Earthline Journal of Mathematical Sciences
ISSN (Online): 2581-8147
Volume 2, Number 2, 2019, Pages 545-555
https://doi.org/10.34198/ejms.2219.545555

L-Fuzzy Ideals in Couple Γ-Semirings

Clement Boateng Ampadu

31 Carrolton Road, Boston, MA 02132-6303, USA
e-mail: drampadu@hotmail.com

Abstract

Let M be a Γ-semiring. In this paper we obtain some properties of *L*-fuzzy ideals in $M \times M$. Our results take inspiration from [1]. The readers are left with a conjecture.

1. Introduction and Preliminaries

Definition 1.1 [1]**.** A *partially ordered set* (*poset*) is a pair $(X, \leq)$, where X is a nonempty set and $\leq$ is a partial order (a reflexive, transitive, and antisymmetric binary relation) on X.

Definition 1.2 [1]**.** For any subset A of X and $x \in X$, we say x is a *lower bound* (*upper bound*) of A if $x \leq a$ ($a \leq x$ respectively) for all $a \in A$.

Definition 1.3 [1]**.** A poset $(X, \leq)$ is called a *lattice* if every nonempty finite subset of X has a greater lower bound (glb or infimum) and a least upper bound (lub or supremum) in X.

Remark 1.4 [1]**.** Let $(X, \leq)$ be a lattice. For any $a, b \in X$, define $a \wedge b = \inf\{a, b\}$ and $a \vee b = \sup\{a, b\}$, then $\wedge$ and $\vee$ are binary operations on X which are commutative, associative, and idempotent and satisfy the absorption law $a \wedge (a \vee b) = a = a \vee (a \wedge b)$.

Received: August 15, 2019; Accepted: October 8, 2019

2010 Mathematics Subject Classification: 16Y60, 03E72.

Keywords and phrases: Γ-semiring, *L*-fuzzy ideals, ring, ideal.

Copyright © 2019 Clement Boateng Ampadu. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Example 1.5 [1]**.** Let $(X, \wedge, \vee)$ be an algebraic system satisfying the properties in the previous remark, in which the partial order is defined by

$$a \leq b \Leftrightarrow a = a \wedge b \Leftrightarrow a \vee b = b,$$

then $(X, \wedge, \vee)$ is a lattice.

Definition 1.6 [1]**.** A lattice $(X, \wedge, \vee)$ is called *distributive* if $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ for all $a, b, c \in X$ (equivalently $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ for all $a, b, c \in X$).

Definition 1.7 [1]**.** A lattice $(X, \wedge, \vee)$ is called a *bounded lattice* if it has the smallest element 0 and largest element 1, that is, there are elements 0 and 1 in *X*, such that $0 \leq x \leq 1$ for all $x \in X$.

Definition 1.8 [1]**.** A partially ordered set in which every subset has an infimum and supremum is called a *complete lattice*

Definition 1.9 [1]**.** Two elements *a*, *b* of a bounded lattice $(L, \wedge, \vee, 0, 1)$ are *complements* if $a \wedge b = 0$ and $a \vee b = 1$. In this case each of *a*, *b* is the complement of the other.

Definition 1.10 [1]**.** A complement lattice is a bounded lattice in which every element has a complement.

Definition 1.11 [2]**.** A set *S* together with two associative binary operations called *addition* and *multiplication* (denoted by + and $\cdot$) is called a *semiring* provided the following holds:

(a) addition is a commutative operation,

(b) multiplication distributes over addition both from the left and from the right,

(c) there exists $0 \in S$ such that $x + 0 = x$ and $x \cdot 0 = 0 \cdot x = 0$ for each $x \in S$.

Definition 1.12 [2]**.** Let $(M, +)$ and $(\Gamma, +)$ be commutative semigroups. If there exists a mapping $M \times \Gamma \times M \mapsto M$ (images to be denoted by $x\alpha y$, $x, y \in M$, $\alpha \in \Gamma$) satisfying the following axioms for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$, then *M* is called a Γ-semiring

(a) $x\alpha(y + z) = x\alpha y + x\alpha z$,

(b) $(x + y)\alpha z = x\alpha z + y\alpha z$,

(c) $x(\alpha + \beta)y = x\alpha y + x\beta y$,

(d) $x\alpha(y\beta z) = (x\alpha y)\beta z$.

Definition 1.13 [2]**.** A Γ-semiring M is said to have a zero element if there exists an element $0 \in M$ such that $0 + x = x = x + 0$ and $0\alpha x = x\alpha 0 = 0$, for all $x \in M$.

Example 1.14 [2]**.** Every semiring M is a Γ-semiring with $\Gamma = M$ and ternary operation as the usual semiring multiplication.

