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New Approximation Operators using Combined Edges Systems

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Abstract

In this paper, we introduce new approximation operators such as first lower approximation and first upper approximation of a sub undirected graph $ƕ \subseteq \Omega$ by using incident edges system and non-incident edges system respectively. Some properties of these concepts are investigated. In addition the first accuracy of lower and upper approximation operators are introduced and some of its characteristics are studied.

1. Introduction and Preliminaries

Combinatorics branch of graph theory is strongly related to other areas of mathematics like topology, group theory, and matrix theory. The second reason is that graphs will be very beneficial in practice when numerous concepts are empirically represented by them. Topological graph theory is a branch of mathematics with extensive applications in both theoretical and practical contexts [1, 2, 3, 4, 5, 8, 9]. We forecast that a significant factor in bridging the gap between topology and applications would be topological graph structure. We cite Harary [6] for all terms and nomenclature related to graph theory, and Moller [7] for all terms and notation related to topology. Here are some fundamental ideas in graph theory [10]. A undirected graph or graph is pair $\Omega = (\mho(\Omega), \mathcal{E}(\Omega))$ where $\mho(\Omega)$ is a non-empty set whose elements are called points or vertices (called vertex set) and $\mathcal{E}(\Omega)$ is the set of unordered pairs of elements of $\mho(\Omega)$ (called edge set). An edge of a graph that joins a vertex to itself is called a loop. If two edges of a graph are joined by a vertex, then these edges are called the edges ϱ incident with the edges ϱ_1. The set of ϱ is $\{\varrho_1 \in \mathcal{E}(\Omega)\text{: } \varrho_1 \text{incident with } \varrho\}$ and the edges ϱ non incident

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with the edges ϱ_1. The set of ϱ is $\{\varrho_1 \in \mathcal{E}(\Omega)\colon \varrho_1 \text{nonincident with } \varrho\}$. A sub graph of a graph Ω is a graph each of whose vertices belong to $\mho(\Omega)$ and each of whose edges belong to $\mathcal{E}(\Omega)$. An empty graph if the vertices set and edge set is empty. A degree of a vertex ʋ in a graph Ω is the number of edges of Ω incident with ʋ. Let $\Omega = (\mho(\Omega), \mathcal{E}(\Omega))$ be und. g. and an edge $\varrho \in \mathcal{E}(\Omega)$. The incident edges set of ϱ is denoted by $\mathrm{I}\mathcal{E}(\varrho)$ and defined by $\mathrm{I}\mathcal{E}(\varrho) = \{\varrho_1 \in \mathcal{E}(\Omega)\colon \varrho_1 \text{incident with } \varrho\}$ and the non-incident edges set of ϱ is denoted by $\mathrm{NI}\mathcal{E}(\varrho)$ and defined by $\mathrm{NI}\mathcal{E}(\varrho) = \{\varrho_1 \in \mathcal{E}(\Omega)\colon \varrho_1 \text{nonincident with } \varrho\}$. An und. g., $\Omega = (\mho(\Omega), \mathcal{E}(\Omega))$ the incident edges system (resp. non incident edges system) of an edge $\varrho \in \mathcal{E}(\Omega)$ is denoted by $\mathrm{I}\mathcal{E}\mathrm{S}(\varrho)$ (resp. $N\mathrm{I}\mathcal{E}\mathrm{S}(\varrho)$) and defined by: $\mathrm{I}\mathcal{E}\mathrm{S}(\varrho) = \{\mathrm{I}\mathcal{E}(\varrho)\}$ (resp. $N\mathrm{I}\mathcal{E}\mathrm{S}(\varrho) = \{\mathrm{NI}\mathcal{E}(\varrho)\}$). The combined edges system of an edge $\varrho \in \mathcal{E}(\Omega)$ is denoted by $C\mathcal{E}\mathrm{S}(\varrho)$ and defined by $C\mathcal{E}\mathrm{S}(\varrho) = \{\mathrm{I}\mathcal{E}\mathrm{S}(\varrho), N\mathrm{I}\mathcal{E}\mathrm{S}(\varrho)\}$. An edge $\varrho \in \mathcal{E}(\Omega)$ is called isolated edge if $\{\varrho \in \mathcal{E}(\Omega);\ \exists C\mathcal{E}\mathrm{S}(\varrho) \cap (\mathcal{E}(ƕ) - \{\varrho\}) = \phi\}$. Let $\Omega = (\mho(\Omega), \mathcal{E}(\Omega))$ be an und. g. and suppose that $Þ_c\colon \mathcal{E}(\Omega) \to \mathrm{P}(\mathrm{P}(\mathcal{E}(\Omega)))$ is a mapping which assigns for each ϱ in $\mathcal{E}(\Omega)$ its combined edges system in $\mathrm{P}(\mathrm{P}(\mathcal{E}(\Omega)))$. The pair $(\Omega, Þ_c)$ is called the C-space.

