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Some Structural Properties of the Generalized Kumaraswamy (GKw) $q_T - X$ Class of Distributions

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Abstract

Ampadu [1] introduced quantile generated probability distributions as a new way to generate continuous distributions. Combining this idea with the two-parameter Kumaraswamy (Kw) distribution, for example, see Wikipedia contributors [15], this paper introduces a so-called (GKw) $q_T - X$ class of distributions, and obtains some of their structural properties. Practicality of sub-models of this new class of distributions is shown to be effective in modeling real life data. Practicality to various disciplines is proposed as further investigation. A bivariate extension of this new class of distribution is also proposed, and the reader is asked to investigate its properties and applications.

1 Introduction

1.1 The Kumaraswamy Distribution

Kumaraswamy [9] introduced a new probability distribution for double bounded random processes with hydrological applications. The Kumaraswamy's distribution (called from now on the Kw distribution) on the interval $(0, 1)$ has CDF and PDF given, respectively, by

$$F_{(a,b)}(x) = 1 - (1 - x^a)^b \tag{1}$$

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$$f_{(a,b)}(x) = abx^{a-1}(1-x^a)^{b-1}, \tag{2}$$

where $a > 0$ and $b > 0$ are two shape parameters.

1.2 The Kumaraswamy-G Class of Distributions

This is a generalization of the Kw distribution, where the CDF of the uniform distribution on $[0, 1]$ in the CDF of the Kw distribution is replaced with an arbitrary parent CDF, $G(x)$, thus the new class of Kw generalized (Kw-G) distributions has CDF

$$F^*_{(a,b)}(x) = 1 - (1 - G(x)^a)^b \tag{3}$$

and PDF

$$f^*_{(a,b)}(x) = abg(x)G(x)^{a-1}(1 - G(x)^a)^{b-1}, \tag{4}$$

where $a, b > 0$ are two additional parameters whose role is to introduce skewness and to vary tail weights. The class of distributions in this section was proposed by Cordeiro and Castro [4].

1.3 The Exponentiated Class of Distributions

This class of distributions was proposed by Gupta et al. [8], and the CDF and PDF are given, respectively, by

$$F^{**}_{(a)}(x) = G(x)^a \tag{5}$$

and PDF

$$f^{**}_{(a)}(x) = ag(x)G(x)^{a-1}, \tag{6}$$

where $a > 0$, $G(x)$ is an arbitrary parent CDF with PDF $g(x)$. Note that this class of distributions can be obtained from the Kw-G class of distributions in the previous section by letting $b = 1$.

1.4 A Beta-G Sub-Class of Distributions

Eugene et al. [7], introduced the class of generalized beta distributions. In particular the PDF of the Beta-G class of distributions is given by

$$f^{***}_{(a,b)}(x) = \frac{1}{B(a,b)} g(x)G(x)^{a-1}\{1 - G(x)\}^{b-1}, \tag{7}$$

where $B(\cdot,\cdot)$ denotes the Beta function, $a, b > 0$, and $G(x)$ is an arbitrary parent CDF with PDF $g(x)$.

Remark 1.4.1. *When a random variable J follows the Kw-G class of distributions we write $J \sim KwG(a,b)$, and when a random variable Q follows the Beta-G class of distributions we write $Q \sim BG(a,b)$.*

The $KwG(1,b)$ class of distributions coincides with the $BG(1,b)$ class of distributions, in particular, observe that we have the following in terms of the PDF's

$$\begin{aligned} f^{***}_{(1,b)}(x) &= \frac{1}{B(1,b)} g(x)\{1 - G(x)\}^{b-1} \\ &= bg(x)\{1 - G(x)\}^{b-1} \\ &= f^{*}_{(1,b)}(x). \end{aligned} \tag{8}$$

1.5 The $q_T - X$ Family of Distributions

This section is inspired by quantile generated probability distributions introduced by Ampadu [1].

Definition 1.5.1. *Let V be any function such that the following hold:*

(*a*) $F(x) \in [V(a), V(b)]$,

(*b*) $F(x)$ *is differentiable and strictly increasing,*

(*c*) $\lim_{x\to -\infty} F(x) = V(a)$ *and* $\lim_{x\to\infty} F(x) = V(b)$, *then the CDF of the* $q_T - X$ ***family induced by V*** *is given by*

$$K(x) = \int_a^{V(F(x))} \frac{1}{r(Q(t))} dt,$$

where $\frac{1}{r(Q(t))}$ *is the quantile density function of random variable* $T \in [a, b]$*, for* $-\infty \le a < b \le \infty$*, and* $F(x)$ *is the CDF of any random variable* X*.*

Theorem 1.5.2. *The CDF of the* $q_T - X$ ***family induced by V*** *is given by*

$$K(x) = Q[V(F(x))].$$

Proof. Follows from the previous definition and noting that $Q' = \frac{1}{r \circ Q}$. □

Theorem 1.5.3. *The PDF of the* $q_T - X$ ***family induced by V*** *is given by*

$$k(x) = \frac{f(x)}{r[Q(V(F(x)))]} V'[F(x)].$$

Proof. $k = K'$, $Q' = \frac{1}{r \circ Q}$, $F' = f$, and K is given by Theorem 1.5.2 □

Remark 1.5.4. *When the support of* T *is* $[a, \infty)$*, where* $a \ge 0$*, we can take* V *as follows*

(a) $V(x) = 1 - e^{-x}$,

(b) $V(x) = \frac{x}{1+x}$,

(c) $V(x) = \left[1 - e^{-x}\right]^{\frac{1}{\alpha}}$, *where* $\alpha > 0$

(d) $V(x) = \left[\frac{x}{1+x}\right]^{\frac{1}{\alpha}}$, *where* $\alpha > 0$.

