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Multivariate Opial-type Inequalities on Time Scales

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Abstract

Opial inequality was developed to provide bounds for integral of functions and their derivatives. It has become an indispensable tool in the theory of mathematical analysis due to its usefulness. A refined Jensen inequality for multivariate functions is employed to establish new Opial-type inequalities for convex functions of several variables on time scale.

1 Introduction

Opial [11] discussed problems involving functions and their derivatives. His work motivated many researchers to obtain a general version of the results and several methods have been used to extend the inequality in the following directions:

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Godunova and Levin [6] extended and provided sharper inequality compare to [11]. Rozanova [14] adopted convex function to improve [6] by using absolutely continuous function. In [12], Qi refined the results in [11] and obtained a more generalized Opial-type inequalities. In [13], Rauf and Anthonio generalized [5] through convexity and reiterated that the absolute value on both sides of the inequalities therein are not necessary. Bohner and Kaymakcalan [4] presented a version of Opial inequality for time scales and pointed out some of its applications to so-called dynamic equations. Such dynamic equations contain both differential and difference equations as special cases. Various extensions of their inequality were provided as well. Saker [15] proved some new Opial dynamic inequalities involving higher order derivatives on time scales. The results were proved by using Holder's inequality, a simple consequence of Keller's chain rule and Taylor monomials on time scales. Some continuous and discrete inequalities were used to derive special cases in their results. Some weighted generalization of Opial type inequalities in two independent variables for two functions was established in [9]. They also obtain weighted Opial-type inequalities by using p-norms.

An interval, in the time-scale context, is always understood as the intersection of a real interval with a given time-scale. We shall write the delta derivatives f^{Δ} for a function f defined on $\mathbb{T}$ and it turns out that:
(i) $f^{\Delta} = f'$ is the usual derivative if $\mathbb{T} = \mathbb{R}$; and
(ii) $f^{\Delta} = \Delta f$ is the usual forward difference operator if $\mathbb{T} = \mathbb{Z}$.

1.1 Some Basic Definitions

The derivative of a function $f : \mathbb{T} \longrightarrow \mathbb{R}$ denoted by f^{Δ} is as follows: Let $t \in \mathbb{T}$, if a number $\alpha \in \mathbb{R}$ such that for all $\epsilon > 0$ there exists a neighborhood U of t with

$$|f(\sigma(t)) - f(s) - \alpha(\sigma(t) - s)| \leq \epsilon \left|\sigma(t) - s\right|. \tag{1.1}$$

The fundamentals of time scales calculus, Opial inequality and its applications can be sourced from [1], [2], [3], [4], [6], [7], [8], [9], [10], [11] [13] & [16].

Throughout the work, refined Jensen inequality of the form (1.2) is employed

$$\left(\int_{\epsilon}^{t} f(x(t))d\lambda(s)\right)^{\varsigma} \leq \left(\int_{\epsilon}^{t} d\lambda(s)\right)^{\varsigma-\zeta} \left(\int_{\epsilon}^{t} \varphi(f(x(t)))^{\frac{1}{\zeta}} d\lambda(s)\right)^{\zeta}. \quad (1.2)$$

2 Some Results on Opial-Type Inequalities

In this section, we shall discuss Opial-type inequalities on time scale. We begin with the following theorem.

Theorem 2.1. *Let $\mathbb{T}$ be a time scale with $x, y, t \in \mathbb{T}$. Suppose ς and ζ are real numbers, $x, y, t \in C_{rd}([0,t]_{\mathbb{T}}, \mathbb{R})$ where $\phi(y)$ is positive rd-continuous function on $[0,t]_{\mathbb{T}}$ such that $\phi(y)^{\frac{1}{\zeta}} \leq \phi(y)$. Let $\phi(x)$ be a convex and increasing function on $[0,\infty)$ with $\phi(0) = 0$ and $t(x)$ be absolutely continuous on $[0,t]$, $x(0) = 0$ with Lebesque-Stieltjes integrable function with respect to $g(x)$. Then, it follows that*

$$\int_0^t \Delta g(x)\phi'(t(x))t'(x) \leq t^{\frac{1-\zeta}{\zeta}} \left(\int_0^t \Delta g(x)t'(x)\right)^{1+\zeta}. \quad (2.1)$$

Proof. By Jensen inequality, we let $\varphi(t) = t^{\zeta}$ in (1.2) and have

$$\varphi^{-1}\left(\int_0^t \Delta g(y)\phi(y)\right) \leq t^{\frac{1-\zeta}{\zeta}} \left(\int_0^t \Delta g(y)\varphi^{-1}(\phi(y))\right)$$

$$\phi'(y) \leq t^{\frac{\zeta-1}{\zeta}} \varphi\left(\int_0^t \Delta g(y)\phi(y)^{\frac{1}{\zeta}}\right)$$

that is

$$\phi'(y) \leq t^{\frac{\zeta-1}{\zeta}} \varphi\left(\int_0^t \Delta g(y)\phi(y)\right). \quad (2.2)$$

