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Outtopological Digraph Space and Some Related Properties

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Abstract

The outtopological digraph space, a novel topological constraint imposed by a subbasis, is introduced in this work. A subbasis $\overleftarrow{\mathrm{I}_{ẹ}}^{v}$ is a set contains one vertex such that the edge ẹ is outdegree of it. As a result, various theorems have been established. A characterization and some examples are provided to describe this new structure.

1. Introduction

For two reasons, graph theory is recognized as a fundamental idea in independent mathematics and is a helpful mathematical tool in many different contexts. Initially, graphs are selected theoretically from a theoretical standpoint. Graphs can perform topological space, collection objects, and many other mathematical groups even if they are merely simple relational combinations. The second justification is that using graphs to represent some ideas makes them more applicable in practical contexts. Regarding the relationship between graph theory and topology, one of the graph's tools, such as transforming a set of edges or a set of vertices to topological space and exploring other topological ideas of this space, can be used to express topological concepts. Topology is one of the most well-known and contemporary topics that has occupied a wide area of mathematicians. Several earlier studies on the subject of topological graphs are included below. Evans and Harary [1] first proposed the concept of topology on digraphs in 1967. Between the collection of all topologies with n vertices, they discovered only one relationship. Bhargava and Ahlborn [2] looked at the topological space connected to digraphs in 1968. They expanded the previous finding to encompass infinite graphs. In 1983, Majumdar [3] created graph topology from continuous multivalued functions that

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was connected between a dense subset of topology. A domination set of a graph and a dense subset of topology were linked by Subramanian in 2001 [4]. A novel idea in topology on a signed graph and topology on transitive products of a signed graph was researched by Subbiah [5] in 2007. Karanakaram [6] established topology on a graph G from a collection of spanning subgraphs of G in the same year. Thomas [7] investigated topology in 2013 and determined the topological numbers of several graphs using set indexers. By using two fixed vertices and determining vertex and edge incidence depending on the distance between them, Shokry [8] described a new technique for creating graph topology in 2015. When applying topology to a digraph in 2018, Abdu and Kilicman [9] associated two topologies with the set of edges dubbed compatible and incompatible edges topologies. In furthermore, Khalid Al'Dzhabri [10] presented new topological space structures connected to digraphs in 2020 by combining new topologies with digraphs that were generated from particular open sets known as DG-topological space. A few more kinds of open sets linked to graphs were also introduced in 2020 by Khalid Al'Dzhabri [11].

2. Preliminaries

In this work, some basic notions of graph theory [12], and topology [13] are presented.

Definition 2.1. A directed graph or digraph $Ɗ = (Ṿ, Ẹ)$ consists of a vertex set $Ṿ$ and an edge set $Ẹ$ of ordered pairs of elements of $Ṿ$.

Definition 2.2. Let $Ɗ = (Ṿ, Ẹ)$ be a digraph. We call H a subdigraph of $Ɗ$ if $Ṿ(H) \subseteq Ṿ(Ɗ), Ẹ(H) \subseteq Ẹ(Ɗ)$. In this case we would write $H \subseteq Ɗ$.

Definition 2.3. Let $Ɗ = (Ṿ, Ẹ)$ be a digraph. We say that two vertices $ṿ$ and w of a graph (resp., digraph $Ɗ$) are adjacent if there is an edge of the form $ṿw$ (resp., $\overrightarrow{wṿ}$ or $\overrightarrow{ṿw}$) joining them, and the vertices $ṿ$ and w are then incident with such an edge.

Definition 2.4. Let Y is a non-empty set. A collection $\tau \subseteq P(Y)$ is called topology on Y if the followings hold:

(1) $Y, \emptyset \in \tau$.

(2) The intersection of a finite number of elements in τ is in τ.

(3) The union of a finite or infinite number of elements of sets in τ belong to τ.

Then (Y, τ) is called a topological space. Any element in (Y, τ) is called open set and its complement is called closed set.

Definition 2.5. Let Y be a non-empty set and let τ be the collection of every subset from Y. Then τ is named the discrete topology on the set Y. The topological space (Y, τ) is called a discrete space. If $\tau = \{Y, \emptyset\}$, then τ is named indiscrete topology and the topological space (Y, τ) is named an indiscrete topological space.

Definition 2.6. Let (Y, τ) be a topological space, $A \subseteq Y$. The closure of A symbolized by $Cl(A)$ is defined as the smallest closed set that includes A. It is thus the intersection of every closed set that include A.

Definition 2.7. Let (Y, τ) be a topological space, $A \subseteq Y$. The interior of A symbolized by $Int(A)$ is defined as the largest open set included in A. It is thus the union of every open set included in A.

Definition 2.8. Let (Y, τ) be a topological space. Then $A \subseteq Y$ is called dense if $\overline{A} = Y$.

3. Outtopological Digraph Space

To create a topology on the collection of vertices Ṿ, we present our novel family of a digraph Ɗ = (Ṿ, Ẹ).