Example 1.15 [2]**.** Let M be a Γ-semiring, and A be a nonempty subset of M. A is called a Γ-subsemiring of M if A is a sub-semigroup of $(M, +)$ and $A\Gamma A \subseteq A$.

Definition 1.16 [2]**.** Let M be a Γ-semiring. A subset A of M is called a *left* (*right*) *ideal* of M if A is closed under addition and $M\Gamma A \subseteq A$ $(A\Gamma M \subseteq A)$. Moreover, we say A is an ideal of M if it is both a left ideal and right ideal.

Definition 1.17 [3]**.** Let M be a nonempty set, a mapping $f : M \mapsto [0, 1]$ is called a *fuzzy subset* of M.

Definition 1.18 [3]**.** Let f be a fuzzy subset of a nonempty set M, for $t \in [0, 1]$, the set

$$f_t = \{x \in M : f(x) \geq t\}$$

is called a *level subset* of M with respect to f.

Definition 1.19 [4]**.** Let M be a Γ-semiring. A fuzzy subset μ of M is said to be a *fuzzy* Γ*-subsemiring* of M if it satisfies the following conditions

(a) $\mu(x + y) \geq \min\{\mu(x), \mu(y)\}$,

(b) $\mu(x\alpha y) \geq \min\{\mu(x), \mu(y)\}$, for all $x, y \in M$, $\alpha \in \Gamma$.

Definition 1.20 [4]**.** Let M be a Γ-semiring. A fuzzy subset μ of M is said to be a *fuzzy left* (*right*) *ideal* of M if for all $x, y \in M$ and $\alpha \in \Gamma$ the following conditions hold

(a) $\mu(x + y) \geq \min\{\mu(x), \mu(y)\}$,

(b) $\mu(x\alpha y) \geq \mu(y)(\mu(x))$.

Definition 1.21 [4]**.** Let M be a Γ-semiring. A fuzzy subset μ of M is said to be a *fuzzy ideal* of M if for all $x, y \in M$ and $\alpha \in \Gamma$ the following conditions hold

(a) $\mu(x + y) \geq \min\{\mu(x), \mu(y)\}$,

(b) $\mu(x\alpha y) \geq \max\{\mu(x), \mu(y)\}$.

Definition 1.22 [2]**.** Let M be a Γ-semiring. An ideal I of M is called a *k-ideal* if for all $x, y \in M$, $x + y \in I$ and $y \in I$ implies $x \in I$.

Definition 1.23 [3]**.** Let M be a Γ-semiring. A fuzzy subset $\mu : M \mapsto [0, 1]$ is *nonempty* if μ is not the constant function.

Definition 1.24 [3]**.** Let M be a Γ-semiring. For any two fuzzy subsets λ, μ of M, $\lambda \subseteq \mu$ means $\lambda(x) \leq \mu(x)$ for all $x \in M$.

Definition 1.25 [4]**.** Let M be a Γ-semiring, and let f, g be fuzzy subsets of M. Then $f \circ g$ is defined as

$$(f \circ g)(z) = \begin{cases} \sup_{z = x\alpha y}\{\min\{f(x), g(y)\}\} & \\ 0 & \text{otherwise} \end{cases}$$

$f + g$ is defined as

$$(f + g)(z) = \begin{cases} \sup_{z = x + y}\{\min\{f(x), g(y)\}\} & \\ 0 & \text{otherwise} \end{cases}$$

$f \bigcup g$ is defined as

$$(f \bigcup g)(z) = \max\{f(z), g(z)\}$$

and $f \bigcap g$ is defined as

$$(f \bigcap g)(z) = \min\{f(z), g(z)\}$$

$x, y \in M$, $\alpha \in \Gamma$, for all $z \in M$.

Definition 1.26 [2]**.** A function $f : R \mapsto M$, where R and M are Γ-semirings is said to be a Γ-*semiring homomorphism* if $f(a + b) = f(a) + f(b)$ and $f(a\alpha b) = f(a)\alpha f(b)$ for all $a, b \in R$, and $\alpha \in \Gamma$.

Definition 1.27 [4]**.** Let A be a nonempty subset of a Γ-semiring M. The *characteristic function* of A is a fuzzy subset of M, and is defined by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases}$$

Definition 1.28 [4]**.** Let M be a Γ-semiring, "0" be the zero element in M, and f be a fuzzy ideal of M. We say f is a k-fuzzy ideal of M if $f(x + y) = f(0)$ and $f(y) = f(0)$ implies $f(x) = f(0)$ for all $x, y \in M$.

