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Iterative Methods and Sensitivity Analysis for Exponential General Variational Inclusions

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Abstract

In this paper, we introduce some new classes of exponentially variational inclusions. Several important special cases are obtained as applications. Using the resolvent operator, it is shown that the exponentially variational inclusions are equivalent to the fixed point problem. This alternative formulation is used to suggest and investigate a wide call of iterative schemes for solving the variational inclusions. Dynamical systems is used to study asymptotic stability of the solution. We study the convergence analysis for proposed iterative methods. Sensitivity analysis is also considered. Our results represent a significant improvement over the existing ones. As special cases, we obtain some new and old results for solving exponentially variational inclusions and related optimization problems.

1 Introduction

Variational inclusion theory contains a wealth of new ideas and techniques, which can be viewed as a novel extension and generalization of the variational

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inequalities. It is amazing that a wide class of unrelated problems can be studied in the unified framework of variational inclusions. The resolvent equations were introduced and studied by Noor [28, 29]. Noor [28, 29] proved that the variational inclusions are equivalent to the resolvent equations using the resolvent operator technique. This equivalent alternative formulation has been used to study the existence of a solution as well as to develop various iterative methods for solving the variational inclusions. In this direction, several numerical methods have been developed for solving the variational inclusions and their variant forms. Noor [29, 31, 32] suggested and analyzed some three-step forward-backward splitting algorithms for solving variational inequalities and quasi variational inclusions by using the updating techniques of the solution. These forward-backward splitting algorithms are similar to those of Glowinski et al. [16–18], which they suggested by using the Lagrangian technique. It is known that three-step schemes are versatile and efficient. These three-step schemes are a natural generalization of the splitting methods for solving partial differential equations. For applications of the splitting techniques to partial differential equations, see Ames [3] and the references therein. For novel applications of the three-step methods, see Ashish et al. [5]. These methods include the Mann and Ishikawa iterative schemes and modified forward-backward splitting methods of Tseng [68], Noor [29, 31, 32] and Noor et al. [54, 56] as special cases.

Related to variational inclusions, we have problem of dynamical systems. Dynamical systems arise naturally in numerous applied and theoretical fields including celestial mechanics financial forecasting, environmental applications, neuroscience, brain modeling. It is known that the variational inequalities are equivalent to the fixed point problems. Dupuis et al. [12] suggested the projected dynamical system using the fixed point technique. This approach is used to study the asymptotic stability of the solution of the variational inequalities. See also Nagurney et al. [22] and Noor et al. [42] for more details. Noor et al. [41, 52, 54, 56] used this technique to suggest some efficient iterative schemes for solving variational inequalities. Noor et al. [52] has proved that variational inclusions are equivalent to the dynamically systems. This equivalence has been used to study the existence and stability of the solution of variational inclusions. Alvarez [2] used the inertial type projection methods for solving variational inequalities, the origin of which can be traced back to Polyak [61]. Noor [34] suggested and investigated inertial type projection methods for solving general variational inequalities. These inertial type methods have been modified in various directions for solving variational inequalities and related optimization problems. Recently Shehu et al. [64], Noor et al. [45, 53, 56, 57] and Jabeen et al. [19] analyzed some inertial projection methods for some classes of general quasi variational inequalities. Convergence analysis of these inertial type methods has been considered under some mild conditions.

In recent years, various extensions and generalizations of convex functions and convex sets have been considered and studied using innovative ideas and techniques. It is known that more accurate and inequalities can be obtained using the logarithmically convex functions than the convex functions. Closely related to the log-convex functions, we have the concept of exponentially convex(concave) functions, the origin of exponentially convex functions can be traced back to Bernstein [9]. Avriel [6] introduced and studied the concept of r-convex functions. For further properties of the r-convex functions, which have important applications in information theory, big data analysis, machine learning and statistics, see Zhao et al. [69] and the references therein. Noor and Noor [36–40, 46, 51, 57] introduced and investigated some new concepts of exponentially convex functions. It is have been shown that the exponentially convex(concave) have nice nice properties which convex functions enjoy. Several new concepts have been introduced and investigated. For more details, see [1,4,6,7,36–41,46,51,54,59,69] and the references therein. Noor and Noor. [36–41] proved that the optimal conditions of the differentiable exponentially convex functions can be characterized by a class of variational inequalities, which is called the exponentially variational inequalities.

We like to mention that sensitivity analysis is important for several reasons. First, estimating problem data often introduces measurement errors, sensitivity analysis helps in identifying sensitive parameters that should be obtained with relatively high accuracy. Second, sensitivity analysis may help to predict the future changes of the equilibrium as a result of changes in the governing system. Third, sensitivity analysis provides useful information for designing or planning various equilibrium systems. Furthermore, from mathematical and engineering point of view, sensitivity analysis can provide new insight regarding problems being studied can stimulate new ideas and techniques for solving exponentially variational inclusions and related problem.

In this paper, we consider some new classes of exponentially variational inclusions. It have been shown that the complementarity problems, general variational inequalities, exponentially variational inequalities, system of absolute value general equations and optimization problems can be obtained as special cases of exponentially variational inclusions. We prove that the exponentially variational inclusions are equivalent to fixed point problems. This alternative formulation is used to suggest and investigate some new three step implicit and explicit iterative methods for solving exponentially variational inclusions. These new iterative methods can be viewed as significant generalization of the three-step methods of Noor [30,34] and Tseng [68]. We have also used the dynamical systems technique coupled with finite difference schemes to propose some new iterative methods for solving the exponentially variational inclusions. The convergence criteria of the proposed implicit methods is discussed under some mild conditions. Several important special cases are discussed as applications of our results. We have only considered the theoretical aspects of the proposed methods.We also study the sensitivity analysis of the exponentially variational inclusion. It is still an open problem to implement these methods and compare with other techniques. It is expected the techniques and ideas of this paper may be starting point for further research.

2 Preliminaries

Let H be a real Hilbert space whose inner product and norm are denoted by $< \cdot, \cdot >$ and $\|.\|$ respectively. Let K be a nonempty closed convex set in H. Let $\varphi : H \to R \cup \{+\infty\}$ be a proper, convex and lower semicontinuous function.

For given nonlinear operators $T, g : H \to H$, and a maximal monotone operator $A : H \to H$, consider the problem of finding $u \in H$ such that

$$0 \in e^{Tu} + A(g(u)), \tag{2.1}$$

which is called the exponentially general variational inclusion.

(i) If $e^{Tu} = \Phi(u)$, then problem (2.1) reduces to finding

$$0 \in \Phi(u) + A(g(u)), \tag{2.2}$$

which is called the general variational inclusion for the sum of two monotone operators.

(ii) Note that if $g \equiv I$, the identity operator, then problem (2.1) is equivalent to finding $u \in H$ such that

$$0 \in e^{Tu} + A(u), \tag{2.3}$$

is called the exponentially variational inclusion.

(iii) Note that for $e^{Tu} = \Phi(u)$, then problem (2.3) reduces to finding $u \in H$ such that

$$0 \in \Phi(u) + A(u), \tag{2.4}$$

Problem (2.4) is known as finding the zero of the sum of two monotone operators. This problem is being studied extensively and has important applications in operations research and engineering sciences. For recent state of the art, see [13, 14, 18, 20, 21, 31, 32, 63, 68] and the references therein.

(iv) If $A(.) = \partial\phi(.)$, where $\partial\varphi(.)$ is the subdifferential of a proper, convex and lower semicontinuous function $\varphi : H \longrightarrow R \cup \{+\infty\}$, then problem (2.1) reduces to finding $u \in H$ such that

$$0 \in e^{Tu} + \partial\phi(g(u)) \tag{2.5}$$

or equivalently, finding $u \in H$ such that

$$\langle e^{Tu}, g(v) - g(u)\rangle + \phi(g(v)) - \phi(g(u)) \geq 0, \qquad \forall g(v) \in H. \tag{2.6}$$

The inequality of type (2.6) is called the mixed exponentially general variational inequality or the exponentially general variational inequality of the second kind. It can be shown that a wide class of linear and nonlinear problems arising in pure and applied sciences can be studied via the mixed exponentially general variational inequalities (2.6).

(v) If $g \equiv I$, the identity operator, then the problem (2.6) is equivalent to finding $u \in H$ such that

$$\langle e^{Tu}, v - u\rangle + \phi(v) - \phi(u) \geq 0, \quad \forall v \in H, \tag{2.7}$$

which are called the mixed exponentially variational inequalities.

(vi) If φ is the indicator function of a closed convex set K in H, that is,

$$\varphi(u) \equiv I_K(u) = \begin{cases} 0, & \text{if } u \in K \\ +\infty, & \text{otherwise,} \end{cases}$$

then the exponentially mixed general variational inequality (2.6) is equivalent to

$$\langle e^{Tu}, g(v) - g(u)\rangle \geq 0, \qquad \forall g(v) \in K. \tag{2.8}$$

The inequality of the type (2.8) is known as the exponentially general variational inequality, which was introduced and studied by Noor and Noor [46].

(vii) If $e^{Tu} = \Phi(u)$, then problem (2.8) collapses to finding $u \in H,\ g(u) \in K$ such that

$$\langle \Phi(u), g(v) - g(u)\rangle \geq 0, \qquad \forall g(v) \in K. \tag{2.9}$$

is known as the general variational inequality, introduced and studied by Noor [24] in 1988.

(viii) For $K = H$, and $\Phi(u) = Mu - N(u)$, where M, N are operators, problem (2.9) is equivalent to finding $u \in H$ such that

$$\langle Mu - N(u), g(v) - g(u)\rangle \geq 0, \qquad \forall v \in H, \tag{2.10}$$

which is called the system of general equations, see [26, 48–50, 55]. It turned out that the odd-order and nonsymmetric free, unilateral, obstacle and equilibrium problems can be studied by the general variational inclusion (2.9), see [26, 30, 34, 53, 54] and the references therein.

(ix) If $K^* = \{u \in H : \langle u, v\rangle \geq 0, \text{ for all } v \in K\}$ is a polar cone of a convex cone K in H, then problem (2.8) is equivalent to finding $u \in H$ such that

$$g(u) \in K, \quad e^{Tu} \in K^*, \quad \langle e^{Tu}, g(u)\rangle = 0, \tag{2.11}$$

which is called the exponentially general complementarity problem. For $g = I$, problem (2.11) is called the exponentially complementarity problems. For the theory, applications and numerical methods of complementarity problems, see [10, 24, 26, 27, 34, 46] and the references therein.

(x) For $g \equiv I$, the identity operator, the exponentially general variational inequality (2.8) collapses to: find $u \in K$ such that

$$\langle e^{Tu}, v - u\rangle \geq 0, \qquad \forall v \in K, \tag{2.12}$$

which is called the exponentially variational inequality, introduced and studied by Noor et al. [51].

(xi) If $e^{Tu} = \Phi(u)$, then problem (2.13 is equivalent to finicking $u \in K$ such that

$$\langle \Phi(u), v - u\rangle \geq 0, \qquad \forall v \in K, \tag{2.13}$$

is called the classical variational inequality studied by Stampacchia [67] in 1964. For the recent state-of-the art, see [8, 10, 12–23, 25–34, 41–44, 46, 47, 51–54, 56, 58, 60, 62–67, 70].