Definition 3.1. Let Ɗ = (Ṿ, Ẹ) be a digraph. Then define $\overleftarrow{\mathrm{I}_{ẹ}}^{v}$ a set contains one vertex such that the edge ẹ is outdegree of it. Also define $\overleftarrow{\mathrm{S}_{Ɗ}}^{v}$ as follows $\overleftarrow{\mathrm{S}_{Ɗ}}^{v} = Ṿ(Ɗ) \cup \left\{\overleftarrow{\mathrm{I}_{ẹ}}^{v} \middle| ẹ \in Ẹ\right\}$. Hence $\overleftarrow{\mathrm{S}_{Ɗ}}^{v}$ forms a subbasis for a topology $\overleftarrow{\tau_{Ɗ}}^{v}$ on Ṿ called outtopological digraph space $\overleftarrow{\tau_{Ɗ}}^{v}$ (briefly outop. digsp.) of Ɗ.

Example 3.2. Let Ɗ = (Ṿ, Ẹ) be digraph as in Figure 1 such that Ṿ = {ṿ$_1$, ṿ$_2$, ṿ$_3$, ṿ$_4$, ṿ$_5$} and Ẹ = {ẹ$_1$, ẹ$_2$, ẹ$_3$, ẹ$_4$, ẹ$_5$}.

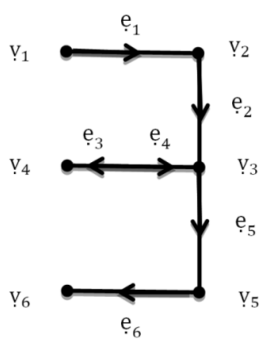

Figure 1

We have

$$\overleftarrow{\text{I}_{e_1}}^{v} = \{v_1\},\ \overleftarrow{\text{I}_{e_2}}^{v} = \{v_2\},\ \overleftarrow{\text{I}_{e_3}}^{v} = \{v_3\},\ \overleftarrow{\text{I}_{e_4}}^{v} = \{v_4\},\ \overleftarrow{\text{I}_{e_5}}^{v} = \{v_3\},\ \overleftarrow{\text{I}_{e_6}}^{v} = \{v_5\}.$$

And

$$\overleftarrow{\text{S}_{D}}^{v} = \{V(D), \{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_5\}\}$$

$$\overleftarrow{\tau_{D}}^{v} = \{V(D), \emptyset, \{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_5\}, \{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_4\}, \{v_1, v_5\}, \{v_2, v_3\},$$
$$\{v_2, v_4\}, \{v_2, v_5\}, \{v_3, v_4\}, \{v_3, v_5\}, \{v_4, v_5\}, \{v_1, v_2, v_3\}, \{v_1, v_2, v_4\}, \{v_1, v_2, v_5\},$$
$$\{v_1, v_3, v_4\}, \{v_1, v_3, v_5\}, \{v_1, v_4, v_5\}, \{v_2, v_3, v_4\}, \{v_2, v_3, v_5\}, \{v_2, v_4, v_5\},$$
$$\{v_3, v_4, v_5\}, \{v_1, v_2, v_3, v_4\}, \{v_1, v_2, v_3, v_5\},$$
$$\{v_1, v_2, v_4, v_5\}, \{v_1, v_3, v_4, v_5\}, \{v_2, v_3, v_4, v_5\}\}.$$

Hence $\overleftarrow{\tau_{D}}^{v}$ is an outop.digsp.

Definition 3.3. Let $D = (V, E)$ be a digraph. Then $\overleftarrow{E_{v}}$ is the set of all edges that outdgree to the vertices v.

Example 3.4. According to Example 3.2, we get

$$\overleftarrow{E_{v_1}} = \{e_1\},\ \overleftarrow{E_{v_2}} = \{e_2\},\ \overleftarrow{E_{v_3}} = \{e_3\},\ \overleftarrow{E_{v_4}} = \{e_4\},\ \overleftarrow{E_{v_5}} = \{e_6\},\ \overleftarrow{E_{v_6}} = \{\emptyset\}.$$

Proposition 3.5. *Let* $\overleftarrow{\tau_{D}}^{v}$ *be an outtop.digsp. of the digraph* $D = (V, E)$. *If* $\overleftarrow{E_{v}} \neq \emptyset$, *then* $\{v\} \in \overleftarrow{\tau_{D}}^{v}$ *for every* $v \in V$.

Proof. Let D be a digraph. Since $\overleftarrow{E_{v}} \neq \emptyset$, we get $\bigcap_{e \in \overleftarrow{E_{v}}} \overleftarrow{\text{I}_{e}}^{v} = \{v\}$ [because $\overleftarrow{\text{I}_{e}}^{v} = \{v\}, \forall\, e \in \overleftarrow{E_{v}}$].