Assuming

$$y(x) = \int_0^t \Delta g(y)t'(x)$$

which can be written as

$$t'(x) = y'(x)^{\zeta}. \quad (2.3)$$

Combining (2.2) and (2.3) and also integrate with respect to $g(x)$ yields

$$\int_0^t \Delta g(x)\phi'(t(x))t'(x) \le t^{\frac{1-\zeta}{\zeta}} \int_0^t \Delta g(x)\phi'(y(t))y'(t)^\zeta = t^{\frac{\zeta-1}{\zeta}}(y(t))^{1+\zeta}$$
$$= t^{\frac{1-\zeta}{\zeta}} \left(\int_0^t \Delta g(x)t'(x)\right)^{\zeta+1} \tag{2.4}$$

which is Godunova and Levin result when $\mathbb{T} = \mathbb{R}$. □

Remark 2.2. If $\int_0^t \Delta g(y)\phi(y)^{\frac{1}{\zeta}} \le \int_0^t \Delta g(y)\phi(y)$, then

$$\phi'(y) \le t^{\frac{1-\zeta}{\zeta}} \varphi\left(\int_0^t \Delta g(y)\phi(y)\right) \tag{2.5}$$

(2.5) is a direct consequence of (2.2). Further simplification yields

$$\int_0^t \Delta g(x)\phi'(t(x))t'(x) \le t^{\frac{1-\zeta}{\zeta}} \int_0^t \Delta g(x)\phi'(y(t))y'(t) = t^{\frac{1-\zeta}{\zeta}}(y(t))^\zeta$$
$$= t^{\frac{1-\zeta}{\zeta}} \left(\int_0^t \Delta g(x)t'(x)\right)^\zeta. \tag{2.6}$$

However, if we choose t so that

$$y(t) = \int_a^t \Delta g(x)t'(x) = \int_t^\Omega \Delta g(x)t'(x) = \frac{1}{2}\int_0^\Omega \Delta g(x)t'(x) \qquad \forall \quad t \in [0,\Omega]. \tag{2.7}$$

Combination of (2.6) and (2.7) gives

$$\int_0^\Omega \Delta g(x)\phi'(t(x))t'(x) \le t^{\frac{\zeta-1}{\zeta}} \left(\frac{1}{2}\int_0^\Omega \Delta g(x)t(x)\right)^\zeta. \tag{2.8}$$

Theorem 2.3. *Let $\mathbb{T}$ be a time scale with $x, y, t \in \mathbb{T}$. If ζ is a real number, $x, y, t \in C_{rd}([0,t]_{\mathbb{T}}, \mathbb{R})$ where $\phi(y)$ is positive rd-continuous function on $[0,t]_{\mathbb{T}}$. Suppose $\phi(t)$ and $\Phi(t)$ are convex and increasing functions on $[0,\infty)$ with $\phi(0) = 0$ and $t(x)$ is absolutely continuous on $[0,t]$ satisfying $t(0) = t(x) = 0$. Then,*

$$\int_0^\Omega \Delta g(x)\phi'(t(x))t'(x) \le 2\left(\left[\int_0^\Omega \Delta g(x)\mu(x)\right]^{\frac{\zeta-1}{\zeta}} \left(\int_0^\Omega \Delta g(x)\left(\frac{t'(x)}{2\mu(x)}\right)^\zeta \mu(x)\right)^{\frac{1}{\zeta}}\right)^\zeta.$$

Proof. From (1.2) we have

$$\left(\frac{\left(\frac{\int_0^\Omega \Delta g(x)t'(x)}{2\mu(x)}\right)\mu(x)}{\int_0^\Omega \Delta g(x)\mu(x)}\right)^\zeta \leq \frac{\int_0^\Omega \Delta g(x)\left(\frac{t'(x)}{2\mu(x)}\right)^\zeta \mu(x)}{\int_0^\Omega \Delta g(x)\mu(x)}, \tag{2.9}$$

$$\left(\left(\frac{\int_0^\Omega t'(x)\Delta g(x)}{2\mu(x)}\right)\mu(x)\right)^\zeta \leq \left[\int_0^\Omega \Delta g(x)\mu(x)\right]^{\zeta-1}\int_0^\Omega \Delta g(x)\left(\frac{t'(x)}{2\mu(x)}\right)^\zeta \mu(x), \tag{2.10}$$

$$\frac{1}{2}\int_0^\Omega \Delta g(x)t'(x) \leq \left[\int_0^\Omega \Delta g(x)\mu(x)\right]^{\frac{\zeta-1}{\zeta}}\left(\int_0^\Omega \Delta g(x)\left(\frac{t'(x)}{2\mu(x)}\right)^\zeta \mu(x)\right)^{\frac{1}{\zeta}}. \tag{2.11}$$

