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Convolution Properties of a Class of Analytic Functions

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Abstract

In this paper, we introduce a new class $\mathcal{R}_m^\alpha(h)$ of functions $F = f * \psi$, defined in the open unit disc E with $F(0) = F'(0) - 1 = 0$ and satisfying the condition

$$F'(z) + \alpha z F''(z) = \left(\frac{m}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{m}{4} - \frac{1}{2}\right) p_2(z),$$

for $\alpha \geq 0$, $m \geq 2$ and $p_i \prec h$, $i = 1, 2$.
Several convolution properties of this class are obtained by using the method of differential subordination. Many relevant connections of the findings here with those in earlier works are pointed out as special cases.

1 Introduction and Preliminaries

Let $\mathcal{A}$ denote the class of analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

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which are analytic in the open unit disc $E = \{z \in \mathbb{C} : |z| <\}$. Let $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha), 0 \leq \alpha < 1$, denote the subclasses of $\mathcal{A}$ which are respectively starlike of order α and convex of α in E. We denote $\mathcal{S}^*(0) \equiv \mathcal{S}^*$ and $\mathcal{C}(0) \equiv \mathcal{C}$. If f and g are analytic in E, we say that f is subordinate to g, written symbolically as $f(z) \prec g(z)$ or $f \prec g\,(z \in E)$, if and only if there exists a Schwarz function w, analytic in E with $w(0) = 0$ and $|w| < 1$ in E such that $f(z) = g(w(z))$ for $z \in E$.

The Hadamard product (or convolution) of two power series $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ is defined as the power series

$$(f * g)(z) = f(z) * g(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

Recently, subordination and convolution techniques have been extensively used in Geometric function theory. Many subclasses of the class $\mathcal{A}$ can be described in term of subordination and convolution. In 1973, Ruscheweyh and Sheil-Small [21] proved the Polya-Schoenberg conjecture that the class $\mathcal{C}$ is preserved under convolution. Several other problems were studied since then, (see [2, 3, 5, 19, 20]) and many applications found in various fields.

Let p be analytic in E with $p(0) = 1$. Then p is called Caratheodory function with Re $p(z) > 0$ in E, and is said to belong to the class P.

Definition 1.1. Let p be analytic in E with $p(0) = 1$. Then $p \in P[A, B; \beta]$ if and only if

$$p(z) \prec \left(\frac{1 + Az}{1 + Bz}\right)^{\beta} := p_{\beta}(A, B; z), \quad -1 \leq B < A \leq 1,\ \beta \in (0, 1],\ z \in E. \quad (1.2)$$

For $\beta = 1$, $P[A, B; 1] = P[A, B]$, see [7]. We note the following.

(i) It can be shown with simple computation, that the function $p_{\beta}(A, B; z)$ is convex and univalent in E.

(ii) Also, it is simple to see that

$$P[A,B;\beta] \subset P(\rho_1) \subset P,$$

where $\rho_1 = \left(\frac{1-A}{1-B}\right)^{\beta}$ and $P(\rho_1)$ is the class of Caratheodory functions of order ρ_1.

(iii) $p \in P(p_k)$, $k \geq 0$, if $p(z) \prec p_k(z)$, where $p_k(z)$ are the extremal functions mapping E onto the conic domain Ω_k defined as:

$$\Omega_k = \left\{ w = u + iv \colon u > k\sqrt{(u-1)^2 + v^2}\,; k \geq 0 \right\} \quad [9,10].$$

The domain Ω_0 is right half plane, Ω_k $(0 < k < 1)$ indicates a region bounded by hyperbola and a parabola for $k = 1$. For $k > 1$, it denotes an elliptic region. The function $p_k(z)$ is given as

$$p_k(z) = \begin{cases} \frac{1+z}{1-z}, \ k = 0, \\ 1 + \frac{2}{\pi^2}\left(\log \frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, \ k = 1, \\ 1 + \frac{2}{1-k^2}\sinh^2\left[\left(\frac{2}{\pi}\arccos k\right)\arctan\sqrt{z}\right], \ 0 < k < 1\,. \end{cases}$$

For $k > 1$, the extremal function can be found in [9,10]. It is clear that

$$P(p_k) \subset P(\rho_2) \subset P, \quad \rho_2 = \frac{k}{1+k}.$$

Also, $p \in P(p_k)$ implies that

$$|\arg p(z)| < \sigma\frac{\pi}{2}, \quad \sigma = \frac{2}{\pi}\arctan\left(\frac{1}{k}\right), \ z \in E.$$

Definition 1.2. Let p be analytic in E with $p(0) = 1$ and let

$$p(z) = \left(\frac{m+2}{4}\right)p_1(z) - \left(\frac{m-2}{4}\right)p_2(z), \ m \geq 2, \ z \in E, \tag{1.3}$$

where $p_i \prec h(z)$, $i = 1,2$ and $h(z)$ is convex univalent in E. Then $p(z)$ is said to belong to the class $P_m(h)$ in E.