Now by the definition of outop.digsp. $\overleftarrow{\tau_{D}}^{v}$, $\{v\}$ is an element in the basis of outop.digsp. $\overleftarrow{\tau_{D}}^{v}$. Hence $\{v\} \in \overleftarrow{\tau_{D}}^{v}$.

Remark 3.6. Let $D = (V, E)$ be a digraph. Then the outop.digsp. $\overleftarrow{\tau_{D}}^{v}$ is not necessary to be discrete topology, in general.

The following example illustrates the above remark.

Example 3.7. Let C_6 be a cyclic digraph such that all edges are not in the same direction as in Figure 2.

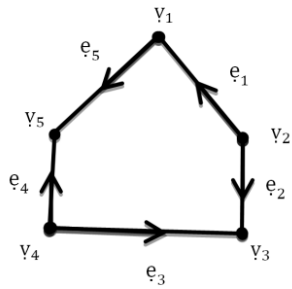

Figure 2. C_5 digraph.

We have

$$\overleftarrow{I_{e_1}}^{v} = \{v_2\},\ \overleftarrow{I_{e_2}}^{v} = \{v_2\},\ \overleftarrow{I_{e_3}}^{v} = \{v_4\},\ \overleftarrow{I_{e_4}}^{v} = \{v_4\},\ \overleftarrow{I_{e_5}}^{v} = \{v_1\}.$$

And

$$\overleftarrow{S_D}^{v} = \{V(D), \{v_1\}, \{v_2\}, \{v_4\}\}$$

$$\overleftarrow{\tau_D}^{v} = \{V(D), \emptyset, \{v_1\}, \{v_2\}, \{v_4\}, \{v_1, v_2\}, \{v_1, v_4\}, \{v_1, v_2, v_4\}\},$$

then we get that the outop.digsp. $\overleftarrow{\tau_D}^{v}$ of C_5 is not discrete.

Corollary 3.8. *Let* $D = (V, E)$ *be a digraph. Then:*

(i) *If* $\overleftarrow{E_v} \neq \emptyset$ *for all* $v \in V$, *then* $\overleftarrow{\tau_D}^{v}$ *is discrete topology.*

(ii) *If* $D = (V, E)$ *is reflexive, then* $\overleftarrow{\tau_D}^{v}$ *is discrete topology.*

(iii) *If* $D = (V, E)$ *is equivalent, then* $\overleftarrow{\tau_D}^{v}$ *is discrete topology.*

(iv) *If* $D = (V, E)$ *is null digraph, then* $\overleftarrow{\tau_D}^{v}$ *is indiscrete topology.*

Proof. Clear.

Proposition 3.9. *The outop.digsp.* $(V, \overleftarrow{\tau_D}^{v})$ *of a digraph* $D = (V, E)$ *satisfies the property of Alexandroff.*

Proof. It is adequate to show that arbitrary intersection of elements of $\overleftarrow{S_D}^{v}$ is open. Let $A \subseteq E$ then either: $\bigcap_{e \in A} \overleftarrow{I_e}^{v} = \emptyset$ is open or $\bigcap_{e \in A} \overleftarrow{I_e}^{v} = \{v\}$ such that $e \in \overleftarrow{E_v}$ for all $e \in A$. This means that $\overleftarrow{E_v} \neq \emptyset$. Then by Proposition 3.5, $\{v\} \in \overleftarrow{\tau_D}^{v}$. Hence $\bigcap_{e \in A} \overleftarrow{I_e}^{v}$ is open. Then the outop.digsp. satisfies the property of Alexandroff.

Definition 3.10. In any digraph $\mathcal{D} = (V, E)$ since $(V, \overleftarrow{\tau_{\mathcal{D}}}^{v})$ is Alexandroff space, for each $v \in V$, the intersection of all open sets containing v is the smallest open set containing v and is denoted by U_v. Also the family $\overleftarrow{M_{\mathcal{D}}}^{v} = \{U_v | v \in V\}$ is the minimal basis for the outop.digsp. $(V, \overleftarrow{\tau_{\mathcal{D}}}^{v})$.

Proposition 3.11. *In any digraph* $\mathcal{D} = (V, E)$, $U_v = \bigcap_{e \in \overrightarrow{E_v}} \overleftarrow{I_e}^{v}$ *for every* $v \in V$.

Proof. Since $\overleftarrow{S_{\mathcal{D}}}^{v}$ is the subbasis of $\overleftarrow{\tau_{\mathcal{D}}}^{v}$ and U_v is the intersection of all open sets containing v, we have $U_v = \bigcap_{e \in A} \overleftarrow{I_e}^{v}$ for some subset A of E, by definition of U_v then $v \in U_v$ and since $U_v = \bigcap_{e \in A} \overleftarrow{I_e}^{v}$ implies $v \in \bigcap_{e \in A} \overleftarrow{I_e}^{v}$ then $v \in \overleftarrow{I_e}^{v}$ for all $e \in A$, since $\overleftarrow{I_e}^{v}$ contains one vertex, then $\overleftarrow{I_e}^{v} = \{v\}$ for all $e \in A$. This leads to $e \in \overleftarrow{E_v}$ for each $e \in A$. Hence $A \subseteq \overleftarrow{E_v}$ and so $v \in \bigcap_{e \in \overleftarrow{E_v}} \overleftarrow{I_e}^{v} \subseteq U_v$. From the definition of U_v the proof is complete.