2. First New Approximation Operators using Combined Edges Systems

In this section, our main goal is to present a set-theoretic framework for granular computing with combined edges systems. There are numerous varieties of arbitrary edges systems in use. Consider the generalized approximation space $\mathfrak{Z} = (\mho(\Omega), \mathcal{E}(\Omega))$, we introduce new definitions of the lower and upper approximation operators using combined edges systems. The properties of the suggested operators are obtained. We also define accuracy for the introduced approximations and investigate its characteristics.

Definition 2.1. Let $\mathfrak{Z} = (\mho(\Omega), \mathcal{E}(\Omega))$ be a generalized approximation space and $ƕ \subseteq \Omega$. Then

a) The first lower and upper approximations of ƕ using incident edges systems are denoted by $L_{\mathrm{i}}^{1}(\mathcal{E}(ƕ))$ and $U_{\mathrm{i}}^{1}(\mathcal{E}(ƕ))$ and defined by:

$$L_{\mathrm{i}}^{1}(\mathcal{E}(ƕ)) = \{\varrho \in \mathcal{E}(ƕ);\ \mathrm{I}\mathcal{E}(\varrho) \subseteq \mathcal{E}(ƕ)\},$$

$$U_{\mathrm{i}}^{1}(\mathcal{E}(ƕ)) = \mathcal{E}(ƕ) \cup \{\varrho \in \mathcal{E}(\Omega) - \mathcal{E}(ƕ);\ \mathrm{I}\mathcal{E}(\varrho) \cap \mathcal{E}(ƕ) \neq \emptyset\},$$

b) The first lower and upper approximations of ƕ using nonincident edges systems are denoted by $L_{n}^{1}(\mathcal{E}(ƕ))$ and $U_{n}^{1}(\mathcal{E}(ƕ))$ and defined by:

$$L_{\mathrm{n}}^{1}(\mathcal{E}(ƕ)) = \{\varrho \in \mathcal{E}(ƕ);\ N\mathrm{I}\mathcal{E}(\varrho) \subseteq \mathcal{E}(ƕ)\},$$

$$U_{\rm n}^{1}(\mathcal{E}(\text{ƕ})) = \mathcal{E}(\text{ƕ}) \cup \{\varrho \in \mathcal{E}(\Omega) - \mathcal{E}(\text{ƕ});\ NI\mathcal{E}(\varrho) \cap \mathcal{E}(\text{ƕ}) \neq \emptyset\},$$

c) The first lower and upper approximations of ƕ using combined edges systems are denoted by $L_{\rm c}^{1}(\mathcal{E}(\text{ƕ}))$ and $U_{\rm c}^{1}(\mathcal{E}(\text{ƕ}))$ and defined by:

$$L_{\rm c}^{1}(\mathcal{E}(\text{ƕ})) = \{\varrho \in \mathcal{E}(\text{ƕ});\ \text{for some } C\mathcal{E}(\varrho) \subseteq \mathcal{E}(\text{ƕ})\},$$

$$U_{\rm c}^{1}(\mathcal{E}(\text{ƕ})) = \mathcal{E}(\text{ƕ}) \cup \{\varrho \in \mathcal{E}(\Omega) - \mathcal{E}(\text{ƕ}); \text{for all } C\mathcal{E}(\varrho) \cap \mathcal{E}(\text{ƕ}) \neq \emptyset\}.$$

Definition 2.2. Let $\mathfrak{Z} = (\mho(\Omega), \mathcal{E}(\Omega))$ be a generalization approximation space and ƕ $\subseteq \Omega$. Then

a) The first boundary, positive and negative regions of ƕ using incident edges systems are denoted by $Bd_{\rm i}^{1}(\mathcal{E}(\text{ƕ}))$, $POS_{\rm i}^{1}(\mathcal{E}(\text{ƕ}))$ and $NEG_{\rm i}^{1}(\mathcal{E}(\text{ƕ}))$ and defined by:

$$Bd_{\rm i}^{1}(\mathcal{E}(\text{ƕ})) = U_{\rm i}^{1}(\mathcal{E}(\text{ƕ})) - L_{\rm i}^{1}(\mathcal{E}(\text{ƕ})),$$