By combining (2.8) and (2.11) yields

$$\int_0^\Omega \Delta g(x)\phi'(t(x))t'(x) \leq 2\left(\left[\int_0^\Omega \Delta g(x)\mu(x)\right]^{\frac{\zeta-1}{\zeta}}\left(\int_0^\Omega \Delta g(x)\left(\frac{t'(x)}{2\mu(x)}\right)^\zeta \mu(x)\right)^{\frac{1}{\zeta}}\right)^\zeta. \tag{2.12}$$

□

Theorem 2.4. *Let $\mathbb{T}$ be a time scale with $x, y, t \in \mathbb{T}$. Suppose ς and ζ are real numbers, $x, y, t \in C_{rd}([0,x]_\mathbb{T}, \mathbb{R})$ where y and $g(y)$ are positive rd-continuous functions on $[0,\Omega]_\mathbb{T}$ such that $\int_{[0,t]} \Delta g(s)\phi(s) < \infty$. If $\phi(t)$ is a convex and increasing function on $[0,\infty)$ with $\phi(0) = 0$, $t_1(x)$ and $t_2(x)$ are absolutely continuous on $[0,x]$ with $t(0) = 0$. Suppose $\phi_1(t)$ and $\phi_2(t)$ are continuous differentiable functions defined on $[0,t]$ and also are positive convex on $[0,\infty)$ Then, the inequality below follows:*

$$\begin{aligned}&\int_0^t \Delta g(x)\left(\phi_1(t_1(x))\phi_2'(t_2(x))t_2'(x) + \phi_2(t_2(x))\phi_1'(t_1(x))t_1'(x)\right)\\ &\leq \left(\left[\int_0^t \Delta g(x)\mu(x)\right]^{\frac{\zeta-1}{\zeta}}\left(\int_0^t \Delta g(x)\left(\frac{t_1'(x)}{\mu(x)}\right)^\zeta \mu(x)\right)^{\frac{1}{\zeta}}\right)^\zeta\\ &\times\left(\left[\int_0^t \Delta g(x)\mu(x)\right]^{\frac{\zeta-1}{\zeta}}\left(\int_0^t \Delta g(x)\left(\frac{t_2'(x)}{\mu(x)}\right)^\zeta \mu(x)\right)^{\frac{1}{\zeta}}\right)^\zeta.\end{aligned} \tag{2.13}$$

Proof. The proof is similar to the latter Theorem.
By applying (1.2), it gives

$$\phi_1(t_1(x))\phi_2'(t_2(x)) \le t^{\frac{\zeta-1}{\zeta}} \left(\int_0^t \Delta g(t)\phi(y_1)^\zeta\right)^{\frac{1}{\zeta}} \tag{2.14}$$

and

$$\phi_2(t_2(x))\phi_1'(t_1(x)) \le t^{\frac{\zeta-1}{\zeta}} \left(\int_0^t \Delta g(t)\phi(y_2)^\zeta\right)^{\frac{1}{\zeta}}. \tag{2.15}$$

Taking addition of (2.14) and (2.15) and integrate over $[0,t]$ with respect to $g(x)$ yields

$$\begin{aligned}
&\int_0^t \Delta g(x)\left(\phi_1(t_1(x))\phi_2'(t_2(x))t_2'(x) + \phi_2(t_2(x))\phi_1'(t_1(x))t_1'(x)\right) \\
&\le \left(\left[\int_0^t \Delta g(x)\mu(x)\right]^{\frac{\zeta-1}{\zeta}} \left(\int_0^t \Delta g(x)\left(\frac{t_1'(x)}{\mu(x)}\right)^\zeta \mu(x)\right)^{\frac{1}{\zeta}}\right)^\zeta \\
&\times \left(\left[\int_0^t \Delta g(x)\mu(x)\right]^{\frac{\zeta-1}{\zeta}} \left(\int_0^t \Delta g(x)\left(\frac{t_2'(x)}{\mu(x)}\right)^\zeta \mu(x)\right)^{\frac{1}{\zeta}}\right)^\zeta.
\end{aligned} \tag{2.16}$$

If $t_1(x) = t_1(x) = t(x), \phi_2((x)) = \phi_1(x)$ and $\zeta = 1$, (2.16) reduces to the following interesting inequality

$$\begin{aligned}
&\int_0^t \Delta g(x)\left(\phi_1(t_1(x))\phi_2'(t_2(x))t_2'(x) + \phi_2(t_2(x))\phi_1'(t_1(x))t_1'(x)\right) \\
&\le \left(\int_0^t \Delta g(x)\left(\frac{t_1'(x)}{\mu(x)}\right)\mu(x)\right)^2.
\end{aligned} \tag{2.17}$$