Remark 1.5.5. *When the support of* T *is* $(-\infty, \infty)$*, we can take* V *as follows*

(a) $V(x) = 1 - e^{-e^x}$,

(b) $V(x) = \frac{e^x}{1+e^x}$,

(c) $V(x) = \left[1 - e^{-e^x}\right]^{\frac{1}{\alpha}}$, *where* $\alpha > 0$

(d) $V(x) = \left[\frac{e^x}{1+e^x}\right]^{\frac{1}{\alpha}}$, *where* $\alpha > 0$.

2 The Generalized Type I (GKw) q_T-X Class of Distributions

2.1 The CDF

Definition 2.1.1. *A random variable* X_1 *will be called generalized type I (GKw)* $q_T - X$ *distributed if the CDF admit the following integral representation*

$$K_{(a,b,\alpha)}(x) = \int_0^{\left[1-e^{-F^*_{(a,b)}(x)}\right]^{\frac{1}{\alpha}}} \frac{1}{(r_T \circ Q_T)(t)} dt,$$

where $F^*_{(a,b)}(x) = 1 - (1 - G(x)^a)^b$, $a, b, \alpha > 0$, *the random variable* X *has CDF* $G(x)$, r_T *and* Q_T *are the PDF and Quantile function, respectively, of the random variable* T.

Theorem 2.1.2. *The CDF of the Generalized Type I (GKw)* q_T-X *class of distributions can be written as*

$$K_{(a,b,\alpha)}(x) = Q_T\left\{[1 - e^{-F^*_{(a,b)}(x)}]^{\frac{1}{\alpha}}\right\},$$

where $F^*_{(a,b)}(x) = 1 - (1 - G(x)^a)^b$, $a, b, \alpha > 0$, *the random variable* X *has CDF* $G(x)$, *and* Q_T *is the Quantile function of the random variable* T.

Proof. It is a direct consequence of Definition 2.1,1 and Theorem 1.5.2. □

Corollary 2.1.3. *The CDF of the Generalized Type I (Kw) q_T-Uniform class of distributions can be written as*

$$U_{(a,b,\alpha)}(x) = Q_T\left\{[1 - e^{-F_{(a,b)}(x)}]^{\frac{1}{\alpha}}\right\},$$

where $F_{(a,b)}(x) = 1 - (1 - x^a)^b$, $a, b, \alpha > 0$, and Q_T is the Quantile function of the random variable T.

Proof. In the previous Theorem, let the random variable X follow the uniform distribution on $[0, 1]$, so that $G(x) = x$. □

Corollary 2.1.4. *The CDF of the Generalized Type I Exponentiated q_T-Uniform class of distributions can be written as*

$$U_{(a,1,\alpha)}(x) = Q_T\left\{[1 - e^{-F_{(a,1)}(x)}]^{\frac{1}{\alpha}}\right\},$$

where $F_{(a,1)}(x) = x^a$, $a, \alpha > 0$, and Q_T is the Quantile function of the random variable T.

Proof. In the previous Corollary, let $b = 1$. □

Corollary 2.1.5. *The CDF of a Generalized Type I Beta q_T-Uniform sub-class of distributions or a Generalized Type I (Kw) q_T-Uniform sub-class of distributions can be written as*

$$U_{(1,b,\alpha)}(x) = Q_T\left\{[1 - e^{-F_{(1,b)}(x)}]^{\frac{1}{\alpha}}\right\},$$

where $F_{(1,b)}(x) = 1 - (1 - x)^b$, $b, \alpha > 0$, and Q_T is the Quantile function of the random variable T.

Proof. In Corollary 2.1.3, let $a = 1$. □

Corollary 2.1.6. *The CDF of the Generalized Type I q_T-Uniform class of distributions can be written as*

$$U_{(1,1,\alpha)}(x) = Q_T\left\{[1 - e^{-F_{(1,1)}(x)}]^{\frac{1}{\alpha}}\right\},$$

where $F_{(1,1)}(x) = x$, $\alpha > 0$, and Q_T is the Quantile function of the random variable T.

Proof. In previous Corollary let $b = 1$. □

Corollary 2.1.7. *The CDF of the Generalized Type I Exponentiated q_T-X class of distributions can be written as*

$$J_{(a,1,\alpha)}(x) = Q_T\left\{[1 - e^{-F^*_{(a,1)}(x)}]^{\frac{1}{\alpha}}\right\},$$

where $F^*_{(a,1)}(x) = G(x)^a$, $a, \alpha > 0$, the random variable X has CDF $G(x)$, and Q_T is the Quantile function of the random variable T.

Proof. Let $b = 1$ in Theorem 2.1.2. □

Corollary 2.1.8. *The CDF of a Generalized Type I Beta q_T-X sub-class of distributions or a Generalized Type I (Kw) q_T-X sub-class of distributions can be written as*

$$J_{(1,b,\alpha)}(x) = Q_T\left\{[1 - e^{-F^*_{(1,b)}(x)}]^{\frac{1}{\alpha}}\right\},$$

where $F^*_{(1,b)}(x) = 1-(1-G(x))^b$, $b, \alpha > 0$, the random variable X has CDF $G(x)$, and Q_T is the Quantile function of the random variable T.

Proof. Let $a = 1$ in Theorem 2.1.2. □

Corollary 2.1.9. *The CDF of the Generalized Type I q_T-X class of distributions can be written as*

$$J_{(1,1,\alpha)}(x) = Q_T\left\{[1 - e^{-F^*_{(1,1)}(x)}]^{\frac{1}{\alpha}}\right\},$$

where $F^*_{(1,1)}(x) = G(x)$, $\alpha > 0$, the random variable X has CDF $G(x)$, and Q_T is the Quantile function of the random variable T.