$$POS_{\rm i}^{1}(\mathcal{E}(\text{ƕ})) = L_{\rm i}^{1}(\mathcal{E}(\text{ƕ})),$$

$$NEG_{\rm i}^{1}(\mathcal{E}(\text{ƕ})) = \mathcal{E}(\Omega) - U_{\rm i}^{1}(\mathcal{E}(\text{ƕ})),$$

b) The first boundary, positive and negative regions of ƕ using nonincident edges systems are denoted by $Bd_{\rm n}^{1}(\mathcal{E}(\text{ƕ})), POS_{\rm n}^{1}(\mathcal{E}(\text{ƕ}))$ and $NEG_{\rm n}^{1}(\mathcal{E}(\text{ƕ}))$ and defined by:

$$Bd_{\rm n}^{1}(\mathcal{E}(\text{ƕ})) = U_{\rm n}^{1}(\mathcal{E}(\text{ƕ})) - L_{\rm n}^{1}(\mathcal{E}(\text{ƕ})),$$

$$POS_{\rm n}^{1}(\mathcal{E}(\text{ƕ})) = L_{\rm n}^{1}(\mathcal{E}(\text{ƕ})),$$

$$NEG_{\rm n}^{1}(\mathcal{E}(\text{ƕ})) = \mathcal{E}(\Omega) - U_{\rm n}^{1}(\mathcal{E}(\text{ƕ})),$$

c) The first boundary, positive and negative regions of ƕ using combined edges systems are denoted by $Bd_{\rm c}^{1}(\mathcal{E}(\text{ƕ})), POS_{\rm c}^{1}(\mathcal{E}(\text{ƕ}))$ and $NEG_{\rm c}^{1}(\mathcal{E}(\text{ƕ}))$ and defined by:

$$Bd_{\rm c}^{1}(\mathcal{E}(\text{ƕ})) = U_{\rm c}^{1}(\mathcal{E}(\text{ƕ})) - L_{\rm c}^{1}(\mathcal{E}(\text{ƕ})),$$

$$POS_{\rm c}^{1}(\mathcal{E}(\text{ƕ})) = L_{\rm c}^{1}(\mathcal{E}(\text{ƕ})),$$

$$NEG_{\rm c}^{1}(\mathcal{E}(\text{ƕ})) = \mathcal{E}(\Omega) - U_{\rm c}^{1}(\mathcal{E}(\text{ƕ})).$$

Definition 2.3. Let $\mathfrak{Z} = (\mho(\Omega), \mathcal{E}(\Omega))$ be a generalized approximation space. Then the first accuracy of the approximation of a sub und. g. ƕ $\subseteq \Omega$ using (incident, nonincident and combined) edges systems are denoted by $(\zeta_{\rm i}^{1}(\mathcal{E}(\text{ƕ})), \zeta_{\rm n}^{1}(\mathcal{E}(\text{ƕ}))$ and $\zeta_{c}^{1}(\mathcal{E}(\text{ƕ})))$ and defined respectively by:

$$\zeta_{\mathrm{i}}^{1}(\mathcal{E}(ƕ)) = 1 - \frac{\left|Bd_{\mathrm{i}}^{1}(\mathcal{E}(ƕ))\right|}{|\mathcal{E}(\Omega)|},$$

$$\zeta_{\mathrm{n}}^{1}(\mathcal{E}(ƕ)) = 1 - \frac{|Bd_{\mathrm{n}}^{1}(\mathcal{E}(ƕ))|}{|\mathcal{E}(\Omega)|},$$

$$\zeta_{\mathrm{c}}^{1}(\mathcal{E}(ƕ)) = 1 - \frac{|Bd_{\mathrm{c}}^{1}(\mathcal{E}(ƕ))|}{|\mathcal{E}(\Omega)|}.$$

It is obvious that $0 \leq \zeta_{\mathrm{i}}^{1}(\mathcal{E}(ƕ)) \leq 1$, $0 \leq \zeta_{\mathrm{n}}^{1}(\mathcal{E}(ƕ)) \leq 1$ and $0 \leq \zeta_{\mathrm{c}}^{1}(\mathcal{E}(ƕ)) \leq 1$. Moreover, if $\zeta_{\mathrm{i}}^{1}(\mathcal{E}(ƕ)) = 1$ or $\zeta_{\mathrm{n}}^{1}(\mathcal{E}(ƕ)) = 1$ or $\zeta_{\mathrm{c}}^{1}(\mathcal{E}(ƕ)) = 1$, then ƕ is called ƕ-definable (ƕ-exact) und. g. otherwise, it is called ƕ-rough.