Furthermore, let $\phi(y)$ and $\Phi(y)$ be convex functions on $[0,\infty)$, $\phi(0) = \Phi(0) = 0$, $\phi'(y) \le \phi(y)$ and by applying (2.8) we have,

$$\phi(t(x))\Phi(t'(x)) \le t^{\frac{\zeta-1}{\zeta}} \left(\int_0^t \phi(y)^\zeta \Delta g(t)\right)^{\frac{1}{\zeta}} \Phi(t'(x)). \tag{2.18}$$

Integrate (2.18) with respect to $g(x)$ yields

$$\int_0^\Omega \Delta g(x)\phi(t(x))\Phi(t'(x)) \le t^{\frac{\zeta-1}{\zeta}} \int_0^\Omega \Delta g(x)\left(\int_0^t \Delta g(x)\phi(t(x))^\zeta\right)^{\frac{1}{\zeta}} \Phi(t'(x)). \tag{2.19}$$

Assuming

$$\Phi\left(\int_0^t \Delta g(x)t'(x)\right) = t\left(\Phi\left(\frac{\int_0^t \Delta g(x)t'(x)}{\int_0^t \Delta g(x)}\right)\right)$$

which is

$$\begin{aligned}&\left(\int_0^t \Delta g(x)t'(x)\right) \leq \left(\int_0^t \Delta g(x)t'(x)^{\zeta}\right)\\&= t\Phi^{-1}\left(\Phi\left(\frac{\int_0^t \Delta g(x)t'(x)^{\zeta}}{\int_0^t \Delta g(x)}\right)\right) \leq t\Phi^{-1}\left(\frac{1}{t}\int_0^t \Delta g(x)\Phi(t'(x))^{\zeta}\right).\end{aligned} \tag{2.20}$$

In view of (2.18) and (2.20) we have

$$\int_0^{\Omega} \Delta g(x)\phi(t(x))\Phi(t'(x)) \leq t^{\frac{\zeta-1}{\zeta}+1}\int_0^{\Omega}\left(\Phi^{-1}\left(\left(\frac{1}{t}\int_0^t \Delta g(x)\Phi(t'(x))^{\zeta}\right)^{\frac{1}{\zeta}}\right)\right) \times \Phi(t'(x))\Delta g(x)$$

if $a = \frac{1}{t}\int_0^t \Phi(t'(x))\Delta g(x)$ and $b = \Phi(t'(x))$ in latter inequality yields

$$\begin{aligned}\int_0^{\Omega} \Delta g(x)\phi(t(x))\Phi(t'(x)) &\leq t^{\frac{\zeta-1}{\zeta}+1}\int_0^{a} \Delta g(t)\phi\left(t\Phi^{-1}(t)\right)\\&= \int_0^{\Omega}\int_0^{b} \Delta g(t)\Delta g(x)\phi\left(t\Phi^{-1}(t)\right)\end{aligned} \tag{2.21}$$

which is Qi's result when $\mathbb{T} = \mathbb{R}$. □

3 Refined Sub-variate and Multi-variate Opial-type Inequalities on Time Scales

We present our main results on Jensen inequality for functions of several variables on time scale as follows:

Theorem 3.1. *Let $\mathbb{T}$ be a time scale with $x, y, t \in \mathbb{T}$. Let ζ be real numbers, let $x, y, t \in C_{rd}([0,t]_{\mathbb{T}}, \mathbb{R})$ where y and $\phi(y)$ are positive rd-continuous functions on $[0,t]_{\mathbb{T}}$. Let $t(x)$ be absolutely continuous function which is non-decreasing on $[0, x]$*

and let $t_i, \mu_i \geq 0 \forall i = 1, 2$ and $\varphi(x)$ be convex function, $\mu(x)$ be Lebesque-Stieltjes integrable function with respect to $g(x)$. Then the following inequality holds:

$$\int_0^x \Delta g(s) \left(\frac{t_1'(x) + t_2'(x)}{(\mu_1'(x) + \mu_2'(x))} \right)^\zeta (\mu_1'(x) + \mu_2'(x)) \varphi' \left(\frac{t_1(x) + t_2(x)}{(\mu_1(x) + \mu_2(x))} (\mu_1(x) + \mu_2(x)) \right)$$
$$\leq \varphi \left(\int_0^x \Delta g(s)(\mu_1'(x) + \mu_2'(x)) \left(\frac{y_1'(s) + y_2'(s)}{\mu_1'(x) + \mu_2'(x)} \right)^\zeta \right).$$

Proof. From (1.2), we have

$$\left(\frac{\int_0^x \Delta g(s) \frac{t_1'(s)+t_2'(s)}{\mu_1'(s)+\mu_2'(s)} (\mu_1'(s) + \mu_2'(s))}{\int_0^x \Delta g(s) \mu_1'(s) + \mu_2'(s)} \right)^\zeta$$
$$\leq \frac{1}{\mu_1(x) + \mu_2(x)} \int_0^x \Delta g(s)(\mu_1'(s) + \mu_2'(s)) \left(\frac{y_1'(s) + y_2'(s)}{\mu_1'(s) + \mu_2'(s)} \right)^\zeta. \tag{3.1}$$