Proof. Let $b = 1$ in previous Corollary. □

2.2 The PDF

Theorem 2.2.1. *The PDF of the Generalized Type I (GKw) q_T-X class of distributions can be written as*

$$k_{(a,b,\alpha)}(x) = \frac{f^*_{(a,b)}(x)e^{-F^*_{(a,b)}(x)}[1-e^{-F^*_{(a,b)}(x)}]^{\frac{1-\alpha}{\alpha}}}{\alpha(r_T \circ Q_T)\left\{[1-e^{-F^*_{(a,b)}(x)}]^{\frac{1}{\alpha}}\right\}},$$

where

$$f^*_{(a,b)}(x) = abg(x)G(x)^{a-1}(1-G(x)^a)^{b-1}$$

$$F^*_{(a,b)}(x) = 1-(1-G(x)^a)^b$$

$a, b, \alpha > 0$, the random variable X has CDF $G(x)$ and PDF $g(x)$, r_T and Q_T are the PDF and Quantile function, respectively, of the random variable T.

Proof. Differentiate the CDF, $K_{(a,b,\alpha)}(x)$. □

Theorem 2.2.2. *The PDF of the Generalized Type I (Kw) q_T-Uniform class of distributions can be written as*

$$u_{(a,b,\alpha)}(x) = \frac{f_{(a,b)}(x)e^{-F_{(a,b)}(x)}[1-e^{-F_{(a,b)}(x)}]^{\frac{1-\alpha}{\alpha}}}{\alpha(r_T \circ Q_T)\left\{[1-e^{-F_{(a,b)}(x)}]^{\frac{1}{\alpha}}\right\}},$$

where

$$f_{(a,b)}(x) = abx^{a-1}(1-x^a)^{b-1}$$

$$F_{(a,b)}(x) = 1-(1-x^a)^b$$

$a, b, \alpha > 0$, r_T and Q_T are the PDF and Quantile function, respectively, of the random variable T.

Proof. Differentiate the CDF, $U_{(a,b,\alpha)}(x)$. □

Corollary 2.2.3. *The PDF of the Generalized Type I Exponentiated q_T-Uniform class of distributions can be written as*

$$u_{(a,1,\alpha)}(x) = \frac{f_{(a,1)}(x)e^{-F_{(a,1)}(x)}[1-e^{-F_{(a,1)}(x)}]^{\frac{1-\alpha}{\alpha}}}{\alpha(r_T \circ Q_T)\left\{[1-e^{-F_{(a,1)}(x)}]^{\frac{1}{\alpha}}\right\}},$$

where

$$f_{(a,1)}(x) = ax^{a-1}$$

$$F_{(a,1)}(x) = x^a$$

$a, \alpha > 0$, r_T and Q_T are the PDF and Quantile function, respectively, of the random variable T.

Proof. Let $b = 1$ in the previous Theorem. □

Corollary 2.2.4. *The PDF of the Generalized Type I q_T-Uniform class of distributions can be written as*

$$u_{(1,1,\alpha)}(x) = \frac{f_{(1,1)}(x)e^{-F_{(1,1)}(x)}[1-e^{-F_{(1,1)}(x)}]^{\frac{1-\alpha}{\alpha}}}{\alpha(r_T \circ Q_T)\left\{[1-e^{-F_{(1,1)}(x)}]^{\frac{1}{\alpha}}\right\}},$$

where

$$f_{(1,1)}(x) = 1$$

$$F_{(1,1)}(x) = x$$

$\alpha > 0$, r_T and Q_T are the PDF and Quantile function, respectively, of the random variable T.

Proof. Let $a = 1$ in the previous Corollary. □

Corollary 2.2.5. *The PDF of a Generalized Type I Beta q_T-Uniform sub-class of distributions or a Generalized Type I (Kw) q_T-Uniform sub-class of distributions* *can be written as*

$$u_{(1,b,\alpha)}(x) = \frac{f_{(1,b)}(x)e^{-F_{(1,b)}(x)}[1-e^{-F_{(1,b)}(x)}]^{\frac{1-\alpha}{\alpha}}}{\alpha(r_T \circ Q_T)\left\{[1-e^{-F_{(1,b)}(x)}]^{\frac{1}{\alpha}}\right\}},$$

where

$$f_{(1,b)}(x) = b(1-x)^{b-1}$$

$$F_{(1,b)}(x) = 1-(1-x)^{b}$$

$b, \alpha > 0$, r_T and Q_T are the PDF and Quantile function, respectively, of the random variable T.

Proof. Let $a = 1$ in Theorem 2.2.2. □

Theorem 2.2.6. *The PDF of the Generalized Type I Exponentiated q_T-X class of distributions can be written as*

$$j_{(a,1,\alpha)}(x) = \frac{f^*_{(a,1)}(x)e^{-F^*_{(a,1)}(x)}[1-e^{-F^*_{(a,1)}(x)}]^{\frac{1-\alpha}{\alpha}}}{\alpha(r_T \circ Q_T)\left\{[1-e^{-F^*_{(a,1)}(x)}]^{\frac{1}{\alpha}}\right\}},$$

where

$$f^*_{(a,1)}(x) = ag(x)G(x)^{a-1}$$

$$F^*_{(a,1)}(x) = G(x)^{a}$$

$a, \alpha > 0$, the random variable X has CDF $G(x)$ and PDF $g(x)$, r_T and Q_T are the PDF and Quantile function, respectively, of the random variable T.

Proof. Differentiate the CDF given by Corollary 2.1.7. □

3 Some Structural Properties

3.1 Renyi Entropies

Let X be a random variable with PDF $f(x)$. A concept of entropy introduced by Renyi [12] was defined as follows

$$I_R(\delta) = \frac{1}{1-\delta} Log\Big(\int_{-\infty}^{\infty} f^{\delta}(x)dx \Big), \tag{9}$$

where $\delta > 0$, and $\delta \neq 1$.

Theorem 3.1.1. *The Renyi entropy of the Generalized Type I Quantile Uniform-Uniform class of distributions is given by*

$$\frac{1}{1-\delta}\Bigg\{ -\delta Log[\alpha] + Log\Bigg[\sum_{k,r=0}^{\infty} \binom{\frac{\delta(1-\alpha)}{\alpha}}{k} \frac{(-1)^{k+r}(\delta+k)^r}{r!(1+r)} \Bigg] \Bigg\},$$

where $\delta > 0$, and $\delta \neq 1$, and $\alpha > 0$.