Definition 1.29 [4]**.** Let M be a Γ-semiring, and f be a fuzzy ideal of M. We say f is a fuzzy k-ideal of M if

$$f(x) \geq \min\{f(x + y), f(y)\}$$

for all $x, y \in M$.

2. Main Results

Notation 2.1. $L = (L, \leq, \wedge, \vee)$ will denote a complemented lattice.

Notation 2.2. M will denote a Γ-semiring, and its zero element will be denoted "0".

Definition 2.3. A mapping $\mu : M \times M \mapsto L$ will be called a *L-fuzzy subset* of M^2.

Definition 2.4. A L fuzzy subset μ of $M \times M$ will be called a *L-fuzzy* Γ*-subsemiring* of M^2 if the following conditions are satisfied:

(a) $\mu(x + y, z + m) \geq \min\{\mu(x, z), \mu(y, m)\}$,

(b) $\mu(x\alpha y, z\alpha m) \geq \min\{\mu(x, z), \mu(y, m)\}$, for all $(x, z), (y, m) \in M \times M$, $\alpha \in \Gamma$.

Definition 2.5. A L-fuzzy Γ-subsemiring of $M \times M$, μ, will be called a *L-fuzzy left* (*right*) *ideal* of M^2 if

$$\mu(x\alpha y, z\alpha m) \geq \mu(y, m)(\mu(x, z)).$$

Moreover if μ is a fuzzy left and fuzzy right ideal of $M \times M$, then μ will be called a L-fuzzy ideal of M^2.

Theorem 2.6. *Let* μ *be a L-fuzzy ideal of* $M \times M$. *Then* $\mu(x, y) \leq \mu(0, 0)$ *for all* $(x, y) \in M \times M$.

Proof. Let $(x, y) \in M \times M$, $\alpha \in \Gamma$. Now observe that $\mu(0, 0) = \mu(0\alpha x, 0\alpha y) \geq \mu(x, y)$, therefore $\mu(x, y) \leq \mu(0, 0)$, for all $(x, y) \in M \times M$.

Theorem 2.7. μ *is a L-fuzzy left ideal of* M^2 *iff for* $t \in L$ *such that* $\mu_t \neq \varnothing$, μ_t *is a left ideal of* $M \times M$.

Proof. ($\Rightarrow$) Let μ be a L-fuzzy left ideal of $M \times M$ and $t \in L$ be such that $\mu_t \neq \varnothing$. Let $(x, z), (y, m) \in \mu_t$, then it follows that

$$\mu(x, z), \mu(y, m) \geq t$$

$\Rightarrow$

$$\mu(x + y, z + m) \geq \min\{\mu(x, z), \mu(y, m)\} \geq t.$$

Let $(x, z) \in M \times M, (y, m) \in \mu_t$, and $\alpha \in \Gamma$, then $\mu(x\alpha y, z\alpha m) \geq \mu(y, m) \geq t$. It follows that $x\alpha y, z\alpha m \in \mu_t$. Therefore μ_t is a left ideal of $M \times M$.

($\Leftarrow$) Suppose that μ_t is a left ideal of $M \times M$. Let $(x, z), (y, m) \in M \times M$ and $t = \min\{\mu(x, z), \mu(y, m)\}$. Then

$$\mu(x, z), \mu(y, m) \geq t$$

$\Rightarrow$

$$(x, z), (y, m) \in \mu_t$$

$\Rightarrow$

$$\mu(x + y, z + m) \geq t$$

$\Rightarrow$

$$\mu(x + y, z + m) \geq \min\{\mu(x, z), \mu(y, m)\}.$$

Now let $(x, z), (y, m) \in M^2$.

$$\mu(y, m) = s$$

$\Rightarrow$

$$(y, m) \in \mu_s$$

$\Rightarrow$

$$x\alpha y,\ z\alpha m \in \mu_s$$

$\Rightarrow$

$$\mu(x\alpha y,\ z\alpha m) \geq s = \mu(y, m).$$

It follows that μ is a *L*-fuzzy left ideal.

Theorem 2.8. *Define* $M_\mu = \{(x,\ y) \in M \times M \mid \mu(x,\ y) \geq \mu(0,\ 0)\}$. *If* μ *is a L-fuzzy ideal of* $M \times M$, *then* M_μ *is an ideal of* M^2.