Remark 3.12. Let $\mathcal{D} = (V, E)$ be a digraph. Then for any $v \in V$

(i) if $\overleftarrow{E_v} \neq \emptyset$, then by Proposition 3.11, $U_v = \bigcap_{e \in \overleftarrow{E_v}} \overleftarrow{I_e}^{v} = \{v\}$.

(ii) if $\overleftarrow{E_v} = \emptyset$, then by Proposition 3.11, $U_v = \bigcap_{e \in \overleftarrow{E_v}} \overrightarrow{I_e}^{v} = V$.

Theorem 3.13. *For any* $u, v \in V$ *in a digraph* $\mathcal{D} = (V, E)$, *we have* $u \in U_v$ *if and only if* $\overleftarrow{E_v} = \emptyset$, *i.e.,* $U_v = \{u \in V | \overleftarrow{E_v} = \emptyset\}$.

Proof. $\Longrightarrow$ To prove $\overleftarrow{E_v} = \emptyset$, let $u \in U_v$. If $\overleftarrow{E_v} \neq \emptyset$, then by Remark 3.12(i), $U_v = \{v\} \Rightarrow u \notin U_v$ is a contradiction with hypothesis, then $\overleftarrow{E_v} = \emptyset$ and hence $\overleftarrow{E_v} = \emptyset$.

$\Longleftarrow$ if $\overleftarrow{E_v} = \emptyset$ and by Remark 3.12(ii), we get $U_v = V$ and hence $u \in U_v$.

Corollary 3.14. *For any* $u, v \in V$ *in a digraph* $\mathcal{D} = (V, E)$, *we have* $u \in U_v$ *if and only if* $\overleftarrow{E_v} \subseteq \overleftarrow{E_u}$, *i.e.,* $U_v = \{u \in V | \overleftarrow{E_v} \subseteq \overleftarrow{E_u}\}$.

Proof. The proof is clear by Theorem 3.13.

4. Properties of Outtopological Digraph Space $\overleftarrow{\tau_{\mathcal{D}}}^{v}$

The necessary condition for topology space to be outop.digsp. $\overleftarrow{\tau_{\mathcal{D}}}^{v}$ is shown in the following.

Proposition 4.1. *Let* $\overleftarrow{\tau_{Ɗ}}^{v}$ *be an outop.digsp. of a digraph* $Ɗ = (\text{V̩}, \text{Ȩ})$. *Then we have the followings:*

(i) *If* $H = \{\text{v̩} \in \text{V̩} | \overleftarrow{\text{Ȩ}_{\text{v̩}}} \neq \emptyset\}$, *then* $H \in \overleftarrow{\tau_{Ɗ}}^{v}$.

(ii) *If* $K = \{\text{v̩} \in \text{V̩} | \overleftarrow{\text{Ȩ}_{\text{v̩}}} = \emptyset\}$, *then* K *is closed in* $\overleftarrow{\tau_{Ɗ}}^{v}$.

Proof. (i) Let $\text{v̩} \in H$ since $\overleftarrow{\text{Ȩ}_{\text{v̩}}} \neq \emptyset$, then by Remark 3.12(i), $\text{U}_{\text{v̩}} = \{\text{v̩}\}$.

As a result $\text{v̩} \in \text{U}_{\text{v̩}} \subseteq H$ and so v̩ is an interior point of H. Hence $H \in \overleftarrow{\tau_{Ɗ}}^{v}$.

(ii) By assumption $K = \bigcup_{\text{v̩} \in k}\{\text{v̩}\}$ and so, $\overline{K} = \overline{\bigcup_{\text{v̩} \in k}\{\text{v̩}\}} = \bigcup_{\text{v̩} \in k}\overline{\{\text{v̩}\}}$ by proposition of closure. Let $u \in \overline{K}$, then $u \in \overline{\{\text{v̩}\}}$ for some $\text{v̩} \in K$. By Corollary 3.14, $\overleftarrow{\text{Ȩ}_{\text{u}}} \subseteq \overleftarrow{\text{Ȩ}_{\text{v̩}}}$ since $\overleftarrow{\text{Ȩ}_{\text{v̩}}} = \emptyset$ and $\overleftarrow{\text{Ȩ}_{\text{u}}} \subseteq \overleftarrow{\text{Ȩ}_{\text{v̩}}}$, then $\overleftarrow{\text{Ȩ}_{\text{u}}} = \emptyset$, and so $u \in K$ hence $\overline{K} \subseteq K$, and the proof is complete.