Proof. Since T is uniform, it follows from Corollary 2.2.4 that the PDF of the Generalized Type I Quantile Uniform-Uniform class of distributions can be written as

$$u(x,\alpha) = \frac{1}{\alpha} e^{-x}[1-e^{-x}]^{\frac{1-\alpha}{\alpha}}. \tag{10}$$

By the binomial series, Wikipedia contributors [13], we can write

$$[1-e^{-x}]^{\frac{\delta(1-\alpha)}{\alpha}} = \sum_{k=0}^{\infty} \binom{\frac{\delta(1-\alpha)}{\alpha}}{k} (-1)^k e^{-kx}. \tag{11}$$

It now follows that

$$e^{-x\delta}[1-e^{-x}]^{\frac{\delta(1-\alpha)}{\alpha}} = \sum_{k=0}^{\infty} \binom{\frac{\delta(1-\alpha)}{\alpha}}{k} (-1)^k e^{-x(\delta+k)}. \tag{12}$$

By the power series representation for the real exponential function, Wikipedia contributors [14], we can write

$$e^{-x(\delta+k)} = \sum_{r=0}^{\infty} \frac{(-1)^r(\delta+k)^r x^r}{r!}. \tag{13}$$

It now follows that

$$u(x,\alpha)^{\delta} = \left(\frac{1}{\alpha}\right)^{\delta} \sum_{k,r=0}^{\infty} \binom{\frac{\delta(1-\alpha)}{\alpha}}{k} \frac{(-1)^{k+r}(\delta+k)^r x^r}{r!}. \tag{14}$$

Thus we have the following

$$\begin{aligned}&\frac{1}{1-\delta} Log\Big(\int_{-\infty}^{\infty} u(x,\alpha)^{\delta}(x)dx\Big)\\ =&\frac{1}{1-\delta}\Bigg\{-\delta Log[\alpha] + Log\Bigg[\sum_{k,r=0}^{\infty} \binom{\frac{\delta(1-\alpha)}{\alpha}}{k} \frac{(-1)^{k+r}(\delta+k)^r}{r!} \int_{-\infty}^{\infty} x^r dx\Bigg]\Bigg\}. \end{aligned}\tag{15}$$

Now let $u = F(x) = x$, then $du = dx$, and by properties of distribution function, clearly,

$$\int_{-\infty}^{\infty} x^r dx = \int_0^1 u^r du = \frac{1}{1+r} \tag{16}$$

and so

$$\begin{aligned}&\frac{1}{1-\delta} Log\Big(\int_{-\infty}^{\infty} u(x,\alpha)^{\delta}(x)dx\Big)\\ =&\frac{1}{1-\delta}\Bigg\{-\delta Log[\alpha] + Log\Bigg[\sum_{k,r=0}^{\infty} \binom{\frac{\delta(1-\alpha)}{\alpha}}{k} \frac{(-1)^{k+r}(\delta+k)^r}{r!(1+r)}\Bigg]\Bigg\}. \end{aligned}\tag{17}$$

□

3.2 Order Statistics

A branch of statistics known as order statistics plays a key role in real life applications involving data relating to life testing studies, and these statistics are required in many fields such as climatology, engineering and industry, among others. A good introduction on order statistics and associated inferences can be found in David and Nagaraja [6]. Let $X_{i:n}$ denote the ith order statistic. In this section we obtain an expansion formula for the density $u_{i:n}(x,\alpha)$ of the ith order statistics, for $i = 1, 2, \cdots, n$, from independent and identically distributed random variables $X_1, \cdots, X_n$ having the Generalized Type I Quantile Uniform-Uniform class of distributions.

Remark 3.2.1. *Since T is uniform on $[0,1]$, it follows from Corollary 2.1.6, that the CDF of the Generalized Type I Quantile Uniform-Uniform class of distributions is given by $U(x,\alpha) = [1-e^{-x}]^{\frac{1}{\alpha}}$, where $\alpha > 0$.*

Now we have the following

Theorem 3.2.2. *The density $u_{i:n}(x,\alpha)$ of the ith order statistics, for $i = 1, 2, \cdots, n$, from independent and identically distributed random variables $X_1, \cdots, X_n$ having the Generalized Type I Quantile Uniform-Uniform class of distributions admit the following expansion*

$$u_{i:n}(x,\alpha) = \sum_{v,f,r,t,q,j,m=0}^{\infty} \binom{\frac{1-\alpha}{\alpha}}{v} \binom{\frac{i-1}{\alpha}}{r} \binom{n-i}{q} \binom{\frac{q}{\alpha}}{j} \times \frac{(-1)^{f+v+t+r+m+j+q} x^{f+t+m} (v+1)^f r^t j^m n!}{\alpha f! t! m! (i-1)!(n-i)!},$$

where $\alpha > 0$.

Proof. First observe we have the following, upon using the binomial series, Wikipedia contributors [13],

$$[1-U(x,\alpha)]^{n-i} = \sum_{q=0}^{\infty} (-1)^q \binom{n-i}{q} U(x,\alpha)^q. \tag{18}$$

On the other hand,

$$\begin{aligned} U(x,\alpha)^q &= [1-e^{-x}]^{\frac{q}{\alpha}} \\ &= \sum_{j=0}^{\infty} (-1)^j \binom{\frac{q}{\alpha}}{j} e^{-xj}. \end{aligned} \tag{19}$$

By the power series representation for the real exponential function, Wikipedia contributors [14], we can write

$$e^{-xj} = \sum_{m=0}^{\infty} \frac{(-1)^m x^m j^m}{m!}. \tag{20}$$

Thus

$$U(x,\alpha)^q = \sum_{j,m=0}^{\infty} \binom{\frac{q}{\alpha}}{j} \frac{(-1)^{m+j} x^m j^m}{m!}. \tag{21}$$