Remark 2.1. *For appropriate and suitable choice of the operators and the spaces, one can obtain several new and known classes of variational inclusions and optimization problems as special cases of the exponentially general variational inclusion (2.1). This shoes that problem (2.1) is quite flexible, general and unified ones.*

We also need the following well known concepts and results.

Definition 2.1. *If A is a maximal monotone operator on H, then, for a constant $\rho > 0$, the resolvent operator associated with A is defined by*

$$J_A(u) = (I + \rho A)^{-1}(u), \quad \text{for all } u \in H,$$

where I is the identity operator. It is well known that a monotone operator is maximal, if and only if, its resolvent operator is defined everywhere. In addition, the resolvent operator is a single-valued and nonexpansive, that is,

$$\|J_A(u) - J_A(v)\| \leq \|u - v\|, \quad \forall u, v \in H.$$

Remark 2.2. *It is well known that the subdifferential $\partial\phi$ of a proper, convex and lower semicontinuous function $\phi : H \to R \cup \{+\infty\}$ is a maximal monotone operator, we denote by*

$$J_\varphi(u) = (I + \rho\partial\phi)^{-1}(u), \quad \forall u \in H,$$

the resolvent operator associated with $\partial\phi$, which is defined everywhere on H.

Lemma 2.1. *For a given $z \in H$, $u \in H$ satisfies the inequality*

$$\langle u - z, v - u\rangle + \rho\phi(v) - \rho\phi(u) \geq 0, \quad \forall v \in H, \tag{2.14}$$

if and only if,

$$u = J_\phi z, \tag{2.15}$$

where $J_\varphi = (I+\rho\partial\phi)^{-1}$ is the resolvent operator and ρ is a constant. This property of the resolvent operator J_ϕ plays an important part in obtaining our results.

We also need the following concepts.

Definition 2.2. *An operator $T : H \to H$ is said to be:*

(i) *exponentially general monotone, if*

$$\langle e^{Tu} - e^{Tv}, g(u) - g(v)\rangle \geq 0, \quad \forall u, v \in H.$$

(ii) *exponentially general pseudomonotone, if*

$$\langle e^{Tu}, g(v) - g(u)\rangle \geq 0 \quad \textit{implies} \quad \langle e^{Tv}, g(v) - g(u)\rangle \geq 0, \quad \forall u, v \in H.$$

(iii) *exponentially general quasi-monotone, if*

$$\langle e^{Tu}, g(v) - g(u)\rangle > 0 \quad \textit{implies} \quad \langle e^{Tv}, g(v) - g(u)\rangle \geq 0, \quad \forall u, v \in H.$$

(iv) *exponentially general Lipschitz continuous, if there exists a constant $\delta > 0$ such that*

$$\langle e^{Tu} - e^{Tv}, g(u) - g(v)\rangle \leq \delta \|g(v) - g(u)\|^2, \quad \forall u, v \in H.$$

Note that for $g \equiv I$, the identity operator, Definition 2.2 reduces to the standard definition of monotonicity, pseudomonotonicity, quasimonotonicity and (relaxed) Lipschitz continuity of the operator T. Note that monotonicity implies pseudomonotonicity and pseudomonotonicity implies quasimonotonicity, but the converse is not true, see [8].

3 Resolvent Method

In this section, we suggest and analyze some new iterative methods for solving the exponentially general variational inclusions (2.1). First of all, we prove that problem (2.1)is equivalent to the fixed point problem by using the definition of the resolvent operator.

Lemma 3.1. *The function $u \in H$ is a solution of the variational inclusion (2.1), if and only if, $u \in H$ satisfies the relation*

$$g(u) = J_A[g(u) - \rho e^{Tu}], \tag{3.1}$$

where $J_A = (I + \rho A)^{-1}$ is the resolvent operator and $\rho > 0$ is a constant.

Proof. Let $u \in H$ be a solution of (2.1). Then, for a constant $\rho > 0$, the exponentially general variational inclusion (2.1) can be written as

$$0 \in -g(u) + \rho e^{Tu} + (I + \rho A)g(u),$$

which is equivalent to finding $u \in H$ such that

$$g(u) = (I + \rho A)^{-1}[g(u) - \rho e^{Tu}] = J_A[g(u) - \rho e^{Tu}],$$

the required result. □

Lemma 3.1 implies that the exponentially general variational inclusion (2.1) is equivalent to the fixed point problem (3.1). This equivalent fixed point formulation was used to suggest some implicit iterative methods for solving the exponentially general variational inclusions. One uses the equivalent fixed point formulation(3.1) to suggest the following iterative methods for solving variational inclusion (2.1).

We rewrite the equation (3.1) as:

$$u = u - g(u) + J_A[g(u) - \rho e^{Tu}],$$

which is another fixed point formulation. This equivalent fixed point formulation is used tp suggest the following iterative methods for solving the exponentially general variational inclusion (2.1).

Algorithm 1. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$u_{n+1} = u_n - g(u_n) + J_A[g(u_n) - \rho e^{Tu_n}], \quad n = 0, 1, 2, ...$$

which is known as the resolvent method and has been studied extensively.

Algorithm 2. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$u_{n+1} = u_n - g(u_n) + J_A[g(u_n) - \rho e^{T(g^{-1}J_A(g(u_n)-\rho Tu_n)}], \quad n = 0, 1, 2, ...$$

which can be viewed as the extraresolvent method for solving the classical exponentially general variational inequalities. Using the technique of Noor [34], one can prove the convergence of the extragradient method for pseudomonotone operators.

Algorithm 3. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$u_{n+1} = u_n - g(u_n) + J_A[g(u_{n+1}) - \rho e^{Tu_{n+1}}], \quad n = 0, 1, 2, ...$$

which is known as the modified resolvent method in the sense of Noor [34].

To implement Algorithm 3, we use the predictor-corrector technique. Consequently, we obtain the following two-step iterative method.

Algorithm 4. *For a given $u_0 \in H$, compute u_{n+1} by the iterative schemes*

$$\begin{aligned} y_n &= u_n - g(u_n) + J_A[g(u_n) - \rho e^{Tu_n}] \\ u_{n+1} &= u_n - g(u_n) + J_A[g(y_n) - \rho e^{Ty_n}], \quad n = 0, 1, 2, ... \end{aligned}$$

We can rewrite the equation (3.1) as:

$$g(u) = J_A[g(\frac{u+u}{2}) - \rho e^{Tu}].$$

This fixed point formulation was used to suggest the following implicit method for solving variational inclusion. We used this equivalent formulation to suggest implicit methods for exponentially general variational inclusion (2.1).

Algorithm 5. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$u_{n+1} = u_n - g(u_n) + J_A[g(\frac{u_n + u_{n+1}}{2}) - \rho e^{Tu_{n+1}}], \quad n = 0, 1, 2, ...$$

For the implementation of this Algorithm 5, one can use the predictor-corrector technique to suggest the following two-step iterative method for solving general variational inclusions.

Algorithm 6. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$\begin{aligned} g(y_n) &= J_A[g(u_n) - \rho e^{Tu_n}] \\ u_{n+1} &= u_n - g(u_n) + J_A[g(\frac{y_n + u_n}{2}) - \rho e^{Ty_n}], \quad \lambda \in [0,1], \quad n = 0,1,2,... \end{aligned}$$

which is a two-step iterative method:

From the equation (3.1), we have

$$g(u) = J_A[g(u) - \rho e^{T(\frac{u+u}{2})}].$$

This fixed point formulation is used to suggest the implicit method for solving the exponentially variational inclusion as

Algorithm 7. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$u_{n+1} = u_n - g(u_n) + J_A[g(u_n) - \rho e^{T(\frac{u_n+u_{n+1}}{2})}], \quad n = 0,1,2,....$$

which is another implicit method.

To implement this implicit method, one can use the predictor-corrector technique to rewrite Algorithm 7 as equivalent two-step iterative method.

Algorithm 8. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$\begin{aligned} g(y_n) &= J_A[g(u_n) - \rho e^{Tu_n}], \\ u_{n+1} &= u_n - g(u_n) + J_A[g(u_n) - \rho e^{T(\frac{u_n+y_n}{2})}], \quad n = 0,1,2,.... \end{aligned}$$

which is known as the mid-point implicit method for solving exponentially general variational inclusion.

Using the techniques of Noor et al. [54], one can investigate the the convergence analysis and other aspects of Algorithm 5.

It is obvious that Algorithm 5 and Algorithm 7 have been suggested using different variant of the fixed point formulations (3.1). It is natural to combine these fixed point formulation to suggest a hybrid implicit method for solving the exponentially general variational inclusion and related optimization problems, which is the main motivation of this paper.

One can rewrite the equation (3.1) as

$$g(u) = J_A[g\big(\frac{u+u}{2}\big) - \rho e^{T(\frac{u+u}{2})}].$$

This equivalent fixed point formulation enables to suggest the following method for solving the exponentially general variational inclusion (2.1).

Algorithm 9. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$u_{n+1} = u_n - g(u_n) + J_A[g\big(\frac{u_n + u_{n+1}}{2}\big) - \rho e^{T(\frac{u_n+u_{n+1}}{2})}], \quad n = 0, 1, 2, ...$$

which is an implicit method.

We would like to emphasize that Algorithm 9 is an implicit method. To implement the implicit method, one uses the predictor-corrector technique. We use Algorithm 1 as the predictor and Algorithm 9 as corrector. Thus, we obtain a new two-step method for solving exponentially general variational inclusion.

Algorithm 10. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$\begin{aligned} g(y_n) &= J_A[g(u_n) - \rho e^{Tu_n}] \\ u_{n+1} &= u_n - g(u_n) + J_A[g\big(\frac{y_n + u_n}{2}\big) - \rho e^{T\left(\frac{y_n+u_n}{2}\right)}], \quad n = 0, 1, 2, ... \end{aligned}$$

which is a two-step method.

For constants $\lambda, \xi \in [0,1]$, we can rewrite the equation (3.1) as:

$$g(u) = J_A\big[(1-\lambda)g(u) + \lambda g(u)) - \rho e^{T((1-\xi)u+\xi u)}\big].$$

This equivalent fixed point formulation enables to suggest the following method for solving the exponentially general variational inclusion (2.1).

Algorithm 11. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$g(u_{n+1}) = J_A\big[(1-\lambda)g(u_n) + \lambda g(u_{n+1})) - \rho e^{T((1-\xi)u_n+\xi u_{n+1})}\big], \quad n = 0, 1, 2, ...$$

which is an implicit method.

Using the prediction-correction technique, Algorithm 11 can be written in the following form.

Algorithm 12. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme.*

$$\begin{aligned} g(y_n) &= J_A[g(u_n) - \rho e^{Tu_n}] \\ g(u_{n+1}) &= J_A\big[(1-\lambda)g(u_n) + \lambda g(y_n)) - \rho e^{T((1-\xi)u_n+\xi y_n)}\big], \quad n = 0, 1, 2, .. \end{aligned}$$

which is two step method.