Proof. Let μ be a *L*-fuzzy ideal of $M \times M$ and $(x,\ z), (y,\ m) \in M_\mu$, then it follows that

$$\mu(x,\ z) \geq \mu(0,\ 0),\ \mu(y,\ m) \geq \mu(0,\ 0)$$

which implies that

$$\mu(x + y,\ z + m) \geq \min\{\mu(x,\ z),\ \mu(y,\ m)\} \geq \mu(0,\ 0)$$

which implies that

$$(x + y,\ z + m) \in M_\mu.$$

Now observe that

$$\mu(x\alpha y,\ z\alpha m) \geq \min\{\mu(x,\ z),\ \mu(y,\ m)\} \geq \mu(0,\ 0)$$

implies that

$$(x\alpha y,\ z\alpha m) \in M_\mu.$$

Now let $(x,\ z) \in M_\mu$, $(y,\ m) \in M$, and $\alpha \in \Gamma$, then it follows that

$$\mu(x,\ z) \geq \mu(0,\ 0)$$

which implies

$$m(y\alpha x,\ m\alpha z) \geq \mu(x,\ z) \geq \mu(0,\ 0)$$

which implies

$$(y\alpha x,\ m\alpha z) \in M_\mu.$$

Similarly, we have

$$(x\alpha y,\ z\alpha m) \in M_\mu.$$

It now follows that M_μ is an ideal of $M \times M$.

Theorem 2.9. *Let* μ *and* γ *be two L-fuzzy ideals of* $M \times M$, *then* $\mu \cap \gamma$ *is a L-fuzzy ideal of* $M \times M$.

Proof. Let $(x, z), (y, m) \in M \times M$, and $\alpha \in \Gamma$, then we have the following

$$\begin{aligned}(\mu \cap \gamma)(x + y,\ z + m) &= \min\{\mu(x + y,\ z + m),\ \gamma(x + y,\ z + m)\}\\ &\geq \min\{\min\{\mu(x, z), \mu(y, m)\}, \min\{\gamma(x, z), \gamma(y, m)\}\}\\ &= \min\{\min\{\mu(x, z), \gamma(x, z)\}, \min\{\mu(y, m), \gamma(y, m)\}\}\\ &= \min\{(\mu \cap \gamma)(x, z), (\mu \cap \gamma)(y, m)\}.\end{aligned}$$

On the other hand

$$\begin{aligned}(\mu \cap \gamma)(x\alpha y,\ z\alpha m) &= \min\{\mu(x\alpha y,\ z\alpha m),\ \gamma(x\alpha y,\ z\alpha m)\}\\ &\geq \min\{\max\{\mu(x, z), \mu(y, m)\}, \max\{\gamma(x, z), \gamma(y, m)\}\}\\ &= \max\{\min\{\mu(x, z), \gamma(x, z)\}, \min\{\mu(y, m), \gamma(y, m)\}\}\\ &= \max\{(\mu \cap \gamma)(x, z), (\mu \cap \gamma)(y, m)\}.\end{aligned}$$

It now follows that $\mu \cap \gamma$ is a L-fuzzy ideal of $M \times M$.

Definition 2.10. Let M be a Γ-semiring, and μ be a L-fuzzy ideal of $M \times M$. We say μ is a *L-fuzzy k ideal* of M^2 if

$$\mu(x, z) \geq \min\{\mu(x + y,\ z + m),\ \mu(y, m)\}$$

for all $x, y, z, m \in M$. Moreover, if $\mu(x + y, z + m) = 0, \mu(y, m) = 0 \Rightarrow \mu(x, z) = 0$, then we say μ is a $L - k$ *fuzzy ideal* of $M \times M$.

Theorem 2.11. *Let M be a Γ-semiring, and let f and g be L-fuzzy k ideals of* M^2. *Then* $f \cap g$ *is a L-fuzzy k ideal of* M^2.

Proof. Let M be a Γ-semiring, and let f and g be L-fuzzy k ideals of M^2. By the previous theorem, $f \cap g$ is a L-fuzzy k ideal of M^2. Let $x, y, z, m \in M$, and observe we have the following

$$\begin{aligned}(f \cap g)(x, z) &= \min\{f(x, z), g(x, z)\} \\ &\geq \min\{\min\{f(x+y, z+m), f(y, m)\}, \min\{g(x+y, z+m), g(y, m)\}\} \\ &\geq \min\{\min\{f(x+y, z+m), g(x+y, z+m)\}, \min\{f(y, m), g(y, m)\}\} \\ &= \min\{(f \cap g)(x+y, z+m), (f \cap g)(y, m)\}.\end{aligned}$$

Hence $f \cap g$ is a L fuzzy k ideal of $M \times M$.