Example 4.2. According to example 3.7, we get $H = \{\text{v̩}_1, \text{v̩}_2, \text{v̩}_4\}$, $K = \{\text{v̩}_3, \text{v̩}_5\}$.

We note that $\{\text{v̩}_1, \text{v̩}_2, \text{v̩}_4\} \in \overleftarrow{\tau_{Ɗ}}^{v} \Rightarrow H \in \overleftarrow{\tau_{Ɗ}}^{v}$ and $\{\text{v̩}_3, \text{v̩}_5\} \notin \overleftarrow{\tau_{Ɗ}}^{v} \Rightarrow K$ is closed in $\overleftarrow{\tau_{Ɗ}}^{v}$ since $\{\text{v̩}_3, \text{v̩}_5\}^c = \{\text{v̩}_1, \text{v̩}_2, \text{v̩}_4\} \in \overleftarrow{\tau_{Ɗ}}^{v}$.

Proposition 4.3. *Let* $Ɗ = (\text{V̩}, \text{Ȩ})$ *be a digraph. Then* $(\text{V̩}, \overleftarrow{\tau_{Ɗ}}^{v})$ *is a compact outop.digsp. if and only if* V̩ *is finite.*

Proof. Let $(\text{V̩}, \overleftarrow{\tau_{Ɗ}}^{v})$ be a compact outop.digsp. Suppose that V̩ is infinite. Then $\overleftarrow{\text{M}_{Ɗ}}^{v} = \{\text{U}_{\text{v̩}} \mid u \in \text{V̩}\}$ is an open covering of $(\text{V̩}, \overleftarrow{\tau_{Ɗ}}^{v})$ which has no finite subcover. Therefore, $(\text{V̩}, \overleftarrow{\tau_{Ɗ}}^{v})$ is not compact which is a contradiction. For the converse, it follows directly that $(\text{V̩}, \overleftarrow{\tau_{Ɗ}}^{v})$ is compact since there are only finitely many open subsets on finite space.

Definition 4.4. Let $Ɗ = (\text{V̩}, \text{Ȩ})$ be a digraph. If the number of components of Ɗ increases by the remove of a vertex v̩, then v̩ is said to be a cut vertex, if $Ɗ - C$ has more than one component such that $C \subseteq \text{V̩}(Ɗ)$ and Ɗ is connected, then C is called a vertex cut, if every proper subset of the vertex cut C of Ɗ is not a vertex cut, then C is called a minimal vertex cut.

Remark 4.5. Let $Ɗ = (\text{V̩}, \text{Ȩ})$ be a digraph (not necessary connected) and v̩ be a cut vertex. Then $\overleftarrow{\text{Ȩ}_{\text{v̩}}} \neq \emptyset$.

Proof. We will prove by a contradiction if $\overleftarrow{\text{Ȩ}_{\text{v̩}}} = \emptyset$, then the deletion of a vertex of

$\overleftarrow{E_v} = \emptyset$ and the out edges of it, does not increase the number of component of a digraph D this contradiction (since v is cut vertex), thus $\overleftarrow{E_v} \neq \emptyset$, the following example is applied to show this remark.

Example 4.6. Let $D = (V, E)$ be a digraph as in Figure 3 such that $V = \{v_1, v_2, v_3, v_4\}, E = \{e_1, e_2, e_3, e_4, e_5\}$.

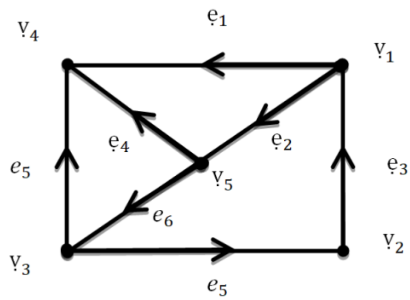

Figure 3

We note that v_1,v_2,v_3 are cut vertices but v_4 is not cut vertex.

$$\overleftarrow{E_{v_1}} \neq \emptyset,\ \overleftarrow{E_{v_2}} \neq \emptyset,\ \overleftarrow{E_{v_3}} \neq \emptyset,\ \overleftarrow{E_{v_5}} \neq \emptyset \ \text{ but } \ \overleftarrow{E_{v_4}} = \emptyset.$$

Lemma 4.7. *Let* $\overleftarrow{\tau_D}^{v}$ *be an outop.digsp. in a digraph* $D = (V, E)$ [*not necessary connected*] *and* v *be a cut vertex. Then* $\{ v \} \in \overleftarrow{\tau_D}^{v}$.

Proof. In a digraph $D = (V, E)$ not necessary connected, if v is cut vertex, then by Remark 4.5, $\overleftarrow{E_v} \neq \emptyset$. Consequently by Proposition 3.5, we get $\{v\} \in \overleftarrow{\tau_D}^{v}$.

Proposition 4.8. *Let* C *be a minimal vertex cut in connected digraph* $D = (V, E)$. *Then* $C \in \overleftarrow{\tau_D}^{v}$.