For a constants $\xi \in [0,1]$, we can rewrite the equation (3.1) as:

$$g(u) = J_A\big[g((1-\xi)u + \xi u)) - \rho e^{T((1-\xi)u+\xi u)}\big].$$

This equivalent fixed point formulation enables to suggest the following method for solving the exponentially general variational inclusion (2.1).

Algorithm 13. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$g(u_{n+1}) = J_A\big[g((1-\xi)u_n + \xi u_{n-1}) - \rho e^{T((1-\xi)u_n+\xi u_{n-1})}\big], \quad n = 0, 1, 2, ...$$

which is an inertial implicit method for solving the exponentially general variational inclusions (2.1).

Algorithm 13 can be rewritten as the following method using the predictor-corrector technique.

Algorithm 14. *For a given $u_0, u_1 \in H$, compute u_{n+1} by the iterative scheme*

$$\begin{aligned} y_n &= (1-\xi)u_n + \xi u_{n-1}) \\ g(u_{n+1}) &= J_A\big[g(y_n) - \rho T y_n\big], \quad n = 0, 1, 2, ... \end{aligned}$$

which is known as the inertial two-step method.

Remark 3.1. *It is worth mentioning that Algorithm 12 is a unified ones. For suitable and appropriate choice of the constant λ and ξ, one can obtain a wide class of iterative methods for solving exponentially general variational inclusions and related optimization problems.*

We now define the resolvent residue vector by the relation

$$R(u) = g(u) - J_A[g(u) - \rho e^{Tu}]. \tag{3.2}$$

From Lemma 3.1, it is clear the $u \in H, g(u) \in K$ is a solution of (2.1), if and only if, $u \in H, g(u) \in H$ is a zero of the equation

$$R(u) = 0. \tag{3.3}$$

For a positive constant γ, we can rewrite equation (3.3) as

$$g(u) + \rho e^{Tu} = g(u) + \rho e^{Tu - \gamma R(u)}.$$

This fixed-point formulation allows us to suggest and analyze the following iterative method.

Algorithm 15. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$g(u_{n+1}) = g(u_n) + \rho e^{Tu_n} - \rho e^{Tu_{n+1} - \gamma R(u_n)},$$

which is known as the implicit iterative method.

In order to implement Algorithm 15, one has to compute the solution implicitly, which is itself a difficult problem. In order to over come this difficulty, we suggest another iterative method, the convergence which also requires monotonicity of the operator.

For a positive step size γ, equation (3.3) can be written as

$$g(u) = g(u) - \gamma R(u). \tag{3.4}$$

This fixed-point formulation allows to suggest the following iterative method for solving the exponentially general variational inequalities (2.1).

Algorithm 16. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes*

$$g(u_{n+1}) = g(u_n) - \gamma_n R(u_n), \quad n = 0, 1, 2 \ldots$$

Note that for $\gamma_n = 1$, Algorithm 16 coincides with Algorithm 12.

It is well known that the convergence analysis of Algorithm 12 requires that both the operators T and g must be strongly monotone and Lipschitz continuous. These strict conditions rule out many important applications of Algorithm 15. To overcome these drawbacks, one uses the technique of updating the solution. Using this technique, we can rewrite the equation (3.3) in the form

$$g(u) = J_A[g(u) - \rho e^{Tg^{-1}J_A[g(u) - \rho Tu]}], \tag{3.5}$$

if g^{-1} exists.

We use this fixed-point formulation to suggest the following extraresolvent-type method for solving exponentially general variational inclusion (2.1).

Algorithm 17. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme.*

Predictor step.

$$g(v_n) = J_A[g(u_n) - \rho_n e^{Tu_n}],$$

where ρ_n *satisfies*

$$\rho_n \langle e^{Tu_n} - e^{Tg^{-1}J_A[g(u_n)-\rho e^{Tu_n}]}, R(u_n)\rangle \leq \sigma\|R(u_n)\|^2, \quad \sigma \in (0,1).$$

Corrector step.

$$g(u_{n+1}) = J_A[g(u_n) - \alpha_n e^{Tv_n}], \quad n = 0, 1, 2 \ldots,$$

where

$$\begin{aligned} \alpha_n &= \frac{(1-\sigma)\|R(u_n)\|^2}{\|e^{Tv_n}\|^2} \\ e^{Tv_n} &= e^{Tg^{-1}J_A[g(u_n)-\rho_n Tu_n]}. \end{aligned}$$

For $g \equiv I$, the identity operator, Algorithm 17 reduces to:

Algorithm 18. *For a given* $u_0 \in H$, *compute* u_{n+1} *by the iterative schemes:*

Predictor step.

$$v_n = J_A[u_n - \rho_n e^{Tu_n}],$$

where ρ_n *satisfies the relation*

$$\rho_n \langle e^{Tu_n} - e^{Tv_n}, R(u_n)\rangle \leq \sigma\|R(u_n)\|^2, \quad \sigma \in (0,1).$$

Corrector step.

$$u_{n+1} = J_A[u_n - \alpha_n e^{Tv_n}],$$

where

$$\begin{aligned} \alpha_n &= \frac{(1-\sigma)\|R(u_n)\|^2}{\|e^{Tv_n}\|^2} \\ e^{Tv_n} &= e^{TJ_A[u_n-\rho_n e^{Tu_n}]}. \end{aligned}$$

Algorithm 18 is an improved version of the extragradient-type method.

Now consider

$$g(w) = (1-\eta)g(u) + \eta J_A[g(u) - \rho e^{Tu}] = g(u) - \eta R(u) \in H, \tag{3.6}$$

from which we have

$$g(u) = J_A[g(u) - \rho e^{Tg^{-1}(g(u)-\eta R(u))}].$$

This fixed-point formulation is used to suggest and analyze the following modified extraresolvent method for exponentially general variational inclusion (2.1).

Algorithm 19. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes*

Predictor step.

$$g(w_n) = g(u_n) - \eta_n R(u_n),$$

where $\eta_n = a^{m_k}$, and m_k is the smallest nonnegative integer m such that

$$\rho_n \eta_n \langle e^{Tu_n} - e^{Tg^{-1}(g(u_n)-a^{m_k}R(u_n))}, R(u_n)\rangle \leq \sigma \|R(u_n)\|^2, \quad \sigma \in (0,1).$$

Corrector step.

$$g(u_{n+1}) = J_A[g(u_n) - \alpha_n e^{Tg^{-1}(g(u_n)-\eta_n R(u_n))}], \quad n = 0,1,2\ldots,$$

where

$$\alpha_n = \frac{(\eta_n - \sigma)\|R(u_n)\|^2}{\|e^{Tg^{-1}(g(u_n)-\eta_n R(u_n))}\|^2}.$$

For $g \equiv I$, where I is the identity operator, we obtain a variant form of the modified extraresolvent-type methods for solving exponentially variational inequalities

For $\eta_n = 1$, Algorithm 19 is exactly Algorithm 17.

For a positive constant α, one can rewrite equation (3.1) as

$$g(u) = J_A[(1-\alpha)g(u) + \alpha g(u) - \rho e^{Tu}], \tag{3.7}$$

which is the fixed-point problem. This equivalent fixed-point formulation allows us to suggest the following iterative method.

Algorithm 20. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$g(u_{n+1}) = J_A[(1-\alpha)g(u_n) + \alpha g(u_{n-1}) - \rho e^{Tu_n}], \quad n = 0, 1, 2, \ldots$$

which can be rewritten in the equivalent form as:

Algorithm 21. *For given $u_0, u_1 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned} g(y_n) &= (1-\alpha)g(u_n) + \alpha g(u_{n-1}) \\ g(u_{n+1}) &= J_A[g(y_n) - \rho e^{Tu_n}], \quad n = 0, 1, 2, \ldots \end{aligned}$$

which is known as the inertial method for solving the exponentially general variational inclusions (2.1) and appears to a new one.

For $g = I$, the identity operator, Algorithm 21 reduces to the inertial method for solving exponentially general variational inclusions. Note that $\alpha_n = 0$, Algorithm 20 is equivalent to Algorithm 12. The process described above is reminiscent to a technique by which two-step methods can be derived as one step method. Compare this method for the heavy-ball method of Polyak [61]. Using the above technique, one can suggest a number of new and improved methods for the exponentially general variational inclusions (2.1) and related problems.

Using this technique, we can suggest the following inertial type methods for solving exponentially general variational inequalities (2.1).

Algorithm 22. *For given $u_0, u_1 \in H$, compute u_{n+1} by the recurrence relation*

$$\begin{aligned} w_n &= u_n - \Theta_n (u_n - u_{n-1}) \\ g(u_{n+1}) &= J_A \left[g(w_n) - \rho e^{Tw_n} \right], \quad n = 1, 2, \ldots, \end{aligned}$$

where $\Theta_n \in [0,1], \; for \; all \; n \geq 1$.

Algorithm 22 is known as modified inertial method for solving inequality (2.1).

Algorithm 23. *For given $u_0, u_1 \in H$, compute u_{n+1} by the recurrence relation*

$$\begin{aligned} w_n &= u_n - \Theta_n (u_n - u_{n-1}) \\ g(y_n) &= J_A \left[g(w_n) - \rho e^{Tw_n} \right], \\ g(u_{n+1}) &= J_A \left[g(y_n) - \rho e^{Ty_n} \right], \quad n = 1, 2, \ldots, \end{aligned}$$

where $\Theta_n \in [0,1], \forall n \geq 1$.

Algorithm 23 is a three-step modified inertial method for solving the exponentially general variational inclusion(2.1).

We now suggest a four-step inertial method for solving the exponentially general variational inclusion (2.1).

Algorithm 24. *For given $u_0, u_1 \in H$, compute μ_{n+1} by the recurrence relation*

$$\begin{aligned} w_n &= u_n - \Theta_n (u_n - u_{n-1}), \\ x_n &= (1-\gamma_n)u_n + \gamma_n \{ w_n - g(w_n) + J_A \left[g(w_n) - \rho e^{Tw_n} \right] \}, \\ y_n &= (1-\beta_n)u_n + \beta_n \{ x_n - g(x_n) + J_A \left[g(x_n) - \rho e^{Tx_n} \right] \}, \\ u_{n+1} &= (1-\alpha_n)u_n + \alpha_n \{ y_n - g(y_n) + J_A \left[g(y_n) - \rho e^{Ty_n} \right] \}, \quad n = 1, 2, \ldots, \end{aligned}$$

where $\alpha_n, \beta_n, \gamma_n, \Theta_n \in [0,1], \quad \forall n \geq 1$.

If $g = I$, the identity, then Algorithm (24) reduces to:

Algorithm 25. *For given $\mu_0, \mu_1 \in H$, compute μ_{n+1} by the recurrence relation*

$$\begin{aligned} w_n &= u_n - \Theta_n (u_n - u_{n-1}), \\ x_n &= (1-\gamma_n)u_n + \gamma_n J_A \left[w_n - \rho e^{Tw_n} \right], \\ u_{n+1} &= (1-\alpha_n)u_n + \alpha_n J_A \left[y_n - \rho e^{Ty_n} \right], \quad n = 1, 2, \ldots, \end{aligned}$$

where $\alpha_n, \beta_n, \gamma_n, \Theta_n \in [0,1], \quad \forall n \geq 1$.