Definition 2.12. Let X be a set and μ be a L-fuzzy subset of $X \times X$, and $a, b \in L$. The mapping $\mu_a^T : X \times X \mapsto L, \mu_b^M : X \times X \mapsto L$, and $\mu_{b,a}^{MT} : X \times X \mapsto L$ will be called a *fuzzy type translation*, a *fuzzy type multiplication*, and a *fuzzy type magnified translation* of μ respectively, if for all $x, z \in M$,

$$\mu_a^T(x, z) = \mu(x, z) \vee a$$

$$\mu_b^M(x, z) = b \wedge \mu(x, z)$$

$$\mu_{b,a}^{MT}(x, z) = (b \wedge \mu(x, z)) \vee a.$$

Theorem 2.13. *Let M be a Γ-semiring, and let μ be a L-fuzzy subset of* $M \times M$, *and let* $a \in L$. *μ is a L-fuzzy ideal of* $M \times M$ *iff* μ_a^T *is a L-fuzzy ideal of* $M \times M$.

Proof. (⇒) Suppose μ is a L-fuzzy ideal of $M \times M$. Let $x, y, z, m \in M$ and $\alpha \in \Gamma$. Now observe we have the following

$$\begin{aligned}\mu_a^T(x+y, z+m) &= \mu(x+y, z+m) \vee a \\ &\geq \min\{\mu(x, z), \mu(y, m)\} \vee a\end{aligned}$$

$$= \min\{\mu(x, z) \vee a, \mu(y, m) \vee a\}$$

$$= \min\{\mu_a^T(x, z), \mu_a^T(y, m)\}.$$

On the other hand

$$\mu_a^T(x\alpha y, z\alpha m) = \mu(x\alpha y, z\alpha m) \vee a$$

$$\geq \min\{\mu(x, z), \mu(y, m)\} \vee a$$

$$= \min\{\mu(x, z) \vee a, \mu(y, m) \vee a\}$$

$$= \min\{\mu_a^T(x, z), \mu_a^T(y, m)\}.$$

It now follows that μ_a^T is a L-fuzzy ideal of $M \times M$.

($\Leftarrow$) Suppose that $a \in L$, μ_a^T is a L-fuzzy ideal of $M \times M$. Let $x, y, z, m \in M$ and $\alpha \in \Gamma$. Now

$$\mu_a^T(x + y, z + m) \geq \min\{\mu_a^T(x, z), \mu_a^T(y, m)\}$$

$\Rightarrow$

$$\mu(x + y, z + m) \vee a \geq \min\{\mu(x, z) \vee a, \mu(y, m) \vee a\}$$

$\Rightarrow$

$$\mu(x + y, z + m) \vee a \geq \min\{\mu(x, z), \mu(y, m)\} \vee a$$

$\Rightarrow$

$$\mu(x + y, z + m) \geq \min\{\mu(x, z), \mu(y, m)\}.$$

On the other hand

$$\mu_a^T(x\alpha y, z\alpha m) \geq \max\{\mu_a^T(x, z), \mu_a^T(y, m)\}$$

$\Rightarrow$

$$\mu(x\alpha y, z\alpha m) \vee a \geq \max\{\mu(x, z) \vee a, \mu(y, m) \vee a\}$$

$\Rightarrow$

$$\mu(x\alpha y, z\alpha m) \vee a \geq \max\{\mu(x, z), \mu(y, m)\} \vee a$$

⇒

$$\mu(x\alpha y,\ z\alpha m) \geq \max\{\mu(x,\ z),\ \mu(y,\ m)\}.$$

It now follows that μ is a *L*-fuzzy ideal of $M \times M$.

3. Open Problem

Conjecture 3.1. *Let M be a* Γ*-semiring, and let* μ *be a L-fuzzy subset of* $M \times M$, *and* $a \in L$. *Then* μ *is a L-fuzzy k ideal of* $M \times M$ *iff* μ_a^T *is a L-fuzzy k ideal of* $M \times M$.

4. Concluding Remarks

The present paper has introduced a concept of *L*-fuzzy ideals in couple Γ-semirings, and investigated some of their properties. Finally, we have a left the reader with an open problem inspired by Theorem 3.20 [1].

References

[1] M. Murali Krishna Rao and B. Vekateswarlu, *L*-fuzzy ideals in Γ-semiring, *Annals of Fuzzy Mathematics and Informatics* 10(1) (2015), 1-16.

[2] M. Murali Krishna Rao, Γ-semirings-I, *Southeast Asian Bull. Math.* 19(1) (1995), 49-54.

[3] L. A. Zadeh, Fuzzy sets, *Information and Control* 8 (1965), 338-353. https://doi.org/10.1016/S0019-9958(65)90241-X

[4] M. Murali Krishna Rao, Fuzzy soft Γ-semiring and fuzzy soft *k* ideal over Γ-semiring, *Ann. Fuzzy Math. Inform.* 9(2) (2015), 12-25.