Thus

$$[1 - U(x,\alpha)]^{n-i} = \sum_{q,j,m=0}^{\infty} \binom{n-i}{q} \binom{\frac{q}{\alpha}}{j} \frac{(-1)^{m+j+q} x^m j^m}{m!}. \tag{22}$$

Now

$$\begin{aligned} U(x,\alpha)^{i-1} &= [1 - e^{-x}]^{\frac{i-1}{\alpha}} \\ &= \sum_{r=0}^{\infty} (-1)^r \binom{\frac{i-1}{\alpha}}{r} e^{-xr} \end{aligned} \tag{23}$$

and

$$e^{-xr} = \sum_{t=0}^{\infty} \frac{(-1)^t x^t r^t}{t!}. \tag{24}$$

Thus

$$U(x,\alpha)^{i-1} = \sum_{r,t=0}^{\infty} \binom{\frac{i-1}{\alpha}}{r} \frac{(-1)^{t+r} x^t r^t}{t!}. \tag{25}$$

So we have

$$U(x,\alpha)^{i-1}[1 - U(x,\alpha)]^{n-i} = \sum_{r,t,q,j,m=0}^{\infty} \binom{\frac{i-1}{\alpha}}{r} \binom{n-i}{q} \binom{\frac{q}{\alpha}}{j} \frac{(-1)^{t+r+m+j+q} x^{t+m} r^t j^m}{t!m!}. \tag{26}$$

From the proof of Theorem 3.1.1, we know that

$$u(x,\alpha) = \frac{1}{\alpha} e^{-x} [1 - e^{-x}]^{\frac{1-\alpha}{\alpha}}. \tag{27}$$

Now we try to find an expansion formula for $u(x,\alpha)$. Observe we have the following

$$[1 - e^{-x}]^{\frac{1-\alpha}{\alpha}} = \sum_{v=0}^{\infty} (-1)^v \binom{\frac{1-\alpha}{\alpha}}{v} e^{-xv}. \tag{28}$$

Thus

$$e^{-x}[1-e^{-x}]^{\frac{1-\alpha}{\alpha}} = \sum_{v=0}^{\infty}(-1)^{v}\binom{\frac{1-\alpha}{\alpha}}{v}e^{-x(v+1)}. \tag{29}$$

On the other hand

$$e^{-x(v+1)} = \sum_{f=0}^{\infty}\frac{(-1)^{f}x^{f}(v+1)^{f}}{f!}. \tag{30}$$

It now follows that

$$u(x,\alpha) = e^{-x}[1-e^{-x}]^{\frac{1-\alpha}{\alpha}} = \sum_{v,f=0}^{\infty}\binom{\frac{1-\alpha}{\alpha}}{v}\frac{(-1)^{f+v}x^{f}(v+1)^{f}}{\alpha f!}. \tag{31}$$

Thus

$$\begin{aligned} u_{i:n}(x,\alpha) &= \frac{n!}{(i-1)!(n-i)!}u(x,\alpha)U(x,\alpha)^{i-1}[1-U(x,\alpha)]^{n-i} \\ &= \sum_{v,f,r,t,q,j,m=0}^{\infty}\binom{\frac{1-\alpha}{\alpha}}{v}\binom{\frac{i-1}{\alpha}}{r}\binom{n-i}{q}\binom{\frac{q}{\alpha}}{j} \\ &\times \frac{(-1)^{f+v+t+r+m+j+q}x^{f+t+m}(v+1)^{f}r^{t}j^{m}n!}{\alpha f!t!m!(i-1)!(n-i)!}. \end{aligned} \tag{32}$$

□

3.3 The nth Incomplete Moment

For empirical purposes, the shape of many distributions can be usefully described by what is known as the incomplete moments. These types of moments play an important role for measuring inequality, for example, income quantiles and Lorenz and Bonferroni curves, which depend upon the incomplete moments. According to Cordeiro et al. [5], the nth incomplete moment of a random variable X is defined by

$$m_{n}(y) = E(X^{n}|X<y) = \int_{-\infty}^{y}x^{n}f(x)dx. \tag{33}$$

In this section we obtain the nth incomplete moment of the Generalized Type I Quantile Uniform-Uniform class of distributions.

Theorem 3.3.1. *The nth incomplete moment of the Generalized Type I Quantile Uniform-Uniform class of distributions is given by*

$$m_n(y,\alpha)=\sum_{v,f=0}^{\infty}\binom{\frac{1-\alpha}{\alpha}}{v}\frac{(-1)^{f+v}(v+1)^f y^{f+n+1}}{\alpha f!(f+n+1)},$$

where $\alpha>0$

Proof. From the proof of Theorem 3.2.2, we know that

$$u(x,\alpha)=\sum_{v,f=0}^{\infty}\binom{\frac{1-\alpha}{\alpha}}{v}\frac{(-1)^{f+v}x^f(v+1)^f}{\alpha f!}. \tag{34}$$

Thus,

$$\begin{aligned}\int_{-\infty}^{y}x^n u(x,\alpha)dx&=\int_{-\infty}^{y}\sum_{v,f=0}^{\infty}\binom{\frac{1-\alpha}{\alpha}}{v}\frac{(-1)^{f+v}x^{f+n}(v+1)^f}{\alpha f!}dx\\&=\sum_{v,f=0}^{\infty}\binom{\frac{1-\alpha}{\alpha}}{v}\frac{(-1)^{f+v}(v+1)^f}{\alpha f!}\int_{-\infty}^{y}x^{f+n}dx.\end{aligned} \tag{35}$$

Let $u=F(x):=x$, then $du=dx$, and using properties of distribution function we have

$$\int_{-\infty}^{y}x^{f+n}dx=\int_{0}^{y}u^{f+n}du=\frac{y^{f+n+1}}{f+n+1}. \tag{36}$$

Thus,

$$\int_{-\infty}^{y}x^n u(x,\alpha)dx=\sum_{v,f=0}^{\infty}\binom{\frac{1-\alpha}{\alpha}}{v}\frac{(-1)^{f+v}(v+1)^f y^{f+n+1}}{\alpha f!(f+n+1)}. \tag{37}$$