Proof. Since C is minimal vertex cut in D, then $\overleftarrow{E_v} \neq \emptyset$ for all $v \in C$. By Proposition 3.11, $v \in U_v \subseteq C$ and so v is interior point of C. Hence $C \in \overleftarrow{\tau_D}^{v}$.

Definition 4.9. Two digraphs $D_1 = (V_1, E_1)$ and $D_2 = (V_2, E_2)$ are said to be isomorphic to each other, and written $D_1 \cong D_2$ if there is a bijection $\mathcal{F} : V_1 \rightarrow V_2$ with $\{x, y\} \in E_1$ if and only if $\{\mathcal{F}(x), \mathcal{F}(y)\} \in E_2$ for all $x, y \in V_1$. The function $\mathcal{F}$ is called an isomorphism.

Example 4.10. Let $D_1 = (V_1, E_1)$, $D_2 = (V_2, E_2)$ be digraphs as in Figure 4 such that $V_1 = \{v_1, v_2, v_3, v_4\}$, $V_2 = \{a, b, c, d\}$.

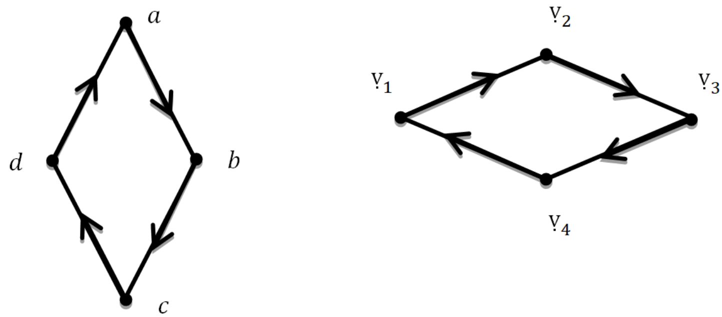

Figure 4

Then the digraph $\mathrm{D}_1, \mathrm{D}_2$ are isomorphic. Since $a \to b \to c \to d$ and $\mathrm{v}_4 \to \mathrm{v}_1 \to \mathrm{v}_2 \to \mathrm{v}_3$ and put $\mathcal{F} : \mathrm{V}_1 \to \mathrm{V}_2$ such that $\mathcal{F}(a) = \mathrm{v}_4$, $\mathcal{F}(b) = \mathrm{v}_1$, $\mathcal{F}(c) = \mathrm{v}_2$, $\mathcal{F}(d) = \mathrm{v}_3$.

Remark 4.11. It is clear that the outop.digsp. $(\mathrm{V}_1, \overleftarrow{\tau_{\mathrm{D}_1}}^{v})$ and $(\mathrm{V}_2, \overleftarrow{\tau_{\mathrm{D}_2}}^{v})$ are homeomorphic, if the digraphs $\mathrm{D}_1 = (\mathrm{V}_1, \mathrm{E}_1)$ and $\mathrm{D}_2 = (\mathrm{V}_2, \mathrm{E}_2)$ are isomorphic but in general the opposite is not true. The following example is applied to show that the opposite is not true.

Example 4.12. Let $\mathrm{D}_1 = (\mathrm{V}_1, \mathrm{E}_1)$, $\mathrm{D}_2 = (\mathrm{V}_2, \mathrm{E}_2)$ be digraphs as in Figure 5 such that $\mathrm{V}_1 = \{\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3, \mathrm{v}_4\}$ and $\mathrm{V}_2 = \{u_1, u_2, u_3, u_4\}$.

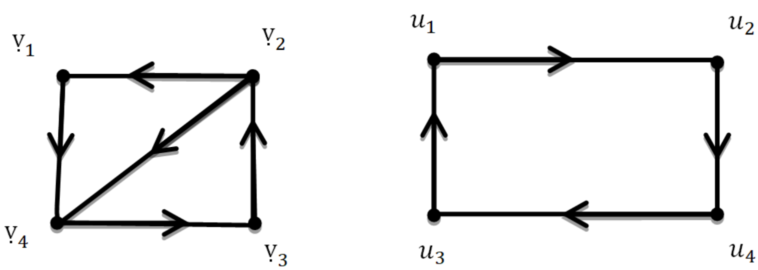

Figure 5

The outop.digsp. $(\mathrm{V}_1, \overleftarrow{\tau_{\mathrm{D}_1}}^{v})$ and $(\mathrm{V}_2, \overleftarrow{\tau_{\mathrm{D}_2}}^{v})$ are homeomorphic (since both are discrete). But they are not isomorphic digraph.

Remark 4.13. Suppose that $(\mathrm{V}_1, \overleftarrow{\tau_{\mathrm{D}_1}}^{v})$ and $(\mathrm{V}_2, \overleftarrow{\tau_{\mathrm{D}_2}}^{v})$ are two intop.dagsp. Then a function $\mathcal{F} : (\mathrm{V}_1, \overleftarrow{\tau_{\mathrm{D}_1}}^{v}) \to (\mathrm{V}_2, \overleftarrow{\tau_{\mathrm{D}_2}}^{v})$ is continuous if and only if $\mathcal{F}(\overline{A}) \subseteq \overline{\mathcal{F}(A)}$ for every subset A of V_1 and closed if and only if $\overline{\mathcal{F}(A)} \subseteq \mathcal{F}(\overline{A})$ for every subset A of V_1 .