(3.1) can be written as

$$\frac{\left(\int_0^x \Delta g(s) \frac{(t_1'(s)+t_2'(s))}{(\mu_1'(s)+\mu_2'(s))} (\mu_1'(s) + \mu_2'(s)) \right)^\zeta}{\left(\int_0^x \Delta g(s)(\mu_1'(s) + \mu_2'(s)) \right)^\zeta}$$
$$\leq \frac{1}{\mu_1(x) + \mu_2(x)} \int_0^x \Delta g(s)(\mu_1'(s) + \mu_2'(s)) \left(\frac{y_1'(s) + y_2'(s)}{\mu_1'(s) + \mu_2'(s)} \right)^\zeta$$
$$= \frac{\left(\int_0^x \Delta g(s)(\mu_1'(s) + \mu_2'(s)) \right)^\zeta}{\mu_1(x) + \mu_2(x)} \int_0^x \Delta g(s)(\mu_1'(s) + \mu_2'(s)) \left(\frac{y_1'(s) + y_2'(s)}{\mu_1'(s) + \mu_2'(s)} \right)^\zeta$$
$$= \frac{(\mu_1(s) + \mu_2(s))^\zeta}{\mu_1(x) + \mu_2(x)} \int_0^x \Delta g(s)(\mu_1'(s) + \mu_2'(s)) \left(\frac{y_1'(s) + y_2'(s)}{\mu_1'(s) + \mu_2'(s)} \right)^\zeta$$
$$\leq \frac{(\mu_1(s) + \mu_2(s))}{\mu_1(x) + \mu_2(x)} \int_0^x \Delta g(s)(\mu_1'(s) + \mu_2'(s)) \left(\frac{y_1'(s) + y_2'(s)}{\mu_1'(s) + \mu_2'(s)} \right)^\zeta \tag{3.2}$$

which can be expressed in the following form with the fact that $y(t) = \int_0^x \mu(x) dg(x)$

$$\int_0^x \Delta g(s) \frac{t_1'(s) + t_2'(s)}{(\mu_1'(s) + \mu_2'(s))} (\mu_1'(s) + \mu_2'(s)) \leq \int_0^x \Delta g(s)(\mu_1'(s) + \mu_2'(s)) \varphi \left(\frac{y_1'(s) + y_2'(s)}{(\mu_1'(s) + \mu_2'(s))} \right) \tag{3.3}$$

further estimation of the inequality yields

$$\begin{aligned}
&\int_0^x \Delta g(s)\left(\frac{t_1'(x)+t_2'(x)}{(\mu_1'(x)+\mu_2'(x))}\right)^{\zeta}(\mu_1'(x)+\mu_2'(x))\varphi'\left(\frac{t_1(x)+t_2(x)}{(\mu_1(x)+\mu_2(x))}(\mu_1(x)+\mu_2(x))\right)\\
&\leq \int_0^x \Delta g(s)\left(\mu_1'(x)+\mu_2'(x)\right)\left(\frac{y_1'(x)+y_2'(x)}{\mu_1'(x)+\mu_2'(x)}\right)^{\zeta}\\
&\times \varphi'\left(\int_0^x \Delta g(s)\left(\mu_1'(s)+\mu_2'(s)\right)\left(\frac{y_1'(s)+y_2'(s)}{\mu_1'(s)+\mu_2'(s)}\right)^{\zeta}\right)\\
&= \int_0^x \Delta g(s)\varphi\left(\int_0^x \Delta g(s)\left(\mu_1'(s)+\mu_2'(s)\right)\left(\frac{y_1'(s)+y_2'(s)}{\mu_1'(s)+\mu_2'(s)}\right)^{\zeta}\right)'\\
&= \varphi\left(\int_0^x \Delta g(x)\left(\mu_1'(x)+\mu_2'(x)\right)\left(\frac{y_1'(x)+y_2'(x)}{\mu_1'(x)+\mu_2'(x)}\right)^{\zeta}\right)
\end{aligned} \tag{3.4}$$

that is

$$\begin{aligned}
&\int_0^x \Delta g(x)\left(\frac{t_1'(x)+t_2'(x)}{(\mu_1'(x)+\mu_2'(x))}\right)^{\zeta}(\mu_1'(x)+\mu_2'(x))\varphi'\left(\frac{t_1(x)+t_2(x)}{(\mu_1(x)+\mu_2(x))}(\mu_1(x)+\mu_2(x))\right)\\
&\leq \varphi\left(\int_0^x \Delta g(x)\left(\mu_1'(x)+\mu_2'(x)\right)\left(\frac{y_1'(s)+y_2'(s)}{\mu_1'(x)+\mu_2'(x)}\right)^{\zeta}\right)
\end{aligned} \tag{3.5}$$