□

3.4 Stress-strength reliability

In reliability theory, a common situation is that the life of a component has a random strength subjected to a random stress. The random strength can be modeled by a random variable X and the random stress can be modeled by a random variable Y. The probability that the component functions satisfactorily is given by $R = P(Y < X)$, which is a well-known measure of component reliability with many applications. According to Bakouch et al. [3], we can calculate R as

$$R = \int_{-\infty}^{\infty} f_X(x)F_Y(x)dx. \tag{38}$$

Remark 3.4.1. *If a random variable X follows the Generalized Type I Quantile Uniform-Uniform class of distributions, write $X \sim GTIQUU(\alpha)$.*

Theorem 3.4.2. *If $X \sim GTIQUU(\alpha_1)$ and $Y \sim GTIQUU(\alpha_2)$, then,*

$$R = \sum_{v,f,r,t=0}^{\infty} \binom{\frac{1-\alpha_1}{\alpha_1}}{v} \binom{\frac{1}{\alpha_2}}{r} \frac{(-1)^{f+r+t+v}(v+1)^f r^t u^{f+t+1}}{\alpha_1 f!t!(f+t+1)}.$$

Proof. One can write

$$R = \int_{-\infty}^{\infty} \sum_{v,f,r,t=0}^{\infty} \binom{\frac{1-\alpha_1}{\alpha_1}}{v} \binom{\frac{1}{\alpha_2}}{r} \frac{(-1)^{f+r+t+v}(v+1)^f r^t x^{f+t}}{\alpha_1 f!t!} dx. \tag{39}$$

Let $u = F(x) = x$, then using properties of distribution function, we have,

$$\int_{-\infty}^{\infty} x^{f+t} dx = \frac{u^{f+t+1}}{f+t+1} \tag{40}$$

and the result follows. □

3.5 Probability Weighted Moments

The probability weighted moments (PWMs) are expectations of certain functions of a random variable whose mean exists. A general theory for the PWMs covers the summarization and description of theoretical probability distributions and observed data samples, nonparametric estimation of the underlying distribution of an observed sample, estimation of parameters, quantiles of probability distributions and hypothesis tests. The PWM method can generally be used for estimating parameters of a distribution whose inverse form cannot be expressed explicitly. According to Ahmed et al. [2], the (k,i)th PWM of X is defined by

$$\rho_{k,i} = E\{X^k F(x)^i\} = \int_{-\infty}^{\infty} x^k f(x) F(x)^i dx. \tag{41}$$

Theorem 3.5.1. *If $X \sim GTIQUU(\alpha)$, then the (k,i)th PWM of X is*

$$\rho_{k,i} = \sum_{v,f,j,m=0}^{\infty} \binom{\frac{1-\alpha}{\alpha}}{v} \binom{\frac{i}{\alpha}}{j} \frac{(-1)^{f+v+m+j}(v+1)^f j^m u^{k+f+m+1}}{\alpha f! m! (k+f+m+1)}.$$

Proof. One can write

$$\rho_{k,i} = \int_{-\infty}^{\infty} \sum_{v,f,j,m=0}^{\infty} \binom{\frac{1-\alpha}{\alpha}}{v} \binom{\frac{i}{\alpha}}{j} \frac{(-1)^{f+v+m+j}(v+1)^f j^m x^{k+f+m}}{\alpha f! m!} dx. \tag{42}$$

Let $u = F(x) = x$, then using properties of distribution function, we have,

$$\int_{-\infty}^{\infty} x^{k+f+m} dx = \frac{u^{k+f+m+1}}{k+f+m+1} \tag{43}$$

and the result follows. □

4 Further Developments

4.1 Practical Significance

As a further development we suggest investigating practical significance of the proposed distribution in this paper to various disciplines. However let the random variable X follow the Logistic distribution with CDF

$$G(x,a,b) = \frac{1}{1+e^{-\frac{-a+x}{b}}} \tag{44}$$

and the random variable T follow the Exponential distribution with quantile function

$$Q(p, r) = -\frac{\text{Log}[1-p]}{r} \tag{45}$$

then from Theorem 2.1.2 we have the following

Theorem 4.1.1. *The CDF of the Generalized Type I (GKw) Quantile Exponential-Logistic distribution has CDF*

$$K_{(a,b,c,d,f,r)}(x) = -\frac{Log\left[1-\left(1-e^{-1+\left(1-\left(\frac{1}{1+e^{-\frac{-a+x}{b}}}\right)^{d}\right)^{c}}\right)^{\frac{1}{f}}\right]}{r}.$$

On the other hand from Corollary 2.1.8 we have the following

Theorem 4.1.2. *The CDF of a Generalized Type I Beta Quantile Exponential-Logistic sub-class of distributions has CDF*

$$J_{(a,b,c,f,r)}(x) = -\frac{Log\left[1-\left(1-e^{-1+\left(1-\frac{1}{1+e^{-\frac{-a+x}{b}}}\right)^{c}}\right)^{\frac{1}{f}}\right]}{r}.$$

Also from Corollary 2.1.7 we have the following

Theorem 4.1.3. *The CDF of the Generalized Type I Exponentiated Quantile Exponential-Logistic distribution has CDF*

$$L_{(a,b,f,r)}(x) = -\frac{Log\left[1-\left(1-e^{-\frac{1}{1+e^{-\frac{-a+x}{b}}}}\right)^{\frac{1}{f}}\right]}{r}.$$

Remark 4.1.4. *By differentiating the CDF, the PDF's in Theorem 4.1.1, Theorem 4.1.2, and Theorem 4.1.3, respectively, can be obtained.*

Notation 4.1.5. *We will use the following throughout this section*

(a) *If a random variable J_1 follows the class of Generalized Type I (GKw) Quantile Exponential-Logistic distributions, write*

$$J_1 \sim GTIGKWQEL(a, b, c, d, f, r).$$

(b) *If a random variable J_2 follows the Generalized Type I Beta Quantile Exponential-Logistic sub-class of distributions, write*

$$J_2 \sim GTIBQEL(a, b, d, f, r).$$

(c) *If a random variable J_3 follows the class of Generalized Type I Exponentiated Quantile Exponential-Logistic distribution , write*

$$J_3 \sim GTIQEL(a, b, f, r).$$

In this section we compare the $GTIGKWQEL(a, b, c, d, f, 1.66667)$ distribution, the $GTIBQEL(a, b, d, f, 3.33333)$ distribution, and the $GTIQEL(a, b, 1.20998, 1.17647)$ distribution in modeling the breaking stress of carbon fibers (in Gba) data which appeared in section six of, Nasiru et al. [11].