For $P_m\left(\frac{1+z}{1-z}\right) = P_m$, we refer to [17]. Also, when we choose $h(z) = (1+sz)^2$, $0 < s \leq \frac{1}{\sqrt{2}}$ and $h(z) = z + \sqrt{1+z^2}$, we obtain the classes introduced by Afis and Noor [1] and Kanwal and Afis [6], respectvely. We note that if $h(z) = \left(\frac{1+Az}{1+Bz}\right)^{\beta}$, then $P_m(h)$ is denoted as $P_m[A, B; \beta]$ and, with $h(z) = p_k(z)$, $k \geq 0$, we have $P_m(p_k)$.

Definition 1.3. Let $\psi, F \in \mathcal{A}$ with $(\psi * F)(z) \neq 0$ and $f'(0) = 1$ in E. Then $F = \psi * f$ is said to belong to the class $\mathcal{R}_m^{\alpha}(h)$ if and only if $F' + \alpha z F'' \in P_m(h)$ in E, where $m \geq 2$, $\alpha \geq 0$.

Special Cases.

(i) Let $\psi(z) = \frac{z}{1-z}$, $h(z) = \frac{1+z}{1-z}$ and $m = 2$. Then $\mathcal{R}_2^{\alpha}$ implies Re $(f'(z) + \alpha z f''(z)) > 0$, $z \in E$.

(ii) By taking $h(z) = p_k(z)$, $\psi(z) = \frac{z}{1-z}$, we have $k - U\mathcal{R}_m^{\alpha}$. Also, $\mathcal{R}_m^{\alpha}\left[\left(\frac{1+Az}{1+Bz}\right)^{\beta}\right] = \mathcal{R}_m^{\alpha}[A, B; \beta]$.

Motivated principally by the works in [9, 10, 17, 19], we study the geometric properties of the class $\mathcal{R}_m^{\alpha}(h)$, which include both the convolution and subordination characterizations. Overall, we give some relevant connections between our findings and the existing ones in the literature.

To prove our main results, we shall need the following Lemmas.

Lemma 1.4. *[14] Let $h(z)$ be convex univalent in E, $h(0) = 1$ and Re $(\delta\psi(z) + t) > 0$. If $p(z)$ is analytic in U with $p(0) = 1$, then*

$$p(z) + \frac{zp'(z)}{\delta p(z) + t} \prec h(z) \quad \Rightarrow \quad p(z) \prec h(z) \quad z \in E,\ \delta, t \in \mathbb{C}. \tag{1.4}$$

Lemma 1.5. *[14] Suppose that the function $\lambda\colon \mathbb{C}^2 \times E \longrightarrow \mathbb{C}$ satisfies the condition Re$[\lambda(ix, y; z)] \leq \eta$ for real $x, y \leq -\frac{(1+x^2)}{2}$ and all $z \in E$. If $p(z) = 1 + c_1 z + \dots$ is analytic in E and*

$$Re\left[\lambda(p(z), zp'(z); z)\right] > \eta,\ for\ z \in E,$$

then Re $p(z) > 0$ in E.

Lemma 1.6. *[25] Let $f, g, h \in \mathcal{A}$, $\alpha, \beta, \gamma \leq 1$. If $f' \in P(\alpha), g' \in P(\beta), h' \in P(\gamma)$, then*

$$\text{Re}\left[\left(f' * g' * h'\right)(z)\right] > 1 - 4(1-\alpha)(1-\beta)(1-\gamma), \quad z \in E.$$

Lemma 1.7. *[22] Let $p(z)$ be analytic in E with $p(0) = 1$ and Re $(p(z)) > \frac{1}{2}$ in E. Then for any function F analytic in E, the function $p * F$ takes values in the convex hull of the image of E under F.*

In the next section, we present the main findings of this work.

2 Main Results

Theorem 2.1. *Let $F_1 = (\phi * f_1) \in \mathcal{R}_m^{\alpha_1}(h)$ and $F_2 = (\phi * f_2) \in \mathcal{R}_m^{\alpha_2}(h)$ in E. Then*

$$F = (F_1 * F_2) \in \mathcal{R}_m^1(\beta),$$

where

$$\beta = 1 - 2(1-\beta_1)(1-\beta_2) \tag{2.1}$$

with

$$\beta_i = 1 - 2(1-\delta_i)(1-\alpha_i), \quad i = 1, 2$$

and

$$\delta_1 = \int_0^1 \frac{dt}{1+t^{\alpha_i}}.$$

Proof. Since $F_1 \in \mathcal{R}_m^{\alpha_1}(h)$, then

$$F_1' + \alpha z F_1'' \in P_m(h).$$

That is

$$(F_1' * \phi_{\alpha_1}) \in P_m(h),$$

where $\phi_{\alpha_1}(z) = 1 + \sum\limits_{n=1}^{\infty}(n\alpha_i + 1)z^n$. Similarly,