Lemma 3.1 implies that the variational inclusions(2.1) are equivalent to the fixed point problems. These alternate equivalent formulations are very useful from the numerical point of view. These fixed point formulations can be used to suggest and analyze the iterative algorithm for solving exponentially general variational inclusion (2.1).

4 Resolvent Equations Technique

In this section, we introduce the exponentially general resolvent equations. To be more precise, let $R_A \equiv I - J_A$, where I is the identity operator and $J_A = (I+\rho A)^{-1}$ is the resolvent operator. Let $g : H \longrightarrow H$ be an invertible operator. For given nonlinear operators $T, g : H \longrightarrow H$, consider the problem of finding $z \in H$ such that

$$e^{Tg^{-1}J_A z} + \rho^{-1} R_A z = 0, \tag{4.1}$$

where $\rho > 0$ is a constant. Equations of type (4.1) are called the exponentially general resolvent equations.

If $g = I$, then general resolvent equations(4.1) collapse to finding $z \in H$ such that

$$e^{TJ_A z} + \rho^{-1} R_A z = 0, \tag{4.2}$$

which are known as the exponentially resolvent equations.

If $A(.) \equiv \partial\varphi(.)$, where $\partial\varphi$ is the subdifferential of a proper, convex and lower semicontinuous function $\varphi : H \longrightarrow R \cup \{+\infty\}$, then exponentially general resolvent equations are equivalent to finding $z \in H$ such that

$$e^{Tg^{-1}J_\varphi z} + \rho^{-1} R_\varphi z = 0, \tag{4.3}$$

which are also called the exponentially general resolvent equations. Using these resolvent equations, one can suggest and analyze a number of iterative methods for solving exponentially general mixed variational inequalities. If $g \equiv I$, the identity operator, then the problem (4.3) reduces to finding $z \in H$ such that

$$e^{TJ_\varphi z} + \rho^{-1} R_\varphi z = 0, \tag{4.4}$$

which are called the resolvent equations. For the applications, formulation and numerical methods of the resolvent equations, see [16–19, 22–27].

We remark that, if φ is the indicator function of a closed convex set K in H, then $J_\varphi \equiv P_K$, the projection of H onto K. Consequently problem (4.4) is equivalent to finding $z \in H$ such that

$$e^{Tg^{-1}P_K z} + \rho^{-1} Q_K z = 0, \tag{4.5}$$

Equations of the type (4.5) are known as the exponentially general Wiener-Hopf equations. For $g = I$ and $e^{Tu} = \Phi(u)$, we obtain the original Wiener-Hopf (normal) equations introduced and studied by Shi [65] and Robinson [62] in connection with the classical variational inequalities. We would like to mention that the Wiener-Hopf equations technique is being used to develop some implementable and efficient iterative algorithms for solving variational inequalities and related fields.

We now prove that the exponentially general variational inclusion (2.1) is equivalent to the resolvent equations (4.1) by invoking Lemma 3.1 and this is the prime motivation of our next result.

Theorem 4.1. *The exponentially general variational inclusion (2.1) has a solution $u \in H$, if and only if, the exponentially resolvent equation (4.1) has a solution $z \in H$, where*

$$g(u) = J_A z \tag{4.6}$$

$$z = g(u) - \rho e^{Tu}, \tag{4.7}$$

where J_A is the resolvent operator and $\rho > 0$ is a constant.

Proof. Let $u \in H$ be a solution of (2.1). Then, by Lemma 3.1, we have

$$g(u) = J_A[g(u) - \rho e^{Tu}]. \tag{4.8}$$

Let $z = g(u) - \rho e^{Tu}$ in (4.8). Then

$$g(u) = J_A z$$

and

$$z = J_A z - \rho e^{Tg^{-1}J_A z},$$

which implies that

$$e^{Tg^{-1}J_A z} + \rho^{-1} R_A z = 0,$$

the required result. □

From Theorem 4.1, we conclude that the exponentially general variational inclusion (2.1) and the exponentially resolvent equations (4.1) are equivalent. This alternative formulation plays an important and crucial part in suggesting and analyzing various iterative methods for solving exponentially variational inclusions and related optimization problems. In this paper, by suitable and appropriate rearrangement, we suggest a number of new iterative methods for solving the exponentially general variational inclusions (2.1).

(I). The equations (4.1) can be written as

$$R_A z = -\rho e^{Tg^{-1}J_A z},$$

which implies that, using (4.6),

$$z = J_A z - \rho e^{Tg^{-1}J_A z} = g(u) - \rho e^{Tu}.$$

This fixed point formulation enables us to suggest the following iterative method for solving the exponentially general variational inclusion (2.1).

Algorithm 26. *For a given $z_0 \in H$, compute u_{n+1} by the iterative schemes*

$$g(u_n) = J_A z_n \tag{4.9}$$

$$z_{n+1} = g(u_n) - \rho e^{Tu_n}, \quad n = 0, 1, 2, \ldots. \tag{4.10}$$

(II). The equations (4.1) may be written as

$$\begin{aligned} z &= J_A z - \rho e^{Tg^{-1}J_A z} + (1-\rho^{-1})R_A z \\ &= u - \rho e^{Tu} + (1-\rho^{-1})R_A z, \quad \text{using (4.6)}. \end{aligned}$$

Using this fixed point formulation, we suggest the following iterative method.

Algorithm 27. *For a given $z_0 \in H$, compute u_{n+1} by the iterative schemes*

$$\begin{aligned} g(u_n) &= J_A z_n \\ z_{n+1} &= u_n - \rho e^{Tu_n} + (1-\rho^{-1})R_A z_n, \quad n = 0, 1, 2, \ldots \end{aligned}$$

(III). If the operator T is linear and T^{-1} exists, then the resolvent equation(4.1) can be written as

$$z = (I - \rho^{-1} g e^{T^{-1})R_A z},$$

which allows us to suggest the iterative method.

Algorithm 28. *For a given $z_0 \in H$, compute z_{n+1} by the iterative scheme*

$$z_{n+1} = (I - \rho^{-1} g e^{T^{-1})R_A z_n}, \quad n = 0, 1, 2 \ldots$$

We remark that if $g \equiv I$, the identity operator, then Algorithms 26, 27 and 28 reduce to the following algorithms for solving variational inclusions(2.3).

Algorithm 29. *For a given $z_0 \in H$, compute z_{n+1} by the iterative schemes*

$$\begin{aligned} u_n &= J_A z_n \\ z_{n+1} &= u_n - \rho e^{Tu_n}, \quad n = 0, 1, 2, \ldots \end{aligned}$$

Algorithm 30. *For a given $z_0 \in H$, compute z_{n+1} by the iterative schemes*

$$\begin{aligned} u_n &= J_A z_n \\ z_{n+1} &= u_n - \rho e^{Tu_n} + (1-\rho^{-1})R_A z_n, \quad n = 0, 1, 2 \ldots\ldots \end{aligned}$$

Algorithm 31. *For a given $z_0 \in H$, compute z_{n+1} by the iterative scheme*

$$z_{n+1} = (I - \rho^{-1} e^{T^{-1})R_A z_n}, \quad n = 0, 1, 2, \ldots \ldots$$

We note that Algorithm 30 and Algorithm 31 are new even for the variational inclusions (2.3).

We now study the convergence analysis of Algorithm 26. One can study the convergence analysis of Algorithms 27-31 in a similar way.

Theorem 4.2. *Let $T, g : H \longrightarrow H$ be both exponentially strongly monotone with constants $\alpha > 0, \sigma > 0$ and exponentially Lipschitz continuous with constants $\beta > 0, \delta > 0$ respectively. If*

$$|\rho - \frac{\alpha}{\beta^2}| < \frac{\sqrt{\alpha^2 - \beta^2 k(2-k)}}{\beta^2} \tag{4.11}$$

$$\alpha > \beta\sqrt{k(2-k)}, \tag{4.12}$$

where

$$k = 2\sqrt{1 - 2\sigma + \delta^2}, \tag{4.13}$$

then there exists $z \in H$ satisfying the exponentially resolvent equation(4.1) and the sequence $\{z_n\}$ generated by Algorithm 26 converges to z in H strongly.

Proof. Let $z \in H$ be a solution of the exponentially resolvent equation(4.1). Then from (4.7) and (4.10), we have

$$\begin{aligned} ||z_{n+1} - z|| &= ||g(u_n) - g(u) - \rho(e^{Tu_n} - e^{Tu})|| \\ &\leq ||u_n - u - (g(u_n) - g(u))|| \\ &\quad + ||u_n - u - \rho(e^{Tu_n} - e^{Tu})||. \end{aligned} \tag{4.14}$$

Since T is exponentially strongly monotone with constant $\alpha > 0$ and exponentially Lipschitz continuous with constant $\beta > 0$, so

$$\begin{aligned} \|u_n - u - \rho(e^{Tu_n} - e^{Tu})\|^2 &= \|u_n - u\|^2 - 2\rho < e^{Tu_n} - e^{Tu}, u_n - u > \\ &\quad + \rho^2 \|e^{Tu_n} - e^{Tu}\|\|^2 \\ &\leq (1 - 2\rho\alpha + \rho^2\beta^2)\|u_n - u\|^2, \end{aligned} \tag{4.15}$$

and similarly

$$\|u_n - u - (g(u_n) - g(u))\|^2 \leq (1 - 2\sigma + \delta^2)\|u_n - u\|^2, \tag{4.16}$$

where $\sigma > 0$ and $\delta > 0$ are the exponentially strongly exponentially monotonicity and exponentially Lipschitz continuity constants of the operator g respectively.

Combining (4.14), (4.15) and (4.16), we have

$$\|z_{n+1} - z\| \leq \{\frac{k}{2} + \sqrt{1 - 2\rho\alpha + \rho^2\beta^2}\}||u_n - u||. \tag{4.17}$$

From (4.15) and (4.17), we obtain

$$\begin{aligned} \|u_n - u\| &\leq \|u_n - u - (g(u_n) - g(u))\| + \|J_A z_n - J_A z\| \\ &\leq \frac{k}{2}\|u_n - u\| + \|z_n - z\|, \end{aligned}$$

which implies that

$$\|u_n - u\| \leq (\frac{1}{1 - \frac{k}{2}})\|z_n - z\|. \tag{4.18}$$

Combining (4.18) and (4.17), we have

$$\|z_{n+1} - z\| \leq \frac{\frac{k}{2} + t(\rho)}{1 - \frac{k}{2}}\|z_n - z\| = \theta\|z_n - z\|, \tag{4.19}$$

where

$$\begin{aligned} t(\rho) &= \sqrt{1 - 2\rho\alpha + \rho^2\beta^2} \\ \theta &= (\frac{k}{2} + t(\rho))/(1 - \frac{k}{2}). \end{aligned}$$

From (4.11) and (4.19), it follows that $\theta < 1$. and consequently it follows that the sequence $\{z_n\}$ generated by Algorithm 26 converges to z strongly in H, the required result. □

From Theorem 4.2, it is clear that the convergence of Algorithms 27–31 requires the operators T, g to be exponentially strongly monotone and exponentially Lipschitz continuous. These strict conditions limit the applications of these algorithms to many important problems arising in pure and applied sciences. These facts motivated Noor [18, 19, 22] to use the exponentially resolvent equations technique to suggest other iterative methods for solving mixed variational inequalities. As special cases, we obtain the modified forward-backward splitting method of Tseng [68], which he suggested using the updating method of the solution in the spirit of the extragradient method.