Comparison with Empirical Distribution

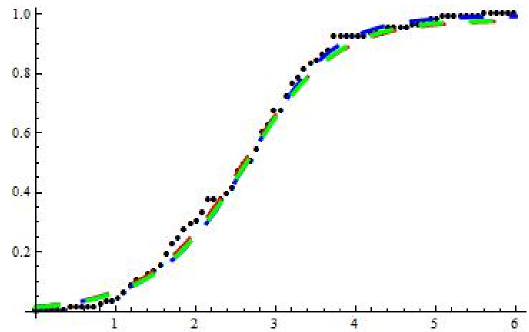

Figure 1: The CDF of the $GTIGKWQEL(\widehat{a},\widehat{b},\widehat{c},\widehat{d},\widehat{f},1.66667)$ distribution (red), the $GTIBQEL(\widehat{a},\widehat{b},\widehat{d},\widehat{f},3.33333)$ distribution (blue), and the $GTIQEL(\widehat{a},\widehat{b},1.20998,1.17647)$ distribution (green), fitted to the empirical distribution of the breaking stress of carbon fibers (in Gba) data.

Table 1: Parameter Estimates for breaking stress of carbon fibers (in Gba) data.

Model	$GTIGKWQEL(a,b,c,d,f,1.66667)$	$GTIBQEL(a,b,d,f,3.33333)$	$GTIQEL(a,b,1.20998,1.17647)$
Parameter Estimate	$\widehat{a}=2.44636$ $\widehat{b}=0.561729$ $\widehat{c}=0.958084$ $\widehat{d}=1.7476$ $\widehat{f}=2.10467$	$\widehat{a}=2.07469$ $\widehat{b}=0.500153$ $\widehat{d}=9.05673$ $\widehat{f}=12.3318$	$\widehat{a}=2.69806$ $\widehat{b}=0.543091$
AIC	302.325	297.866	295.77
AICC	302.963	298.287	295.893
BIC	315.351	308.287	300.98
-2(Log-likelihood)	292.325	289.866	291.77

Comparison with Histogram

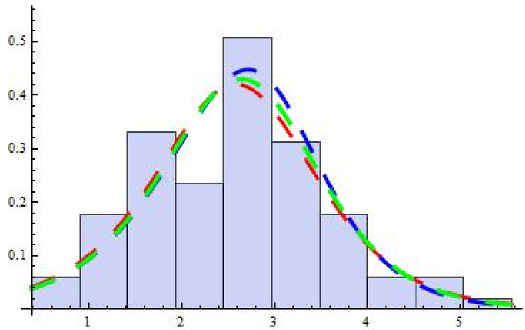

Figure 2: The PDF of the $GTIGKWQEL(\widehat{a}, \widehat{b}, \widehat{c}, \widehat{d}, \widehat{f}, 1.66667)$ distribution (red), the $GTIBQEL(\widehat{a}, \widehat{b}, \widehat{d}, \widehat{f}, 3.33333)$ distribution (blue), and the $GTIQEL(\widehat{a}, \widehat{b}, 1.20998, 1.17647)$ distribution (green), fitted to the histogram of the breaking stress of carbon fibers (in Gba) data.

From the figures above, and the table, it appears the $GTIQEL$ distribution is the best among the three sub-models of the generalized type I (GKw) q_T-X class of distributions in fitting the carbon fibers data.

4.2 Bivariate Extension

In this section we introduce a bivariate generalized type I (GKw) q_T-X class of distributions and ask the reader to further develop the properties and applications of this new class of distributions. It is well known that every bivariate distribution function, F_{X_1,X_2} with continuous marginals F_{X_1} and F_{X_2} correspond to a unique function $C : [0,1]^2 \mapsto [0,1]$ called a copula such that

$$F_{X_1,X_2}(x_1, x_2) = C\{F_{X_1}(x_1), F_{X_2}(x_2)\}. \tag{46}$$

We consider the Clayton copula

$$C_\alpha(u, v) = (u^{\frac{-1}{\alpha}} + v^{\frac{-1}{\alpha}} - 1)^{-\alpha}, \tag{47}$$

where $0 < u, v < 1$. We should remark that this copula was used by Kundu and Gupta [10], to construct the bivariate power normal distribution. The following properties of the random vector (U, V) with joint CDF (47) is well known

Theorem 4.2.1. *If (U, V) has joint CDF (47) for $\alpha > 0$, then*

(a) $U, V \sim uniform[0, 1]$.