which is Rozanova's result. In general, the above can be extended to n^{th} term as

$$\begin{aligned}
&\int_0^x \Delta g(x)\left(\frac{t_1'(x)+...+t_n'(x)}{(\mu_1'(x)+...+\mu_n'(x))}\right)^{\zeta}(\mu_1'(x)+\cdots+\mu_2'(x))\\
&\times \varphi'\left(\frac{t_1(x)+\cdots+t_n(x)}{(\mu_1(x)+\cdots+\mu_n(x))}(\mu_1(x)+\cdots+\mu_n(x))\right)\\
&\leq \varphi\left(\int_0^x \Delta g(x)\mu_1'(x)+\cdots+\mu_n'(x)\left(\frac{\mu_1'(x)+\cdots+\mu_n'(x)}{\mu_1'(x)+\cdots+\mu_n'(x)}\right)^{\zeta}\right)
\end{aligned} \tag{3.6}$$

which implies

$$\begin{aligned}
&\int_0^x \Delta g(x) \prod_{i=1}^{n} \left(\frac{\sum_{i=1}^n t_n'(x)}{\sum_{i=1}^n \mu_n'(x)} \right)^{\zeta} \left(\sum_{i=1}^{n} \mu_n'(x) \right) \\
&\times \varphi' \left(\frac{\sum_{i=1}^n t_n(x)}{(\sum_{i=1}^n \mu_n(x))} (\sum_{i=1}^{n} \mu_n(x)) \right) \\
&\le \varphi \left(\int_0^x \Delta g(x) \left(\sum_{i=1}^{n} \mu_n'(x) \right) \left(\frac{\sum_{i=1}^n y_n'(x)}{\sum_{i=1}^n \mu_n'(x)} \right)^{\zeta} \right).
\end{aligned} \tag{3.7}$$

□

Theorem 3.2. *Let $\mathbb{T}$ be a time scale with $v, \mu, x, y, t \in \mathbb{T}$. Let ζ be real numbers, let $x, y, t \in C_{rd}([0,t]_{\mathbb{T}}, \mathbb{R})$ where y and $\phi(y)$ are positive rd-continuous functions on $[0,t]_{\mathbb{T}}$. Let $\varphi_i(t) = t^{\zeta_i}$ are non-negative, convex and increasing functions on $(0,\infty)$. If $t_i(x)$ are absolutely continuous defined on $[0,\Omega]$, satisfying $t_i(0) = 0$ $\forall i \in \mathbb{N}$. Then,*

$$\begin{aligned}
&\int_0^{\Omega} \Delta g(x) \mu_2'(s) \left(\frac{t_2'(s)}{\mu_2'(s)} \right)^{\zeta_2} v_1(s) \left(\mu_1(s) \left(\frac{t_1(s)}{\mu_1(s)} \right)^{\zeta_1} \right) v_2'(s) \left(\mu_2(s) \left(\frac{t_2(s)}{\mu_2(s)} \right)^{\zeta_2} \right) \\
&+ \int_0^{\Omega} \Delta g(x) \mu_1'(s) \left(\frac{t_1'(s)}{\mu_1'(s)} \right)^{\zeta_1} v_2(s) \left(\mu_2(s) \left(\frac{t_2(s)}{\mu_2(s)} \right)^{\zeta_2} \right) v_1'(s) \left(\mu_1(s) \left(\frac{t_1(s)}{\mu_1(s)} \right)^{\zeta_1} \right) \\
&+ \int_0^{\Omega} \Delta g(x) \mu_0'(s) \left(\frac{t_0'(s)}{\mu_0'(s)} \right)^{\zeta_0} v_3(s) \left(\mu_3(s) \left(\frac{t_3(s)}{\mu_3(s)} \right)^{\zeta_2} \right) v_0'(s) \left(\mu_0(s) \left(\frac{t_0(s)}{\mu_0(s)} \right)^{\zeta_1} \right) \\
&+ \cdots + \int_0^{\Omega} \Delta g(x) \mu_{3-n}'(s) \left(\frac{t_{3-n}'(s)}{\mu_{3-n}'(s)} \right)^{\zeta_{3-n}} v_n(s) \left(\mu_n(s) \left(\frac{t_n(s)}{\mu_n(s)} \right)^{\zeta_n} \right) \\
&\times v_{3-n}'(s) \left(\mu_{3-n}(s) \left(\frac{t_0(s)}{\mu_{3-n}(s)} \right)^{\zeta_{3-n}} \right) \\
&\le v_1 \left(\int_0^{\Omega} \Delta g(s) \mu_1'(s) \left(\frac{t_1'(t)}{\mu_1'(s)} \right)^{\zeta_1} \right) \times v_2 \left(\int_0^{\Omega} \mu_2'(s) \Delta g(x) \left(\frac{t_2'(s)}{\mu_2'(s)} \right)^{\zeta_2} \right) \\
&\times v_3 \left(\int_0^{\Omega} \Delta g(s) \mu_3'(s) \left(\frac{t_3'(t)}{\mu_3'(s)} \right)^{\zeta_3} \right) \times \cdots \times v_n \left(\int_0^{\Omega} v \mu_n' \left(\frac{t_n'(s)}{\mu_n'(s)} \right)^{\zeta_n} \right).
\end{aligned} \tag{3.8}$$