$$(F_2' * \phi_{\alpha_2}) \in P_m(h) \quad \text{in} \quad E.$$

Now, $P_m(h) \subset P_m(\rho_i)$, with $\rho_1 = \left(\frac{1-A}{1-B}\right)^{\beta}$ and $\rho_2 = \frac{k}{k+1}$. Thus, $F_1 \in \mathcal{R}_m^{\alpha_1}(\rho_1)$ and $F_2 \in \mathcal{R}_m^{\alpha_2}(\rho_2)$ and

$$F_1' + \alpha_1 z F_1'' = F_1' * \phi_{\alpha_1} \qquad F_2' + \alpha_2 z F_2'' = F_2' * \phi_{\alpha_2}\,,$$

where

$$\phi_{\alpha_i}(z) = 1 + \sum_{n=1}^{\infty} (n\alpha_i + 1) z^n\,, \quad i = 1, 2.$$

We define $\psi_{\alpha_i} = [\phi_{\alpha_i}]^{(-1)} = 1 + \sum\limits_{n=1}^{\infty} \frac{z^n}{n\alpha_1 + 1} = \int_0^1 \frac{dt}{1+zt^{\alpha_i}}$, $i = 1, 2$ such that

$$(\psi_{\alpha_i} * \phi_{\alpha_i})(z) = \frac{z}{1-z}\,, \quad z \in E.$$

Then

$$F_i' = \left(\frac{m}{4} + \frac{1}{2}\right)[p_1 * \psi_{\alpha_i}] - \left(\frac{m}{4} - \frac{1}{2}\right)[p_2 * \psi_{\alpha_i}], \quad m \geq 2,\, i = 1, 2,$$

where $p_1, p_2 \in P(\rho_i)$. It is known [22] that for $\alpha_i > 0$, ψ_{α_i} is convex and

$$\operatorname{Re} \psi_{\alpha_i}(z) \geq \int_0^1 \frac{dt}{1 + t^{\alpha_i}} = \delta_i,\, \delta_i \in \left[\frac{1}{2}, 1\right).$$

Therefore,

$$F_1' = \left(\frac{m}{4} + \frac{1}{2}\right)[p_1 * \psi_{\alpha_1}] - \left(\frac{m}{4} - \frac{1}{2}\right)[p_2 * \psi_{\alpha_1}],$$

where $(p_i * \psi_{\alpha_1}) \in P(\delta_1)$, $\delta_1 \in \left[\frac{1}{2}, 1\right)$ and

$$F_2' = \left(\frac{m}{4} + \frac{1}{2}\right)[p_1 * \psi_{\alpha_2}] - \left(\frac{m}{4} - \frac{1}{2}\right)[p_2 * \psi_{\alpha_2}],$$

with $(p_i * \psi_{\alpha_2}) \in P(\delta_2)$, $\delta \in \left[\frac{1}{2}, 1\right)$. Hence,

$$F_1' \in P_m(\beta_1) \quad \text{and} \quad F_2' \in P_m(\beta_2), \tag{2.2}$$

where

$$\beta_1 = 1 - 2(1-\delta_1)(1-\alpha_1), \quad \beta_2 = 1 - 2(1-\delta_2)(1-\alpha_2).$$

Combining these relation, we have

$$F'(z) = (F_1(z) * F_2(z))' = F_1(z) * zF_2''(z)$$

and

$$(zF'(z))' = F'(z) + zF''(z) = (F_1'(z) * F_2'(z)). \tag{2.3}$$

From (2.2) and (2.3), we have that $F \in \mathcal{R}_m^1(\beta)$, where β is given by (2.1). □

Remark 2.2. From (2.1) and Lemma 1.4 with $\delta = 0$, we have

$$F' \in P_m(\gamma_1), \quad \gamma_1 = 1 + 4(1-\beta_1)(1-\beta_2)(\ln 2 - 1) \tag{2.4}$$

and repeating the same procedure, it follows that

$$\frac{F(z)}{z} \in P_m(\gamma_2), \quad \gamma_2 = 1 - 8(1-\beta_1)(1-\beta_2)(\ln 2 - 1)^2. \tag{2.5}$$

Theorem 2.3. *Let F_1 and F_2 belong to $\mathcal{R}_2^1(\rho_i)$, $i = 1, 2$ and let $F(z) = (F_1 * F_2)(z)$. Then $F \in \mathcal{S}^*$, provided*

$$(1-\beta_1)(1-\beta_2) < \frac{3}{8(\ln 2 - 1)^2 + 4}, \tag{2.6}$$

where β_i, $i = 1, 2$ are as given in Theorem 2.1.