Example 2.4. Let $\Omega = (Ʊ(\Omega), \mathcal{E}(\Omega))$ such that $Ʊ(\Omega) = \{ꝩ_1, ꝩ_2, ꝩ_3, ꝩ_4\}$ and $\mathcal{E}(\Omega) = \{\varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_5\}$.

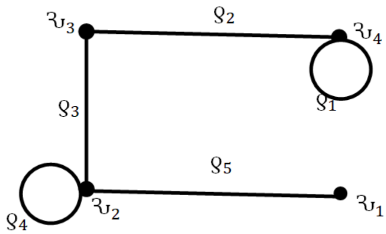

Figure 2.1. und. g. Ω given in Example (2.4).

We get:

$$\mathrm{I}\mathcal{E}(\varrho_1) = \{\varrho_1, \varrho_2\}, \quad \mathrm{I}\mathcal{E}(\varrho_2) = \{\varrho_1, \varrho_3\}, \quad \mathrm{I}\mathcal{E}(\varrho_3) = \{\varrho_2, \varrho_4, \varrho_5\},$$

$$\mathrm{I}\mathcal{E}(\varrho_4) = \{\varrho_3, \varrho_4, \varrho_5\}, \quad \mathrm{I}\mathcal{E}(\varrho_5) = \{\varrho_3, \varrho_4\}.$$

Also we have

$$\mathrm{NI}\mathcal{E}(\varrho_1) = \{\varrho_3, \varrho_4, \varrho_5\}, \quad \mathrm{NI}\mathcal{E}(\varrho_2) = \{\varrho_4, \varrho_5\}, \quad \mathrm{NI}\mathcal{E}(\varrho_3) = \{\varrho_1\},$$

$$\mathrm{NI}\mathcal{E}(\varrho_4) = \{\varrho_1, \varrho_2\}, \quad \mathrm{NI}\mathcal{E}(\varrho_5) = \{\varrho_1, \varrho_2\}.$$

Then we obtain

$$C\mathcal{E}(\varrho_1) = \{\{\varrho_1, \varrho_2\}, \{\varrho_3, \varrho_4, \varrho_5\}\}, \quad C\mathcal{E}(\varrho_2) = \{\{\varrho_1, \varrho_3\}, \{\varrho_4, \varrho_5\}\},$$

$$C\mathcal{E}(\varrho_3) = \{\{\varrho_2, \varrho_4, \varrho_5\}, \{\varrho_1\}\}, \quad C\mathcal{E}(\varrho_4) = \{\{\varrho_3, \varrho_4, \varrho_5\}, \{\varrho_1, \varrho_2\}\},$$

$$C\mathcal{E}(\varrho_5) = \{\{\varrho_3, \varrho_4\}, \{\varrho_1, \varrho_2\}\}.$$

According to Definition 2.3 we get the following table.

Table 2.1. $L_{\mathrm{i}}^{1}(\mathcal{E}(\text{ƕ}))$, $L_{\mathrm{n}}^{1}(\mathcal{E}(\text{ƕ}))$ and $L_{\mathrm{c}}^{1}(\mathcal{E}(\text{ƕ}))$ for all ƕ ⊆ Ω.