5. Stipulations on Topological Space to be Outtopological Digraph Space $\overleftarrow{\tau_{Ɗ}}^{v}$

The necessary condition for topology space to be outtopological digraph space is shown in the part.

Definition 5.1. Any topological space $(Ά, \mathcal{T})$ is called outop.digsp. if $\mathcal{T} = \overleftarrow{\tau_{Ɗ}}^{v}$ for some digraph Ɗ with vertex set Ά.

Remark 5.2.

(i) If $\mathcal{T}$ is a discrete topology on Ά, then by Corollary 3.8, $\mathcal{T} = \overleftarrow{\tau_{Ɗ}}^{v}$ for some digraph Ɗ with vertex set Ά, such that $\overleftarrow{Ẹ_{ṿ}} \neq \emptyset$ for all ṿ ∈ Ṿ. Hence $\mathcal{T}$ is an outop.digsp.

(ii) If $\mathcal{T}$ is not a discrete topology on Ά, by definition outop.digsp. $\overleftarrow{\tau_{Ɗ}}^{v}$, all open sets $U_i \in \overleftarrow{\tau_{Ɗ}}^{v}$ for some i that contain one element are the outdegree edges of the digraph Ɗ and form a subbasis for the outop.digsp. $\overleftarrow{\tau_{Ɗ}}^{v}$. Therefore, if there exist open sets $U_i \in \overleftarrow{\tau_{Ɗ}}^{v}$ forsome i that contain one element such that these open sets are the outdegree edges of a digraph Ɗ and form a subbasis for $\mathcal{T}$, then $\mathcal{T}$ is an outop.digsp.$\overleftarrow{\tau_{Ɗ}}^{v}$ on Ά.

Example 5.3. Let Ά $= \{ṿ_1, ṿ_2, ṿ_3, ṿ_4\}$ such that

$$\mathcal{T}_1 = \{\emptyset, Ά, \{ṿ_1\}, \{ṿ_2\}, \{ṿ_3\}, \{ṿ_1, ṿ_2\}, \{ṿ_1, ṿ_3\}, \{ṿ_2, ṿ_3\}, \{ṿ_1, ṿ_2, ṿ_3\}\}$$

and

$$\mathcal{T}_2 = \{\emptyset, Ά, \{ṿ_1\}, \{ṿ_2\}, \{ṿ_1, ṿ_2\}, \{ṿ_2, ṿ_4\}, \{ṿ_1, ṿ_2, ṿ_4\}\}.$$

According to this example $\mathcal{T}_2$ is an outtopological since $\{ṿ_1\}, \{ṿ_2\}$ and $\{ṿ_4\}$ are outdegree edges of a digraph as in Figure 6. And these outdegree edges form a subbasis for $\mathcal{T}_2$. But $\mathcal{T}_1$ is not an outtopology because $\{ṿ_3\}$ does not belong in a subbasis for $\mathcal{T}_1$.

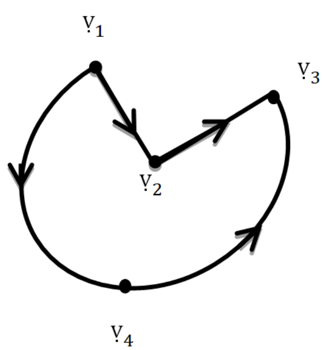

Figure 6

6. Density in Outtopological Digraph Space

Some necessary conditions for dense subsets of outtopological digraph space $\overleftarrow{\tau_{D}}^{v}$ associated to digraph are investigated in this part. The only dense subset in $(V, \overleftarrow{\tau_{D}}^{v})$ of every digraph $D = (V, E)$, such that $\overleftarrow{E_v} \neq \emptyset$ for all $v \in V$ is V since, $\overleftarrow{\tau_{D}}^{v}$ is a discrete topology.

Remark 6.1. It is known that in $(V, \overleftarrow{\tau_{D}}^{v})$ the subset $K \subseteq V$ is dense in V if and only if the complement of K has empty interior.

Proposition 6.2. *Let* $D = (V, E)$ *be a digraph with at least one vertex* $v \in V$ *such that* $\overleftarrow{E_v} = \emptyset$ *and* $n(E) \geq 1$. *Then the set* $K = \{v \in V |\ \overleftarrow{E_v} \neq \emptyset\}$ *is dense in* $(V, \overleftarrow{\tau_{D}}^{v})$.