Proof. $F_1 = \phi * f_1$, $F_2 = \phi * f_2$ and $F_i \in \mathcal{R}_2^1(h) \subset \mathcal{R}_2^1(\rho_i)$. Using Theorem 2.1 with $m = 2$, we have

$$F' + zF'' \in P(\beta).$$

Let $\frac{zF'}{F} = h$ and $\frac{F}{z} = \eta$. Then h and η are analytic in E with $h(0) = \eta(0) = 1$.

Now,

$$F' + zF'' = \eta(z)\left(h^2 + zh'\right) = \lambda\left(h, zh', z\right),$$

where $\lambda(u,v,z)=\eta(z)(u^2+v)$ and Re $[\lambda(h,zh',z)]>\beta$, $z\in E$. For real x,y and $y\le -\frac{1+x^2}{2}$, so we get

$$\begin{aligned}\text{Re }\left[\lambda(h,zh',z)\right] =&(y-x^2)\text{Re}\,\eta(z)\\ \le&-\frac{1+3x^2}{2}\text{Re}\,\eta(z)\\ \le&-\frac{\text{Re}\,\eta(z)}{2}\\ <&\beta,\quad (z\in E)\end{aligned}$$

provided $(1-\beta_1)(1-\beta_2)<\frac{3}{8(\ln 2-1)^2+4}$, where we have used (2.3), (2.4) and (2.5). Hence, from Lemma 1.5, it follows that Re $h(z)>0$ in E and $F\in\mathcal{S}^*$. This completes the proof. □

Theorem 2.4. *Let $\phi(z)=\frac{z}{1-z}$ and $F_i=\phi*f_i=f_i$. Let $f_i\in\mathcal{R}_2^{\alpha_i}(p_k)$, $i=1,2$, and define*

$$F=I(f_1*f_2)=\frac{2}{z}\int_0^z(f_1*f_2)(t)dt.$$

Then $F\in\mathcal{S}^*(\delta_0)$ provided

$$(1-\delta_1)(1-\delta_2)<\frac{3}{8(\ln 2-1)^2+4}\quad \textit{and}\quad \delta_0=\left(\frac{1}{2(2\ln 2-1)}-1\right)=0.294,$$

where

$$\delta_i=\int_0^1\frac{dt}{1-t^{\alpha_i}},\quad i=1,2.$$

Proof. Using Theorem 2.3, it follows that $(f_1*f_2)\mathcal{S}^*$ in E. It is known [14, Theorem 3.3g, p. 116] that the function F defined as Libera integral [12] of (f_1*f_2) is starlike of order $\delta_0=\left(\frac{1}{2(2\ln 2-1)}-1\right)\approx 0.294$.

As a special case, we take $k=1$. Then $\delta_i=\frac{1}{2}$ and Re $(F'(z)+zF''(z))>\frac{1}{2}$, $z\in E$. □

Theorem 2.5. *The class $\mathcal{R}_m^{\alpha}(\rho_i)$ is invariant under convex convolution in E.*

Proof. Let $\psi \in \mathcal{C}, F = f * \phi \in \mathcal{R}_m^\alpha(\rho_i)$. Consider

$$\begin{aligned}(F*\psi)' + \alpha z(F*\psi)'' &= \frac{\psi}{z} * \left(F' + \alpha z F''\right)\\ &= \frac{\psi}{z} * p, \quad p \in P_m(\rho_i)\\ &= \left(\frac{m}{4}+\frac{1}{2}\right)\left(\frac{\psi}{z} * p_1\right) - \left(\frac{m}{4}-\frac{1}{2}\right)\left(\frac{\psi}{z} * p_2\right),\\ &\qquad p_1, p_2 \in P(\rho_i).\end{aligned}$$

Since $\psi \in \mathcal{C}$, so $\mathrm{Re}\left(\frac{\psi}{z}\right) > \frac{1}{2}$ and by applying Lemma 1.7, $(\psi * p_i) \in P(\rho_i)$ and $(\psi * p) \in P_m(\rho_i)$. This proves that $F \in \mathcal{R}_m^\alpha(\rho_i)$ in E.

As an application of Theorem 2.5, we deduce that the class $\mathcal{R}_m^\alpha(\rho_i)$ is invariant under Libera integral operator. □

Remark 2.6. By choosing suitable and permissible values of parameter, we obtain several known and new results. Moreover, the class $\mathcal{R}_m^\alpha(h)$ can be studied further by involving several linear operators such as Carlson-Shaffer [4], Dziok-Srivastava operator [13], which includes Ruscheweyh [18, 19] and Noor [15] operators as special cases. Also see [8, 11, 16, 18, 24].

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