$\mathcal{E}(\text{ƕ})$	$L_{\mathrm{i}}^{1}(\mathcal{E}(\text{ƕ}))$	$L_{\mathrm{n}}^{1}(\mathcal{E}(\text{ƕ}))$	$L_{\mathrm{c}}^{1}(\mathcal{E}(\text{ƕ}))$
$\{\varrho_1\}$	ϕ	ϕ	ϕ
$\{\varrho_2\}$	ϕ	ϕ	ϕ
$\{\varrho_3\}$	ϕ	ϕ	ϕ
$\{\varrho_4\}$	ϕ	ϕ	ϕ
$\{\varrho_5\}$	ϕ	ϕ	ϕ
$\{\varrho_1,\varrho_2\}$	$\{\varrho_1\}$	ϕ	$\{\varrho_1\}$
$\{\varrho_1,\varrho_3\}$	ϕ	$\{\varrho_3\}$	$\{\varrho_3\}$
$\{\varrho_1,\varrho_4\}$	ϕ	ϕ	ϕ
$\{\varrho_1,\varrho_5\}$	ϕ	ϕ	ϕ
$\{\varrho_2,\varrho_3\}$	ϕ	ϕ	ϕ
$\{\varrho_2,\varrho_4\}$	ϕ	ϕ	ϕ
$\{\varrho_2,\varrho_5\}$	ϕ	ϕ	ϕ
$\{\varrho_3,\varrho_4\}$	ϕ	ϕ	ϕ
$\{\varrho_3,\varrho_5\}$	ϕ	ϕ	ϕ
$\{\varrho_4,\varrho_5\}$	ϕ	ϕ	ϕ
$\{\varrho_1,\varrho_2,\varrho_3\}$	$\{\varrho_1,\varrho_2\}$	$\{\varrho_3\}$	$\{\varrho_1,\varrho_2,\varrho_3\}$
$\{\varrho_1,\varrho_2,\varrho_4\}$	$\{\varrho_1\}$	$\{\varrho_4\}$	$\{\varrho_1,\varrho_4\}$
$\{\varrho_1,\varrho_2,\varrho_5\}$	$\{\varrho_1\}$	$\{\varrho_5\}$	$\{\varrho_1,\varrho_5\}$
$\{\varrho_2,\varrho_3,\varrho_4\}$	ϕ	ϕ	ϕ
$\{\varrho_2,\varrho_3,\varrho_5\}$	ϕ	ϕ	ϕ
$\{\varrho_3,\varrho_4,\varrho_1\}$	ϕ	$\{\varrho_3\}$	$\{\varrho_3\}$
$\{\varrho_3,\varrho_4,\varrho_5\}$	$\{\varrho_4,\varrho_5\}$	ϕ	$\{\varrho_4,\varrho_5\}$
$\{\varrho_4,\varrho_5,\varrho_1\}$	ϕ	ϕ	ϕ
$\{\varrho_4,\varrho_5,\varrho_2\}$	ϕ	$\{\varrho_2\}$	$\{\varrho_2\}$
$\{\varrho_1,\varrho_3,\varrho_5\}$	ϕ	$\{\varrho_3\}$	$\{\varrho_3\}$
$\{\varrho_1,\varrho_2,\varrho_3,\varrho_4\}$	$\{\varrho_1,\varrho_2\}$	$\{\varrho_3,\varrho_4\}$	$\{\varrho_1,\varrho_2,\varrho_3,\varrho_4\}$
$\{\varrho_1,\varrho_2,\varrho_3,\varrho_5\}$	$\{\varrho_1,\varrho_2\}$	$\{\varrho_3,\varrho_5\}$	$\{\varrho_1,\varrho_2,\varrho_3,\varrho_5\}$
$\{\varrho_2,\varrho_3,\varrho_4,\varrho_5\}$	$\{\varrho_3,\varrho_4,\varrho_5\}$	$\{\varrho_2\}$	$\{\varrho_2,\varrho_3,\varrho_4,\varrho_5\}$

$\{\varrho_1,\varrho_3,\varrho_4,\varrho_5\}$	$\{\varrho_4,\varrho_5\}$	$\{\varrho_1,\varrho_3\}$	$\{\varrho_1,\varrho_3,\varrho_4,\varrho_5\}$
$\{\varrho_1,\varrho_2,\varrho_4,\varrho_5\}$	$\{\varrho_1\}$	$\{\varrho_2,\varrho_4,\varrho_5\}$	$\{\varrho_1,\varrho_2,\varrho_4,\varrho_5\}$
$\mathcal{E}(\Omega)$	$\mathcal{E}(\Omega)$	$\mathcal{E}(\Omega)$	$\mathcal{E}(\Omega)$
ϕ	ϕ	ϕ	ϕ

Table 2.2. $U_{\mathrm{i}}^{1}(\mathcal{E}(ƕ))$, $U_{\mathrm{n}}^{1}(\mathcal{E}(ƕ))$ and $U_{\mathrm{c}}^{1}(\mathcal{E}(ƕ))$ for all $ƕ \subseteq \Omega$.