Proof. The proof is similar to the proof of **Theorem 3.1** by combining (1.2) and (3.1), it follows that

$$\left(\frac{\int_0^x \Delta g(s)\frac{t_1'(s)}{\mu_1'(s)}\mu_1'(s)}{\int_0^x \Delta g(s)\mu_1'(s)}\right)^{\zeta} \leq \frac{1}{\mu_1(x)}\int_0^x \Delta g(s)\mu_1'(s)\left(\frac{y_1'(s)}{\mu_1'(s)}\right)^{\zeta}. \tag{3.9}$$

By simplification, (3.9) yields

$$\begin{aligned}
&\int_0^{\Omega} \Delta g(x)\mu_2'(s)\left(\frac{t_2'(s)}{\mu_2'(s)}\right)^{\zeta_2} v_1(s)\left(\mu_1(s)\left(\frac{t_1(s)}{\mu_1(s)}\right)^{\zeta_1}\right) v_2'(s)\left(\mu_2(s)\left(\frac{t_2(s)}{\mu_2(s)}\right)^{\zeta_2}\right)\\
&\quad+\int_0^{\Omega} \Delta g(x)\mu_1'(s)\left(\frac{t_1'(s)}{\mu_1'(s)}\right)^{\zeta_1} v_2(s)\left(\mu_2(s)\left(\frac{t_2(s)}{\mu_2(s)}\right)^{\zeta_2}\right) v_1'(s)\left(\mu_1(s)\left(\frac{t_1(s)}{\mu_1(s)}\right)^{\zeta_1}\right)\\
&\quad+\int_0^{\Omega} \Delta g(x)\mu_0'(s)\left(\frac{t_0'(s)}{\mu_0'(s)}\right)^{\zeta_0} v_3(s)\left(\mu_3(s)\left(\frac{t_3(s)}{\mu_3(s)}\right)^{\zeta_2}\right) v_0'(s)\left(\mu_0(s)\left(\frac{t_0(s)}{\mu_0(s)}\right)^{\zeta_1}\right)\\
&\quad+\cdots+\int_0^{\Omega} \Delta g(x)\mu_{3-n}'(s)\left(\frac{t_{3-n}'(s)}{\mu_{3-n}'(s)}\right)^{\zeta_{3-n}} v_n(s)\left(\mu_n(s)\left(\frac{t_n(s)}{\mu_n(s)}\right)^{\zeta_n}\right)\\
&\quad\times v_{3-n}'(s)\left(\mu_{3-n}(s)\left(\frac{t_0(s)}{\mu_{3-n}(s)}\right)^{\zeta_{3-n}}\right)\\
&\leq \int_0^{\Omega} \Delta g(x)\mu_2'(s)\left(\frac{y_2'(s)}{\mu_2'(s)}\right)^{\zeta_2} v_1(s)\left(\int_0^t \Delta g(s)\mu_1'(s)\left(\frac{y_1'(s)}{\mu_1'(s)}\right)^{\zeta_1}\right) v_2'(s)\left(\int_0^t \Delta g(s)\mu_2(s)\left(\frac{y_2'(s)}{\mu_2'(s)}\right)^{\zeta_2}\right)\\
&\quad+\int_0^{\Omega} \Delta g(s)\mu_1'(s)\left(\frac{y_1'(s)}{\mu_1'(s)}\right)^{\zeta_1} v_2(s)\left(\int_0^t \Delta g(s)\mu_2'(s)\left(\frac{y_2'(s)}{\mu_2'(s)}\right)^{\zeta_2}\right) v_1'(s)\left(\int_0^t \Delta g(s)\mu_1(s)\left(\frac{y_1'(s)}{\mu_1'(s)}\right)^{\zeta_1}\right)\\
&\quad+\int_0^{\Omega} \Delta g(x)\mu_0'(s)\left(\frac{y_0'(s)}{\mu_0'(s)}\right)^{\zeta_0} v_3(s)\left(\int_0^t dg(s)\mu_3'(s)\left(\frac{y_3'(s)}{\mu_3'(s)}\right)^{\zeta_3}\right) v_0'(s)\left(\int_0^t dg(s)\mu_0(s)\left(\frac{y_0'(s)}{\mu_0'(s)}\right)^{\zeta_0}\right)\\
&\quad+\cdots+\int_0^{\Omega} dg(x)\mu_{3-n}'(s)\left(\frac{y_{3-n}'(s)}{\mu_{3-n}'(s)}\right)^{\zeta_{3-n}} v_n(s)\left(\int_0^t dg(s)\mu_n'(s)\left(\frac{y_n'(s)}{\mu_n'(s)}\right)^{\zeta_n}\right)\\
&\quad\times v_{3-n}'(s)\left(\int_0^t dg(s)\mu_{3-n}(s)\left(\frac{y_{3-n}'(s)}{\mu_{3-n}'(s)}\right)^{\zeta_{3-n}}\right)\\
&=\int_0^{\Omega} \Delta g(x)\left[v_1\left(\mu_1'(s)\int_0^x \Delta g(s)\left(\frac{y_1'(s)}{\mu_1'(s)}\right)^{\zeta_1}\right)\times v_2\left(\mu_2'(s)\int_0^x \Delta g(s)\left(\frac{y_2'(s)}{\mu_2'(s)}\right)^{\zeta_2}\right)\right.\\
&\quad\left.\times v_3\left(\mu_3'(s)\int_0^x \Delta g(s)\left(\frac{y_3'(s)}{\mu_3'(s)}\right)^{\zeta_3}\right)\times\cdots.\times v_n\left(\mu_n'(s)\int_0^x \Delta g(s)\left(\frac{y_n'(s)}{\mu_n'(s)}\right)^{\zeta_n}\right)\right]'\\
&=v_1\left(\int_0^{\Omega} \Delta g(x)\mu_1'(s)\left(\frac{t_1'(t)}{\mu_1'(s)}\right)^{\zeta_1}\right)\times v_2\left(\int_0^{\Omega} \Delta g(x)\mu_2'(s)\left(\frac{t_2'(s)}{\mu_2'(s)}\right)^{\zeta_2}\right)\\
&\quad\times v_3\left(\int_0^{\Omega} \Delta g(x)\mu_3'(s)\left(\frac{t_3'(t)}{\mu_1'}\right)^{\zeta_3}\right)\times\cdots.\times v_n\left(\int_0^{\Omega} \Delta g(x)\mu_n'\left(\frac{t_n'(s)}{\mu_n'(s)}\right)^{\zeta_n}\right).
\end{aligned} \tag{3.10}$$