We define the residue vector $R(u)$ by the relation

$$R(u) = g(u) - J_A[g(u) - \rho e^{Tu}]. \tag{4.20}$$

From Lemma 3.1, it follows that $u \in H$ is a solution of exponentially general variational inclusion (2.1), if and only if, $u \in H$ is a zero of the equation

$$R(u) = 0. \tag{4.21}$$

Using the fact that $R_A = I - J_A$, the exponentially resolvent equations (4.1) can be written as

$$z - J_A z + \rho e^{Tg^{-1}J_A z} = 0.$$

Thus, for a positive stepsize γ, we can write the above equation as

$$\begin{aligned} g(u) &= g(u) - \gamma\{z - J_A z + \rho e^{Tg^{-1}J_A z}\} \\ &= g(u) - \gamma D(u), \end{aligned}$$

where

$$\begin{aligned} D(u) &= z - J_A z + \rho e^{Tg^{-1}J_A z} \\ &= R(u) - \rho e^{Tu} + \rho e^{Tg^{-1}J_A[g(u)-\rho e^{Tu}]}, \quad \text{using (3.16)}. \end{aligned} \tag{4.22}$$

This fixed point formulation enables us to suggest the following iterative methods for solving the exponentially variational inclusion (2.1).

Algorithm 32. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes*

$$\begin{aligned} z_n &= g(u_n) - \rho e^{Tu_n} \\ D(u_n) &= z_n - \rho J_A z_n + \rho e^{Tg^{-1}J_A z_n} \\ g(u_{n+1} &= g(u_n) - \gamma D(u_n), \quad n = 0, 1, 2 \ldots \end{aligned}$$

Algorithm 33. *For a given $u_0 \in H$, compute u_{n+1} by the iterative schemes*

$$\begin{aligned} z_n &= g(u_n) - \rho e^{Tu_n} \\ D(u_n) &= z_n - J_A z_n + \rho e^{Tg^{-1}J_A z_n} \\ g(u_{n+1}) &= J_A[g(u_n) - \gamma D(u_n)], \quad n = 0, 1, 2, \ldots \end{aligned}$$

If $A(.) \equiv \partial\varphi(.)$, where $\partial\varphi$ is the subdifferential of a proper, convex and lower semicontinuous function $\varphi : H \longrightarrow R \cup \{+\infty\}$, then $J_A \equiv J_\varphi = (I + \partial\varphi)^{-1}$, the exponentially resolvent operator and consequently Algorithm 33 collapses to:

Algorithm 34. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes*

$$\begin{aligned} z_n &= g(u_n) - \rho e^{Tu_n} \\ D(u_n) &= z_n - J_\varphi z_n + \rho e^{Tg^{-1}J_\varphi z_n} \\ g(u_{n+1}) &= g(u_n) - \gamma D(u_n), \quad n = 0, 1, 2 \ldots \end{aligned}$$

For $g \equiv I$, the identity operator, Algorithm 34 collapses to the following new iterative method for solving the exponentially variational inclusion(2.3).

Algorithm 35. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes*

$$\begin{aligned} z_n &= u_n - \rho e^{Tu_n} \\ D(u_n) &= z_n - J_A z_n + \rho e^{TJ_A z_n} \\ u_{n+1} &= J_A[u_n - \rho e^{Tu_n}], \quad n = 0, 1, 2, \ldots . \end{aligned}$$

Note that for $\gamma = 1$, Algorithm 35 reduces to:

Algorithm 36. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$u_{n+1} = J_A[J_A[u_n - \rho e^{Tu_n}] - \rho e^{TJ_A[u_n - \rho Tu_n]} + \rho e^{Tu_n}], \quad n = 0, 1, 2 \ldots.$$

Algorithm 36 would coincide with the modified forward-backward splitting method of Tseng [68] for $J_A = P_K$, the projection operator from H onto K. Note that Tseng [68] used the technique of updating the solution to suggest his method, whereas we have used the resolvent equations technique to suggest. In brief, for a suitable and proper choice of the operators T, g, J_A, and P_K, one can obtain a number of new and old methods for solving the variational inclusions and related problems.

We now study the convergence analysis of Algorithm 34. In a similar way, one can study the convergence criteria of Algorithms 32, 33, 35 and 36. For this purpose, we need the following results.

Lemma 4.1. *Let $\bar{u} \in H$ be a solution of (2.6) and $T : H \longrightarrow H$ be exponentially general quasimonotone and exponentially g-Lipschitz continuous with constant $\delta > 0$. Then*

$$\langle g(u) - g(\bar{u}), D(u) \rangle \geq \{1 - \rho\delta\} ||R(u)||^2, \quad \forall u \in H. \tag{4.23}$$

Proof. Since T is exponentially general quasi-monotone, for all $v, \bar{u} \in H$,

$$\langle e^{T\bar{u}}, g(v) - g(\bar{u}) \rangle + \varphi(g(v)) - \varphi(g(\bar{u})) > 0.$$

implies that

$$\langle e^{Tv}, g(v) - g(\bar{u}) \rangle + \varphi(g(v)) - \varphi(g(\bar{u})) \geq 0. \tag{4.24}$$

Taking $g(v) = J_\varphi[g(u) - \rho Tu]$ in (4.24), we obtain

$$\langle e^{Tg^{-1}J_\varphi[g(u)-\rho Tu]}, J_\varphi[g(u) - \rho e^{Tu}] - g(\bar{u}) \rangle + \varphi(J_\varphi[g(u) - \rho e^{Tu}]) - \varphi(g(\bar{u})) \geq 0. \tag{4.25}$$

Letting $z = g(u) - \rho e^{Tu}, u = J_\varphi[g(u) - \rho e^{Tu}],$ and $v = g(\bar{u})$ in (2.6), we have

$$\langle g(u) - \rho e^{Tu} - J_\varphi[g(u) - \rho e^{Tu}], J_\varphi[g(u) - \rho e^{Tu}] - g(\bar{u})\rangle + \varphi(g(\bar{u})) \\ -\varphi(J_\varphi[g(u) - \rho e^{Tu}]) \geq 0,$$

which implies that

$$\langle R(u) - \rho e^{Tu}, J_\varphi[g(u) - \rho e^{Tu}] - g(\bar{u})\rangle + \varphi(g(\bar{u})) - \varphi(J_\varphi[g(u) - \rho e^{Tu}]) \geq 0 \quad (4.26)$$

Adding (4.25), (4.26) and using (4.17), we have

$$\langle R(u) - \rho e^{Tu} + \rho e^{Tg^{-1}J_\varphi[g(u)-\rho Tu]}, g(u) - g(\bar{u}) - R(u)\rangle \geq 0. \quad (4.27)$$

From the above equations, we

$$\langle D(u), g(u) - g(\bar{u})\rangle \geq \langle R(u), D(u)\rangle. \quad (4.28)$$

Now using the exponentially g-Lipschitz continuity of T with constant $\delta > 0$, we obtain

$$\begin{aligned}\langle R(u), D(u)\rangle &= \langle R(u), R(u) - \rho e^{Tu} + \rho e^{Tg^{-1}J_\varphi[g(u)-\rho Tu]}\rangle \\ &= \|R(u)\|^2 - \rho\langle R(u), e^{Tu} - e^{Tg^{-1}J_\varphi[g(u)-\rho Tu]}\rangle \\ &\geq \{1 - \rho\delta\}\|R(u)\|^2. \quad (4.29)\end{aligned}$$

Combining (4.28) and (4.29), we have

$$\langle g(u) - g(\bar{u}), D(u)\rangle \geq \{1 - \rho\delta\}\|R(u)\|^2,$$

the required result. □

Lemma 4.2. *The sequence $\{u_n\}$ generated by Algorithm 35 for exponentially variational inclusion (2.6) satisfies the inequality*

$$\|g(u_{n+1}) - g(\bar{u})\|^2 \leq \|g(u_n) - g(\bar{u})\|^2 - \gamma(1 - \rho\delta)(2 - \gamma(1 - \rho\delta))\|R(u_n)\|^2, \\ \forall \bar{u} \in H. \quad (4.30)$$

Proof. From (4.20) and Algorithm 35, we have

$$\begin{aligned} \|g(u_{n+1}) - g(\bar{u})\|^2 &= \|g(u_n) - g(\bar{u}) - \gamma D(u_n)\|^2 \\ &\leq ||g(u_n) - g(\bar{u})||^2 - 2\gamma < g(u_n) - g(\bar{u}), D(u_n) > +\gamma^2\|D(u_n)\|^2 \\ &\leq \|g(u_n) - g(\bar{u})\|^2 - \gamma(1-\rho\delta)(2-\gamma(1-\rho\delta))\|R(u_n)\|^2. \end{aligned}$$

□

Theorem 4.3. *Let g be invertible. Then the approximate solution u_{n+1} obtained from Algorithm 35 converges to a solution $\bar{u}$ of the exponentially variational inclusion (2.6).*

Proof. Let $u^* \in H$ be a solution of (2.6). From (4.26), it follows that the sequence $\{u_n\}$ is bounded and

$$\sum_{n=0}^{\infty} \gamma(1-\rho\delta)(2-\gamma(1-\rho\delta))\|R(u_n)\|^2 \leq \|g(u_0) - g(\bar{u})\|^2,$$

and consequently

$$\lim_{n\to\infty} R(u_n) = 0.$$

Let $\bar{u}$ be the cluster point of $\{u_n\}$ and the subsequence $\{u_{n_j}\}$ of the sequence $\{u_n\}$ converge to $\bar{u}$. Since R is continuous, so

$$R(\bar{u}) = \lim_{j\to\infty} R(u_{n_j}) = 0,$$

and $\bar{u}$ is the solution of the exponentially variational inclusion (2.6) by invoking Lemma 4.1 and

$$\|g(u_{n+1}) - g(\bar{u}\|^2 \leq \|g(u_n) - g(\bar{u})\|^2.$$

Thus it follows from the above inequality that the sequence $\{u_n\}$ hasexactly one cluster point and

$$\lim_{n\to\infty} g(u_n) = g(\bar{u}).$$

Since g is invertible, so

$$\lim_{n\to\infty} (u_n) = \bar{u},$$

which is the solution of the exponentially variational inclusion (2.4).