(b) *The joint PDF of (U, V) for $0 \leq u, v \leq 1$ is*

$$f_{U,V}(u, v) = \frac{\alpha + 1}{\alpha(uv)^{\frac{1+\alpha}{\alpha}}(u^{\frac{-1}{\alpha}} + v^{\frac{-1}{\alpha}} - 1)^{\alpha+2}}.$$

(c) *The joint survival function of (U, V) for $0 < u, v < 1$ is*

$$S_{U,V}(u, v) = P(U \geq u, V \geq v) = 1 - u - v + (u^{\frac{-1}{\alpha}} + v^{\frac{-1}{\alpha}} - 1)^{-\alpha}.$$

(d) $U|V = v$ *has the CDF*

$$P(U \leq u|V = v) = \frac{1}{v^{\frac{1+\alpha}{\alpha}}(u^{\frac{-1}{\alpha}} + v^{\frac{-1}{\alpha}} - 1)^{\alpha+1}}.$$

For $a_1, a_2, b_1, b_2, \alpha_1, \alpha_2 > 0$, consider the random variables

$$X_1 = G^{-1}\left(\left\{1 - \left(1 - ln\left[\frac{1}{1 - R_T(U)^{\alpha_1}}\right]\right)^{\frac{1}{b_1}}\right\}^{\frac{1}{a_1}}\right) \tag{48}$$

and

$$X_2 = G^{-1}\left(\left\{1 - \left(1 - ln\left[\frac{1}{1 - R_T(V)^{\alpha_2}}\right]\right)^{\frac{1}{b_2}}\right\}^{\frac{1}{a_2}}\right). \tag{49}$$

The joint CDF of X_1 and X_2 becomes

$$\begin{aligned}
&F_{X_1,X_2}(x_1,x_2)\\
=&P\left(U \leq Q_T\left\{[1-e^{-(1-(1-G(x_1)^{a_1})^{b_1})}]^{\frac{1}{\alpha_1}}\right\}, V \leq Q_T\left\{[1-e^{-(1-(1-G(x_2)^{a_2})^{b_2})}]^{\frac{1}{\alpha_2}}\right\}\right)\\
=&C_\alpha\left(Q_T\left\{[1-e^{-(1-(1-G(x_1)^{a_1})^{b_1})}]^{\frac{1}{\alpha_1}}\right\}, Q_T\left\{[1-e^{-(1-(1-G(x_2)^{a_2})^{b_2})}]^{\frac{1}{\alpha_2}}\right\}\right)\\
=&\left(\left(Q_T\left\{[1-e^{-(1-(1-G(x_1)^{a_1})^{b_1})}]^{\frac{1}{\alpha_1}}\right\}\right)^{\frac{-1}{\alpha}} + \left(Q_T\left\{[1-e^{-(1-(1-G(x_2)^{a_2})^{b_2})}]^{\frac{1}{\alpha_2}}\right\}\right)^{\frac{-1}{\alpha}} - 1\right)^{-\alpha}.
\end{aligned} \tag{50}$$

Now we leave the reader with the following

Definition 4.2.2. *The bivariate random variables* (X_1, X_2) *will be said to have a bivariate generalized type I (GKw)* q_T*-X distribution, denoted by* $BGTIGKWQTX(\alpha, a_1, a_2, b_1, b_2, \alpha_1, \alpha_2)$*, if for* $a_1, a_2, b_1, b_2, \alpha_1, \alpha_2 > 0$*,* (X_1, X_2) *has the joint CDF (50).*

5 Concluding Remarks

In this paper, we introduced the (GKw) $q_T - X$ class of distributions of type I, and obtained some of their structural properties. We also showed sub-models of this new class of distributions is effective in modeling real life data. Finally, we have asked the reader to investigate practicality of the new class of distributions to various discplines, and also to investigate properties and applications of a bivariate model.

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References

[1] C. B. Ampadu, Quantile-generated family of distributions: a new method for generating continuous distributions, *Fundamental Journal of Mathematics and Mathematical Sciences* 9(1) (2018), 13-34.

[2] Z. A. Ahmed, E. Altun, M. Alizadeh, G. Ozel and G. G. Hamedani, The odd exponentiated half-logistic-G family: properties, characterizations and applications, *Chilean Journal of Statistics* 8(2) (2017), 65-91.

[3] H. Bakouch, F. Jamal, C. Chesneau and A. Nasir, A new transmuted family of distributions: Properties and estimation with applications, 2017. hal-01570370v3

[4] G. M. Cordeiro and M. Castro, A new family of generalized distributions, *J. Stat. Comput. Simul.* 81 (2011), 883-893.

[5] G. M. Cordeiro, M. Alizadeh, A. Nascimento and M. Rasekhi, The exponentiated Gompertz generated family of distributions: properties and applications, *Chilean Journal of Statistics* 7(2) (2016), 29-50.

[6] H. A. David and H. N. Nagaraja, *Order Statistics*, John Wiley and Sons, New Jersey 2003.

[7] N. Eugene, C. Lee and F. Famoye, Beta-normal distribution and its applications, *Communications in Statistics - Theory and Methods* 31(4) (2002), 497-512. `https://doi.org/10.1081/sta-120003130`

[8] R. C. Gupta, P. L. Gupta and R. D. Gupta, Modeling failure time data by Lehman alternatives, *Communications in Statistics - Theory and Methods* 27(4) (1998), 887-904. `https://doi.org/10.1080/03610929808832134`

[9] P. Kumaraswamy, A generalized probability density function for double-bounded random-processes, *Journal of Hydrology* 462 (1980), 79-88. `https://doi.org/10.1016/0022-1694(80)90036-0`

[10] D. Kundu and R. D. Gupta, Power-normal distribution, *Statistics* 47(1) (2013), 110-125.

[11] S. Nasiru, P. N. Mwita and O. Ngesa, Exponentiated generalized transformed-transformer family of distributions, *Journal of Statistical and Econometric Methods* 6(4) (2017), 1-17.

[12] A. Renyi, On measures of entropy and information, *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability* 1 (1961), 547-561.

[13] Wikipedia contributors, Binomial series, In: *Wikipedia, The Free Encyclopedia*, November 9, 2018. Retrieved 13:35, November 28, 2018, from `https://en.wikipedia.org/wiki/Binomial_series`

[14] Wikipedia contributors, Exponential function, In: *Wikipedia, The Free Encyclopedia*, November 22, 2018. Retrieved 13:37, November 28, 2018, from `https://en.wikipedia.org/wiki/Exponential_function`

[15] Wikipedia contributors, Kumaraswamy distribution, In: *Wikipedia, The Free Encyclopedia*, June 26, 2017. Retrieved 13:38, November 28, 2018, from `https://en.wikipedia.org/wiki/Kumaraswamy_distribution`