$ , let

$$\left( \frac{t_1 D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta+1} f(z) + t_2 D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta} f(z)}{(t_1 + t_2)z} \right)^\delta \in \mathcal{H}[1, 1] \cap \mathcal{Q}$$

and  $M_4(f, \rho, \gamma, \mu, \varepsilon, \sigma, \tau, \delta, \alpha, \beta, k, \eta, \lambda, t_1, t_2; z)$  as defined by (2.17) be univalent in  $U$ . If

$$\begin{aligned} & \rho + \gamma q_1(z) + \mu q_1^2(z) + \frac{\varepsilon}{q_1(z)} + \left( \frac{\sigma}{q_1(z)} + \frac{\tau}{q_1^2(z)} \right) z q_1'(z) \\ & < M_4(f, \rho, \gamma, \mu, \varepsilon, \sigma, \tau, \delta, \alpha, \beta, k, \eta, \lambda, t_1, t_2; z) \\ & < \rho + \gamma q_2(z) + \mu q_2^2(z) + \frac{\varepsilon}{q_2(z)} + \left( \frac{\sigma}{q_2(z)} + \frac{\tau}{q_2^2(z)} \right) z q_2'(z), \end{aligned}$$

then

$$q_1(z) < \left( \frac{t_1 D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta+1} f(z) + t_2 D_z^{-\lambda} W_{\alpha, \beta}^{k, \eta} f(z)}{(t_1 + t_2)z} \right)^\delta < q_2(z)$$

and  $q_1, q_2$  are respectively the best subordinator and the best dominant.

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