□

5 Dynamical Systems Technique

Using the fixed-point formulation of the variational inequalities, Dupuis and Nagurney [12] introduced and considered the projected dynamical systems, where the right hand side of the ordinary differential equation is a projected operator associated with variational inequalities. The innovative and novel feature of a projected dynamical system is that its set of stationary points corresponds to the set of solutions of the corresponding variational inequality problem. Hence, equilibrium and nonlinear problems arising in various branches in pure and applied sciences, which can be formulated in the setting of the variational inequalities, can now be studied in the more general setting of dynamical systems. It has been shown [12, 22, 34, 52, 54] that the dynamical systems are useful in developing some efficient numerical techniques for solving variational inequalities and related optimization problems. In recent years, much attention has been given to study the globally asymptotic stability of these projected dynamical systems. We use this equivalent fixed point formulation to suggest and analyze the resolvent dynamical system associated with the exponentially general variational inclusions (2.1).

$$\frac{du}{dt} = \lambda\{J_A[g(u) - \rho e^{Tu}] - g(u)\}, \quad u(t_0) = u_0 \in H, \tag{5.1}$$

where λ is a parameter. The system of type (5.1) is called the resolvent exponentially general dynamical system. Here the right hand side is related to the resolvent operator and is discontinuous on the boundary. It is clear from the definition that the solution to (5.1) always stays in the constraint set. This implies that the qualitative results such as the existence, uniqueness and continuous dependence of the solution on the given data can be studied.

Using the fixed-point formulation (3.1), we can suggest the following resolvent dynamical system

$$\frac{du}{dt} = \lambda\{J_A[g(u) - \alpha\{\eta R(u) - \rho e^{Tg^{-1}(g(u)-\eta R(u))}\} - g(u)\}, u(t_0) = u_0 \in H, \tag{5.2}$$

where α and η are positive constants. For $\alpha = 1$ and $\eta = 1$, we can obtain several new resolvent-type dynamical systems associated with exponentially variational inclusions, which are quite different from the previously known ones.

From Lemma 4.1, it follows that the exponentially general variational inclusions are equivalent to the exponentially resolvent equations (4.1). This equivalence is used to suggest the following dynamical system associated with the exponentially general variational inclusions (2.1) as:

$$\frac{du}{dt} = \lambda\{J_A[g(u)-\rho e^{Tu}] - \rho e^{Tg^{-1}J_A[g(u)-\rho e^{Tu}]} + \rho e^{Tu} - g(u)\}, u(t_0) = u_0 \in H, \quad (5.3)$$

which is called the exponentially general resolvent dynamical system.

The equilibrium points of the exponentially dynamical system (5.1) are naturally defined as follows.

Definition 5.1. *An element $u \in H$, $g(u) \in H$ is an equilibrium point of the exponentially dynamical system (5.1), if $\frac{du}{dt} = 0$, that is,*

$$J_A[g(u)-\rho e^{Tu}] - g(u) = 0.$$

Thus it is clear that $u \in H$, $g(u) \in H$ is a solution of the exponentially general variational inclusion (2.1) if and only if $u \in H$, $g(u) \in H$ is an equilibrium point. In a similar way, one can define the concept of equilibrium points for other exponentially dynamical systems.

Definition 5.2. *The exponentially dynamical system is said to converge to the solution set S^* of (2.1), if, irrespective of the initial point, the trajectory of the dynamical system satisfies*

$$\lim_{t\to\infty} dist(u(t), S^*) = 0, \quad (5.4)$$

where

$$dist(u, S^*) = inf_{v\in S^*}\|u-v\|.$$

It is easy to see, if the set S^* has a unique point u^*, then (5.14) implies that

$$\lim_{t\to\infty} u(t) = u^*.$$

If the dynamical system is still stable at u^* in the Lyapunov sense, then the dynamical system is globally asymptotically stable at u^*.

Definition 5.3. *The exponentially dynamical system is said to be globally exponentially stable with degree η at u^*, if, irrespective of the initial point, the trajectory of the system satisfies*

$$\|u(t) - u^*\| \leq \mu_1 \|u(t_0) - u^*\| exp(-\eta(t - t_0)), \quad \forall t \geq t_0,$$

where μ_1 and η are positive constants independent of the initial point.

It is clear that the globally exponentially stability is necessarily globally asymptotically stable and the exponentially dynamical system converges arbitrarily fast.

We now show that the trajectory of the solution of the general dynamical system (5.1) converges to the unique solution of the exponentially general variational inequality (2.1). The analysis is in the spirit of Noor [34]. In a similar way, one can consider the other exponentially dynamical systems.

Theorem 5.1. *Let the operators $T, g : H \longrightarrow H$ be both Lipschitz continuous with constants $\beta > 0$ and $\mu > 0$ respectively. Then, for each $u_0 \in H$, there exists a unique continuous solution $u(t)$ of the exponentially dynamical system (5.1) with $u(_0) = u_0$ over $[t_0, \infty)$.*

Proof. Let

$$G(u) = \lambda\{J_A[g(u) - \rho e^{Tu}] - g(u)\},$$

where $\lambda > 0$ is a constant. For all $u, v \in H$, we have

$$\begin{aligned} \|G(u) - G(v)\| &\leq \lambda\{\|J_A[g(u) - \rho e^{Tu}] - J_A[g(v) - \rho e^{Tv}]\| + \|g(u) - g(v)\|\} \\ &\leq 2\lambda\|g(u) - g(v)\| + \lambda\rho\|e^{Tu} - e^{Tv}\| \\ &\leq \lambda\{2\mu + \beta\rho\}\|u - v\|. \end{aligned}$$

This implies that the operator $G(u)$ is a Lipschitz continuous in H, and for each $u_0 \in H$, there exists a unique and continuous solution $u(t)$ of the dynamical system (5.1), defined on an interval $t_0 \leq t < T_1$ with the initial condition $u(t_0) = u_0$. Let $[t_0, T_1)$ be its maximal interval of existence. Then we have to show that $T_1 = \infty$. Consider, for any $u \in H$,

$$\begin{aligned} \|G(u)\| &= \lambda \|J_A[g(u) - \rho e^{Tu}] - g(u)\| \\ &\leq \lambda\{\|J_A[g(u) - \rho e^{Tu}] - J_A[g(0)]\| + \|J_A[g(0)] - g(u)\|\} \\ &\leq \lambda\{\rho\|e^{Tu}\| + \|J_A[g(u)] - J_A[0]\| + \|J_A[0] - g(u)\|\} \\ &\leq \lambda\{(\rho\beta + 2\mu)\|u\| + \|J_A[0]\|\}. \end{aligned}$$

Then

$$\begin{aligned} \|u(t)\| &\leq \|u_0\| + \int_{t_0}^{t} \|Tu(s)\| ds \\ &\leq (\|u_0\| + k_1(t - t_0)) + k_2 \int_{t_0}^{t} \|u(s)\| ds, \end{aligned}$$

where $k_1 = \lambda\|J_A[0]\|$ and $k_2 = \lambda(\rho\beta + 2\mu)$. Hence by the Gronwall lemma [41], we have

$$\|u(t)\| \leq \{\|u_0\| + k_1(t - t_0)\}e^{k_2(t-t_0)}, \quad t \in [t_0, T_1).$$

This shows that the solution is bounded on $[t_0, T_1)$. So $T_1 = \infty$. □

Theorem 5.2. *Let the operators $T, g : H \longrightarrow H$ be both exponentially Lipschitz continuous with constants $\beta > 0$ and $\mu > 0$ respectively. If the operator $g : H \longrightarrow H$ is strongly monotone with constant $\gamma > 0$ and $\lambda < 0$, then the dynamical system (5.1) converges globally exponentially to the unique solution of the exponentially general variational inclusion(2.1).*

Proof. Since the operators T, g are both Lipschitz continuo, it follows from Theorem 5.1 that the exponentially dynamical system (5.1) has unique solution

$u(t)$ over $[t_0, T_1)$ for any fixed $u_0 \in H$. Let $u(t)$ be a solution of the initial value problem (5.1) For a given $u^* \in H$ satisfying (2.1), consider the Lyapunov function

$$L(u) = \lambda \|g(u(t)) - g(u)^*\|^2, \quad u(t) \in H. \tag{5.5}$$

From (5.1) and (5.5), we have

$$\begin{aligned} \frac{dL(u)}{dt} &= 2\lambda \langle g(u(t)) - g(u)^*, J_A[g(u(t)) - \rho e^{Tu(t)}] - g(u(t))\rangle \\ &= -2\lambda \langle g(u(t)) - g(u)^*, g(u(t)) - g(u^*)\rangle \\ &\quad +2\lambda \langle g(u(t)) - g(u)^*, J_A[g(u(t)) - \rho e^{Tu(t)}] - g(u^*)\rangle \\ &\leq -2\lambda\gamma \|g(u(t)) - g(u)^*\|^2 \\ &\quad +2\lambda \langle g(u(t)) - g(u)^*, J_A[g(u(t)) - \rho e^{Tu(t)}] - g(u^*)\rangle, \end{aligned} \tag{5.6}$$

where $u^* \in H$ is a solution of (2.1). Thus

$$g(u^*) = J_A[g(u^*) - \rho e^{Tu^*}].$$

Using the exponentially Lipschitz continuity of the operators T, g, we have

$$\begin{aligned} \|J_A[g(u) - \rho e^{Tu}] - J_A[g(u^*) - \rho e^{Tu^*}]\| &\leq \|g(u) - g(u^*) - \rho(e^{Tu} - e^{Tu^*})\| \\ &\leq (\mu + \rho\beta)\|u - u^*\|. \end{aligned} \tag{5.7}$$

From (5.6) and (5.7), we have

$$\frac{du}{dt}\|u(t) - u^*\| \leq 2\alpha\lambda \|u(t) - u^*\|,$$

where

$$\alpha = \mu + \rho\beta - \gamma.$$

Thus, for $\lambda = -\lambda_1$, where λ_1 is a positive constant, we have

$$\|u(t) - u^*\| \leq \|u(t_0) - u^*\| e^{-\alpha\lambda_1(t - t_0)},$$

which shows that the trajectory of the solution of the exponentially dynamical system (5.1) converges globally exponentially to the unique solution of the exponentially general variational inclusions (2.1). □

We now suggest a second order exponentially resolvent dynamical system associated with the exponentially general variational inequalities (2.1) as:

$$\frac{d^2u}{dt^2} + \gamma \frac{du}{dt} = J_A[g(u) - \rho e^{Tu}] - g(u), \quad u(t_0) = u_0, \quad u'(t_0) = v_0 \in H. \quad (5.8)$$

For $g \equiv I$, it can be shown that the solution of the second order exponentially resolvent dynamical system (5.8) converges weakly to the solution of the classical variational inequality (2.8). It is an open problem to study the asymptotic and stability analysis of the such type of the exponentially resolvent dynamical system in the context of exponentially variational inclusions.

We now consider exponentially resolvent dynamical system associated with the exponentially variational inclusion. Using the equivalent formulation (3.1), we suggest a new class of exponentially resolvent dynamical systems as

$$\frac{dg(u)}{dt} = \lambda\{J_A[g(u) - \rho e^{Tu}] - g(u)\}, \quad u(t_0) = u_0 \in H, \quad (5.9)$$

where λ is a parameter. The system of type (5.9) is called the exponentially resolvent dynamical system associated with the exponentially general variational inclusion (2.1).