Proof. By Remark 6.1, it is enough to prove that the complement of K has empty interior. For every $v \in K^c$, v is a vertex such that $\overleftarrow{E_v} = \emptyset$. Then if $\{v\} \notin \overleftarrow{\tau_{D}}^{v}$, for every $v \in K^c$. As a result, $B \subseteq K^c$, B cannot be written as a union of finitely intersection of elements of $\overleftarrow{S_{D}}^{v}$, i.e. $B \notin \overleftarrow{\tau_{D}}^{v}$. Hence $\text{int}(K^c) = \emptyset$ and this means K is dense subset in $(V, \overleftarrow{\tau_{D}}^{v})$.

Corollary 6.3. *Let* $D = (V, E)$ *be a digraph such that outop.digsp.* $\overleftarrow{\tau_{D}}^{v}$ *is not a discrete topology and* $n(E) \geq 1$. *Then a subset* B *of* V *is dense in* $(V, \overleftarrow{\tau_{D}}^{v})$ *if and only if* $K \subseteq B$ *such that* $K = \{v \in V \big| \overleftarrow{E_v} \neq \emptyset\}$.

Proof. $\Longrightarrow$ If B is dense in $(V, \overleftarrow{\tau_{D}}^{v})$, then by Remark 6.1, B^c has empty interior. By Proposition 3.5, $\{v\} \in \overleftarrow{\tau_{D}}^{v}$ for every $v \in K$ and so $K \in \overleftarrow{\tau_{D}}^{v}$. Hence $K \subseteq B$ because B^c has empty interior.

$\Longleftarrow$ By Proposition 6.2, $\overline{K} = V$. From assumption, $K \subseteq B$. Hence $\overline{B} = V$ and so B is dense in $(V, \overleftarrow{\tau_{D}}^{v})$.

Remark 6.4. Let $D = (V, E)$ be a digraph such that outop.digsp. $\overleftarrow{\tau_{D}}^{v}$ is not a discrete topology and $n(E) \geq 1$. By Corollary 6.2, $\overleftarrow{I_{e}}^{v}$ in $\overleftarrow{S_{D}}^{v}$ is dense such that for every $e \in E$ if and only if $K \subseteq \overleftarrow{I_{e}}^{v}$, such that $K = \{v \in V |\ \overleftarrow{E_v} \neq \emptyset\}$.

The next proposition gives the topological property for topological space to be outop.digsp. $\overleftarrow{\tau_{D}}^{v}$.

Proposition 6.5. *Let* $\overleftarrow{\tau_{\text{Ɗ}}}^{v}$ *be the outtopological of a digraph* $\text{Ɗ} = (\text{Ṿ}, \text{Ẹ})$. *Then the topological space* $(\text{Ṿ}^*, \mathcal{T})$ *is an outtopology if it is homeomorphic to* $(\text{Ṿ}, \overleftarrow{\tau_{\text{Ɗ}}}^{v})$.

Proof. Suppose that $\mathcal{F}: (\text{Ṿ}, \overleftarrow{\tau_{\text{Ɗ}}}^{v}) \to (\text{Ṿ}^*, \mathcal{T})$ is a homeomorphism. Since $(\text{Ṿ}, \overleftarrow{\tau_{\text{Ɗ}}}^{v})$ is an Alexandroff space and $(\text{Ṿ}, \overleftarrow{\tau_{\text{Ɗ}}}^{v}) \cong (\text{Ṿ}^*, \mathcal{T})$, $(\text{Ṿ}^*, \mathcal{T})$ is an Alexandroff space. To construct on Ṿ^* a digraph $\text{Ɗ}^* = (\text{Ṿ}^*, \text{Ẹ}^*)$ we put $\{\mathcal{F}(u), \mathcal{F}(\text{ṿ})\}$ is outdegree edges in Ẹ^* if and only if $\{u, \text{ṿ}\}$ is indegree in Ẹ for every $u, \text{ṿ} \in \text{Ṿ}$. Then we have $\mathcal{F}(\{u, \text{ṿ}\}) = \{\mathcal{F}(u), \mathcal{F}(\text{ṿ})\}$ and so $\mathcal{T} = \overleftarrow{\tau_{\text{Ɗ}}}^{v}$. As resulted, ${\text{U}^*}_u = M_u$ such that ${\text{U}^*}_u, M_u$ are the smallest open sets containing u in $(\text{Ṿ}^*, \overleftarrow{\tau_{\text{Ɗ}^*}}^{v})$ and $(\text{Ṿ}^*, \mathcal{T})$ respectively. Since $\mathcal{F}$ is a homeomorphism, $\mathcal{F}({\text{U}^*}_u) = M_{\mathcal{F}(\text{ṿ})}$ such that ${\text{U}^*}_u$ is the smallest open set containing u in $(\text{Ṿ}^*, \overleftarrow{\tau_{\text{Ɗ}^*}}^{v})$. Also $\mathcal{F}$ is an isomorphism between Ɗ and Ɗ^*, then $\mathcal{F}(\text{U}_u) = {\text{U}^*}_{\mathcal{F}(u)}$.

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