In conclusion, if $\mathbb{R} = \mathbb{T}$, $\mu \geq 2$, $\mu_1(x) = \mu_2(x) = \mu(x)$, $\varphi_1(x) = \varphi_2(x) = \varphi(x)$, $t_1(x) = t_2(x) = t(x)$ and $v_1(x) = v_2(x) = v(x) = 1$, then (3.10) yields

$$\begin{aligned}
&\int_0^\Omega \Delta g(x)\, \mu_2'(s) \left(\frac{t_2'(s)}{\mu_2'(s)}\right)^{\zeta_2} v_2'(s) \left(\mu_2(s) \left(\frac{t_2(s)}{\mu_2(s)}\right)^{\zeta_2}\right) \left(\mu_1(s) \left(\frac{t_1(s)}{\mu_1(s)}\right)^{\zeta_1}\right)\Bigg) \\
&+ \int_0^\Omega \Delta g(x) \mu_1'(s) \left(\frac{t_1'(s)}{\mu_1'(s)}\right)^{\zeta_1} v_1'(s) \left(\mu_1(s) \left(\frac{t_1(s)}{\mu_1(s)}\right)^{\zeta_1}\right) \left(\mu_2(s) \left(\frac{t_2(s)}{\mu_2(s)}\right)^{\zeta_2}\right) \\
&\leq v_1 \left(\int_0^\Omega \Delta g(x) \mu_1'(s) \left(\frac{t_1'(t)}{\mu_1'(s)}\right)^{\zeta_1}\right) v_2 \left(\int_0^\Omega \Delta g(x) \mu_2'(s) \left(\frac{t_2'(s)}{\mu_2'(s)}\right)^{\zeta_2}\right).
\end{aligned} \tag{3.11}$$

Also if $\mathbb{R} = \mathbb{T}$ and $v_1(x) = v_2(x) = v(x)$ in (3.11), we obtain

$$\begin{aligned}
&\int_0^\Omega \Delta g(x) \mu'(s) \left(\frac{t'(s)}{\mu'(s)}\right)^{\zeta} v'(s) \left(\mu(s) \left(\frac{t(s)}{\mu(s)}\right)^{\zeta}\right) \left(\mu(s) \left(\frac{t(s)}{\mu(s)}\right)^{\zeta}\right) \\
&\leq \frac{1}{2} \left(v \left(\int_0^\Omega \Delta g(x) \mu_1'(s) \left(\frac{t'(s)}{\mu'(s)}\right)\right)\right)^2.
\end{aligned} \tag{3.12}$$

□

Remark 3.3. Inequality (1.2) was employed to obtain the generalized Opial-type inequalities for convex functions of several variables on time scales.

Conflicts of Interest

The authors declare that they have no competing interests.

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