We use the exponentially resolvent dynamical system (5.9) to suggest some iterative for solving the exponentially general variational inclusion (2.1). These methods can be viewed in the sense of Noor [42] involving the double resolvent operator.

For simplicity, we consider the dynamical system

$$\frac{dg(u)}{dt} + g(u) = J_A[g(u) - \rho e^{Tu}], \quad u(t_0) = \alpha. \quad (5.10)$$

We construct the implicit iterative method using the forward difference scheme. Discretizing the equation (5.10), we have

$$\frac{g(u_{n+1}) - g(u_n)}{h} + g(u_{n+1}) = J_A[g(u_n) - \rho e^{Tu_{n+1}}], \quad (5.11)$$

where h is the step size. Now, we can suggest the following implicit iterative method for solving the variational inclusion (2.1).

Algorithm 37. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$g(u_{n+1}) = J_A\left[g(u_n) - \rho e^{Tu_{n+1}} - \frac{g(u_{n+1}) - g(u_n)}{h}\right], \quad n = 0, 1, 2, \ldots.$$

This is an implicit method and is quite different from the known implicit method.

If $A(g(u)) = \partial\phi(g(u)$, then $J_A = J_\phi$ and Algorithm 37 is equivalent to the following iterative method:

Algorithm 38. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$g(u_{n+1}) = J_\phi\left[g(u_n) - \rho e^{Tu_{n+1}} - \frac{g(u_{n+1}) - g(u_n)}{h}\right], \quad n = 0, 1, 2, \ldots.$$

Using Lemma 3.1, Algorithm 38 can be rewritten in the equivalent form as:

Algorithm 39. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$\begin{aligned}&\langle \rho e^{Tu_{n+1}} + \frac{1+h}{h}(g(u_{n+1}) - g(u_n)), g(v) - g(u_{n+1})\rangle \\ &+\phi(g(v) - \phi(g(u_{n+1}) \geq 0, \quad \forall v \in H.\end{aligned} \tag{5.12}$$

We now study the convergence analysis of algorithm 39 under some mild conditions.

Theorem 5.3. *Let $u \in H : g(v) \in H$ be a solution of the exponentially general variational inequality (2.1). Let u_{n+1} be the approximate solution obtained from (5.12). If T is g-monotone, then*

$$\|g(u) - g(u_{n+1})\|^2 \leq \|g(u) - g(u_n)\|^2 - \|g(u_n) - g(u_{n+1})\|^2. \tag{5.13}$$

Proof. Let $u \in H : g(v) \in H$ be a solution of (2.6). Then

$$\langle e^{Tv}, g(v) - g(u)\rangle + \phi(g(v)) - \phi(g(u)) \geq 0, \quad \forall v \in H, \tag{5.14}$$

since T is a exponentially general monotone operator.

Set $v = u_{n+1}$ in (5.14), to have

$$\langle e^{Tu_{n+1}}, g(u_{n+1}) - g(u)\rangle + \phi(g(v)) - \phi(g(u_{n+1})) \geq 0. \tag{5.15}$$

Taking $v = u$ in (5.12), we have

$$\langle \rho e^{Tu_{n+1}} + \{\frac{(1+h)g(u_{n+1}) - (1+h)g(u_n)}{h}\}, g(u) - g(u_{n+1})\rangle \geq 0. \tag{5.16}$$

From (5.15) and (5.16), we have

$$\langle (1+h)(g(u_{n+1}) - g(u_n)), g(u) - g(u_{n+1})\rangle \geq 0. \tag{5.17}$$

From (5.17) and using $\quad 2\langle a, b\rangle = \|a+b\|^2 - \|a\|^2 - \|b\|^2, \quad \forall a, b \in H$, we obtain

$$\|g(u_{n+1}) - g(u)\|^2 \leq \|g(u) - g(u_n)\|^2 - \|g(u_{n+1}) - g(u_n)\|^2, \tag{5.18}$$

the required result. □

Theorem 5.4. *Let $u \in H$ be the solution of the exponentially general variational inequality (2.6). Let u_{n+1} be the approximate solution obtained from (5.12). If T is a exponentially general monotone operator and g^{-1} exists, then u_{n+1} converges to $u \in H$ satisfying (2.6).*

Proof. Let T be a exponentially general monotone operator. Then, from (5.13), it follows the sequence $\{u_i\}_{i=1}^{\infty}$ is a bounded sequence and

$$\sum_{i=1}^{\infty} \|g(u_n) - g(u_{n+1})\|^2 \leq \|g(u) - g(u_0)\|^2,$$

which implies that

$$\lim_{n\to\infty} \|u_{n+1} - u_n\|^2 = 0, \tag{5.19}$$

since g^{-1} exists.

Since sequence $\{u_i\}_{i=1}^{\infty}$ is bounded, so there exists a cluster point $\hat{u}$ to which the subsequence $\{u_{ik}\}_{k=1}^{\infty}$ converges. Taking limit in (5.12) and using (5.19), it follows that $\hat{u} \in K$ satisfies

$$\langle e^{T\hat{u}}, g(v) - g(\hat{u})\rangle + \phi(g(v)) - \phi(g(\hat{g(u)}) \geq 0, \quad \forall v \in H : g(v) \in H,$$

and

$$\|g(u_{n+1}) - g(u)\|^2 \leq \|g(u) - g(u_n)\|^2.$$

Using this inequality, one can show that the cluster point $\hat{u}$ is unique and

$$\lim_{n\to\infty} u_{n+1} = \hat{u}.$$

□

We now suggest another implicit iterative method for solving (2.1).

Discretizing (5.10), we have

$$\frac{g(u_{n+1}) - g(u_n)}{h} + g(u_{n+1}) = J_A[g(u_{n+1}) - \rho e^{Tu_{n+1}}], \tag{5.20}$$

where h is the step size.

This formulation enable us to suggest the following iterative method.

Algorithm 40. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$g(u_{n+1}) = J_A\left[g(u_{n+1}) - \rho e^{Tu_{n+1}} - \frac{g(u_{n+1}) - g(u_n)}{h}\right].$$

Using Lemma 3.1, Algorithm 40 can be rewritten in the equivalent form as:

Algorithm 41. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$\begin{aligned}&\langle \rho e^{Tu_{n+1}} + \{\frac{g(u_{n+1}) - g(u_n)}{h}\}, g(v) - g(u_{n+1})\rangle \\ &+\rho\phi(g(v)) - \rho\phi(g(u_{n+1}) \geq 0, \quad \forall v \in H : g(v) \in H.\end{aligned} \tag{5.21}$$

Again using the exponentially resolvent dynamical systems, we can suggested some iterative methods for solving the exponentially general variational inequalities and related optimization problems.

Algorithm 42. *For a given $u_0 \in K$, compute u_{n+1} by the iterative scheme*

$$g(u_{n+1}) = P_K\left[\frac{(h+1)(g(u_n) - g(u_{n+1}))}{h} - \rho e^{Tu_n}\right], \quad n = 0, 1, 2, \ldots,$$

Discretizing (5.10), we have

$$\frac{g(u_{n+1}) - g(u_n)}{h} + g(u_n) = J_A[g(u_{n+1}) - \rho e^{Tu_{n+1}}], \tag{5.22}$$

where h is the step size.

This formulation enable us to suggest the following iterative method for $h = 1$.

Algorithm 43. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$g(u_{n+1}) = J_A\left[g(u_{n+1}) - \rho e^{Tu_{n+1}}\right]$$

which is well known double resolvent iterative method for solving the exponentially general variational inclusion (2.1).

In a similar way, one can suggest a wide class of implicit iterative methods for solving variational inclusions and related optimization problems. the comparison of these methods with other methods is an interesting problem for future research.

6 Sensitivity Analysis

In recent years variational inequalities are being used as mathematical programming models to study a large number of equilibrium problems arising in finance, economics, transportation, operations research and engineering sciences.

The behaviour of such equilibrium problems as a result of changes in the problem data is always of concern. In this section, we study the sensitivity analysis of the exponentially variational inclusion, that is, examining how solutions of such problems change when the data of the problems are changed. We like to mention that sensitivity analysis is important for several reasons. First, estimating problem data often introduces measurement errors, sensitivity analysis helps in identifying sensitive parameters that should be obtained with relatively high accuracy. Second, sensitivity analysis may help to predict the future changes of the equilibrium as a result of changes in the governing system. Third, sensitivity analysis provides useful information for designing or planning various equilibrium systems. Furthermore, from mathematical and engineering point of view, sensitivity analysis can provide new insight regarding problems being studied can stimulate new ideas and techniques for problem solving. Sensitivity analysis for variational inequalities has been studied by many authors including Noor and Noor [35], Moudafi and Noor [21] using quite different techniques. The techniques suggested so far vary with the problem being studied. The equivalence between the variational inequalities and the fixed-point problem to study the sensitivity analysis of the classical variational inequalities. This technique has been modified and extended by many authors for studying the sensitivity analysis of various other classes of variational inequalities. This approach has strong geometrical flavor. It is well known that the variational inequalities are equivalent to the Wiener-Hopf equations, see Noor [34]. This fixed-point equivalence is obtained by a suitable and appropriate rearrangement of the Wiener-Hopf equations. The Wiener-Hopf equation approach is quite general, flexible unified and provides us with a new technique to study the sensitivity analysis of variational inequalities without assuming the differentiability of the given data. Our analysis is in the spirit of Noor and Noor [35].

We now consider the parametric versions of the problem (2.1). To be more precise, let M be an open subset of H in which the parameter λ takes values. Let $T(u, \lambda)$ be a given operator defined on $H \times M$ and takes values in H. From now onward, we denote $T_\lambda(.) := T(., \lambda)$ unless otherwise specified. The parametric exponentially general variational inclusion problem is to find $(u, \lambda) \in H \times M$ such that

$$0 \in e^{T_\lambda u} + A_\lambda(g(u)). \tag{6.1}$$

We also assume that the parametric exponentially general variational inclusion (6.1) has a unique solution $\bar{u}$ for some $\bar{\lambda} \in M$.

Related to the parametric exponentially general variational inclusion (6.1), we consider the parametric exponentially resolvent equations. We consider the problem of finding $(z, \lambda) \in H \times M$ such that

$$e^{T_\lambda g^{-1} J_{A_\lambda} z} + \rho^{-1} R_{A_\lambda} z = 0, \tag{6.2}$$

where $\rho > 0$ ia constant and $R_{A_\lambda} \equiv I - J_{A_\lambda}$, is defined on the set of (z, λ) with $\lambda \in M$ and takes values in H. The equations of the type (6.2) are called the exponentially parametric resolvent equations.

Using Lemma 4.1, one can easily establish the equivalence between problems (6.1) and (6.2).

Lemma 6.1. *The parametric exponentially general variational inclusion (6.1) has solution $(u, \lambda) \in H \times M$ if and only if the parametric exponentially resolvent equation (6.1) has a solution (z, λ), if*

$$\begin{aligned} g(u) &= P_{K_\lambda} z, & (6.3)\\ z &= g(u) - \rho e^{T_\lambda(u)}. & (6.4) \end{aligned}$$

From Lemma 4.1, we see that the problems (6.1) and (6.2) are equivalent. We use this equivalence to study the sensitivity analysis of the exponentially general variational inclusion (2.1). We assume that for some $\bar{\lambda} \in M$, problem (6.2) has a unique solution $\bar{z}$ and X is a closure of a ball in H centered at $\bar{z}$. We want to investigate those conditions under which for each λ in a neighborhood of $\bar{z}$, problem (6.2) has a unique solution $z(\lambda)$ near $\bar{z}$ and the function $z(\lambda)$ is Lipschitz continuous and differentiable.

First of all, we recall the following well known concepts.

Definition 6.1. *Let $T_\lambda(.)$ be an operator on $X \times M$. Then, $\lambda \in M, \quad \forall u, v \in X$, the operator T_λ is said to be:*

(a) locally strongly exponentially exponentially monotone, if there exists a constant $\alpha > 0$ such that

$$\langle T_\lambda(u) - T_\lambda(v), u - v\rangle \geq \alpha\|u - v\|^2.$$

(b) locally exponentially Lipschitz continuous, if there exists a constant $\beta > 0$ such that

$$\|e^{T_\lambda(u)} - e^{T_\lambda(v)}\| \leq \beta\|u - v\|.$$

From (a) and (b), it follows that $\alpha \leq \beta$.

We now consider the case, when the solutions of the parametric exponentially resolvent equations (6.2) lie in the interior of X. Following the ideas and technique of Noor and Noor [35], we consider the map

$$\begin{aligned} F_\lambda(z) &= J_{A_\lambda} z - \rho e^{T_\lambda(u)}, \quad \forall (z, \lambda) \in X \times M, \\ &= g(u) - \rho e^{T_\lambda(u)}, \end{aligned} \tag{6.5}$$

where

$$g(u) = J_{A_\lambda} z. \tag{6.6}$$

We have to show that the map $F_\lambda(z)$ defined by (6.5) has a fixed point, which is solution of the exponentially resolvent equation (6.2). We have to show that the map $F_\lambda(z)$ defined by (6.5) is a contraction map with respect to z uniformly in $\lambda \in M$.

Lemma 6.2. *Let $T_\lambda(.)$ be a locally strongly exponentially monotone with constant $\alpha > 0$ and locally exponentially Lipschitz continuous with constant $\beta > 0$. If the operator g is strongly monotone with constant $\sigma > 0$ and Lipschitz continuous with constant $\delta > 0$, then*

$$\|F_\lambda(z_1) - F_\lambda(z_2)\| \leq \theta\|z_1 - z_2\|,$$

$$\theta = \frac{k + \sqrt{1 - 2\alpha\rho + \beta^2\rho^2}}{1 - k}, \tag{6.7}$$

for

$$\begin{aligned} |\rho - \frac{\alpha}{\beta^2}| &< \frac{\sqrt{\alpha^2 - 4k(1-k)\beta^2}}{\beta^2}, & (6.8)\\ \alpha &> 2\beta\sqrt{k(1-k)} & (6.9)\\ k &= \sqrt{1 - 2\delta + \sigma^2} < 1. & (6.10) \end{aligned}$$

Proof. $\forall z_1, z_2 \in H$, and $\lambda \in M$, we have

$$\begin{aligned} \|F_\lambda(z_1) - F_\lambda(z_2)\| &\leq \|u_1 - u_2 - (g(u_1) - g(u_2))\| \\ &\quad + \|u_1 - u_2 - \rho(e^{T_\lambda(u_1)} - e^{T_\lambda(u_2)})\|. \end{aligned} \tag{6.11}$$

Using the strongly monotonicity and Lipschitz continuity of the operator g, we have

$$\begin{aligned} \|u_1 - u_2 - (g(u_1) - g(u_2))\|^2 &\leq \|u_1 - u_2\|^2 - 2\langle(u_1 - u_2, g(u_1) - g(u_2)\rangle \\ &\quad + \|g(u_1) - g(u_2)\|^2 \\ &\leq (1 - 2\delta + \sigma^2)\|u_1 - u_2\|^2. \end{aligned} \tag{6.12}$$

In a similar way, we have

$$\|u_1 - u_2 - \rho(e^{T_\lambda(u_1)} - e^{T_\lambda(u_2)})\|^2 \leq (1 - 2\rho\alpha + \beta^2\rho^2)\|u_1 - u_2\|^2. \tag{6.13}$$

From (6.6) and (6.12), we obtain

$$\begin{aligned} \|u_1 - u_2\| &\leq \|u_1 - u_2 - (g(u_1) - g(u_2))\| + \|J_{A_\lambda} z_1 - J_{A_\lambda} z_2\| \\ &\leq k\|u_1 - u_2\| + \|z_1 - z_2\|, \end{aligned}$$

which implies

$$\|u_1 - u_2\| \leq \frac{1}{1 - k}\|z_1 - z_2\|. \tag{6.14}$$

Combining (6.14) and (6.11), we have

$$\begin{aligned}\|F_\lambda(z_1) - F_\lambda(z_2)\| &\leq \{\frac{k + \sqrt{1 - 2\rho\alpha + \rho^2\beta^2}}{1-k}\}\|z_1 - z_2\| \\ &= \theta\|z_1 - z_2\|, \quad \text{using (6.7)}.\end{aligned}$$

From (6.8)-(6.10), it follows that $\theta < 1$ and consequently, the map $F_\lambda(z)$ defined by (6.5) is a contraction map and has a fixed point $z(\lambda)$, which is a solution of the resolvent equation (6.2). □

Remark 6.1. *From Lemma 6.2, we see that the map $F_\lambda(z)$ defined by (6.5) has a unique fixed point $z(\lambda)$, that is, $z(\lambda) = F_\lambda(z)$. Also, by assumption, the function $\bar{z}$, for $\lambda = \bar{\lambda}$ is a solution of the parametric resolvent equation (6.2). Again using Lemma 6.2, we see that $\bar{z}$, for $\lambda = \bar{\lambda}$, is a fixed point of $F_\lambda(z)$ and it is also a fixed point of $F_{\bar{\lambda}}(z)$. Consequently, we conclude that*

$$z(\bar{\lambda}) = \bar{z} = F_{\bar{\lambda}}(z(\bar{\lambda})).$$

Using Lemma 6.2 and technique of Noor and Noor [35], we can prove the continuity of the solution $z(\lambda)$ of the parametric resolvent equation (6.2). We include its proof to convey an idea of the technique.

Lemma 6.3. *Assume that the operator $T_\lambda(.)$ is locally Lipschitz continuous with respect to the parameter λ. If the operators $T_\lambda(.)$ is locally exponentially Lipschitz continuous and the map $\lambda \longrightarrow J_{A_\lambda}$ is continuous (or Lipschitz continuous), then the function $z(\lambda)$ satisfying (6.2) is (Lipschitz) continuous at $\lambda = \bar{\lambda}$.*

Proof. $\forall \lambda \in M$, invoking Lemma 6.2 and the triangle inequality, we have

$$\begin{aligned}\|z(\lambda) - z(\bar{\lambda})\| &\leq \|F_\lambda(z(\lambda)) - F_{\bar{\lambda}}(z(\bar{\lambda}))\| + \|F_\lambda(z(\bar{\lambda}) - F_{\bar{\lambda}}(z(\bar{\lambda}))\| \\ &\leq \theta\|z(\lambda) - z(\bar{z})\| + \|F_\lambda(z(\bar{\lambda})) - F_{\bar{\lambda}}(z(\bar{\lambda}))\|. \qquad (6.15)\end{aligned}$$

From (6.15) and the fact that the operator $T_\lambda(.)$ is a locally Lipschitz continuous with respect to the parameter λ, we have

$$\begin{aligned} \|F_\lambda(z(\bar{\lambda})) - F_{\bar{\lambda}}(z(\bar{\lambda}))\| &= \|u(\bar{\lambda}) - u(\bar{\lambda}) + \rho(e^{T_\lambda(u(\bar{\lambda}))} - e^{T_{\bar{\lambda}}(u(\bar{\lambda}))})\| \\ &\leq \rho\mu\|\lambda - \bar{\lambda}\|. \end{aligned} \tag{6.16}$$

Combining (6.15) and (6.16), we obtain

$$\|z(\lambda) - z(\bar{\lambda})\| \leq \frac{\rho\mu}{1-\theta}\|\lambda - \bar{\lambda}\|, \quad \forall \lambda, \bar{\lambda} \in M,$$

from which the required result follows. □

We now state and prove the main result of this section and is the motivation of our next result.

Theorem 6.1. *Let $\bar{u}$ be the solution of the parametric exponentially general variational inclusion (6.1) and $\bar{z}$ be the solution of the exponentially parametric resolvent equation (6.2) for $\lambda = \bar{\lambda}$. Let $T_\lambda(.)$ be the locally strongly exponentially monotone Lipschitz continuous operator. Let the operator g be also strongly monotone Lipschitz continuous operator. If the map $\lambda \longrightarrow P_{K_\lambda}$ is (Lipschitz) continuous at $\lambda = \bar{\lambda}$, then there exists a neighborhood $N \subset K$ of $\bar{\lambda}$ such that for $\lambda \in N$, the parametric exponentially resolvent equation (6.2) has a unique solution $z(\lambda)$ in the interior of X, $z(\bar{\lambda}) = \bar{z}$ and $z(\lambda)$ is (Lipschitz) continuous at $\lambda = \bar{\lambda}$.*

Proof. Its proof follows from Lemma 6.2, Lemma 6.3 and Remark 6.1. □

Conclusion

In this paper, we have introduced and studied some new classes of exponentially general variational inclusions. Some interesting and important known and new classes of exponentially general variational inequalities and optimizations are discussed. We have proved that the exponentially general variational inclusions are equivalent to this fixed point problems, resolvent equations and dynamical systems. These alternative formulations are used to discuss the existence of a solution of the exponentially general variational inclusions and suggest some new iterative methods for solving the exponentially general variational inclusions. These new methods include extraresolvent method, modified double resolvent methods and inertial type are suggested using the techniques of resolvent method, resolvent equations and dynamical systems. Convergence analysis of the proposed method is discussed for monotone operators. We have given only the glimpse of the applications of the exponentially dynamical systems. This technique is quite flexible and unified one. Using the ideas and techniques of this paper, one can suggest and investigate several new implicit methods for solving various classes of exponentially general variational inclusions and related problems. The implementation and comparison of these methods with other methods needs further efforts. We have also discussed the sensitivity analysis of the exponentially general variational inclusions. Since the exponentially general variational inclusions include the classical variational inequalities, mixed (quasi) variational inequalities, and complementarity problems as special cases, the technique developed in this paper can be used to study the sensitivity analysis of these problems. The fixed formulation of the exponentially general variational inclusions allows us to study the Holder and Lipschitz continuity of the solution of the parametric exponentially problems. It is worth mentioning that the resolvent equations technique does not require the differentiability of the given data. In fact, our results represents a refinement and significant improvement of previous known results.

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