$\mathcal{E}(ƕ)$	$U_{\mathrm{i}}^{1}(\mathcal{E}(ƕ))$	$U_{\mathrm{n}}^{1}(\mathcal{E}(ƕ))$	$U_{\mathrm{c}}^{1}(\mathcal{E}(ƕ))$
$\{\varrho_1\}$	$\{\varrho_1,\varrho_2\}$	$\{\varrho_1,\varrho_3,\varrho_4,\varrho_5\}$	$\{\varrho_1\}$
$\{\varrho_2\}$	$\{\varrho_1,\varrho_2,\varrho_3\}$	$\{\varrho_2,\varrho_4,\varrho_5\}$	$\{\varrho_2\}$
$\{\varrho_3\}$	$\{\varrho_2,\varrho_3,\varrho_4,\varrho_5\}$	$\{\varrho_1,\varrho_3\}$	$\{\varrho_3\}$
$\{\varrho_4\}$	$\{\varrho_3,\varrho_4,\varrho_5\}$	$\{\varrho_1,\varrho_2,\varrho_4\}$	$\{\varrho_4\}$
$\{\varrho_5\}$	$\{\varrho_3,\varrho_4,\varrho_5\}$	$\{\varrho_1,\varrho_2,\varrho_5\}$	$\{\varrho_5\}$
$\{\varrho_1,\varrho_2\}$	$\{\varrho_1,\varrho_2,\varrho_3\}$	$\mathcal{E}(\Omega)$	$\{\varrho_1,\varrho_2,\varrho_3\}$
$\{\varrho_1,\varrho_3\}$	$\mathcal{E}(\Omega)$	$\{\varrho_1,\varrho_3,\varrho_4,\varrho_5\}$	$\{\varrho_1,\varrho_3,\varrho_4,\varrho_5\}$
$\{\varrho_1,\varrho_4\}$	$\mathcal{E}(\Omega)$	$\mathcal{E}(\Omega)$	$\mathcal{E}(\Omega)$
$\{\varrho_1,\varrho_5\}$	$\mathcal{E}(\Omega)$	$\mathcal{E}(\Omega)$	$\mathcal{E}(\Omega)$
$\{\varrho_2,\varrho_3\}$	$\mathcal{E}(\Omega)$	$\mathcal{E}(\Omega)$	$\mathcal{E}(\Omega)$
$\{\varrho_2,\varrho_4\}$	$\mathcal{E}(\Omega)$	$\{\varrho_1,\varrho_2,\varrho_4,\varrho_5\}$	$\{\varrho_1,\varrho_2,\varrho_4,\varrho_5\}$
$\{\varrho_2,\varrho_5\}$	$\mathcal{E}(\Omega)$	$\{\varrho_1,\varrho_2,\varrho_4,\varrho_5\}$	$\{\varrho_1,\varrho_2,\varrho_4,\varrho_5\}$
$\{\varrho_3,\varrho_4\}$	$\{\varrho_2,\varrho_3,\varrho_4,\varrho_5\}$	$\{\varrho_1,\varrho_2,\varrho_3,\varrho_4\}$	$\{\varrho_2,\varrho_3,\varrho_4\}$
$\{\varrho_3,\varrho_5\}$	$\{\varrho_2,\varrho_3,\varrho_4,\varrho_5\}$	$\{\varrho_1,\varrho_2,\varrho_3,\varrho_5\}$	$\{\varrho_2,\varrho_3,\varrho_4\}$
$\{\varrho_4,\varrho_5\}$	$\{\varrho_3,\varrho_4,\varrho_5\}$	$\{\varrho_1,\varrho_2,\varrho_4,\varrho_5\}$	$\{\varrho_4,\varrho_5\}$
$\{\varrho_1,\varrho_2,\varrho_3\}$	$\mathcal{E}(\Omega)$	$\mathcal{E}(\Omega)$	$\mathcal{E}(\Omega)$
$\{\varrho_1,\varrho_2,\varrho_4\}$	$\mathcal{E}(\Omega)$	$\mathcal{E}(\Omega)$	$\mathcal{E}(\Omega)$
$\{\varrho_1,\varrho_2,\varrho_5\}$	$\mathcal{E}(\Omega)$	$\mathcal{E}(\Omega)$	$\mathcal{E}(\Omega)$
$\{\varrho_2,\varrho_3,\varrho_4\}$	$\mathcal{E}(\Omega)$	$\mathcal{E}(\Omega)$	$\mathcal{E}(\Omega)$
$\{\varrho_2,\varrho_3,\varrho_5\}$	$\mathcal{E}(\Omega)$	$\mathcal{E}(\Omega)$	$\mathcal{E}(\Omega)$
$\{\varrho_3,\varrho_4,\varrho_1\}$	$\mathcal{E}(\Omega)$	$\mathcal{E}(\Omega)$	$\mathcal{E}(\Omega)$
$\{\varrho_3,\varrho_4,\varrho_5\}$	$\{\varrho_2,\varrho_3,\varrho_4,\varrho_5\}$	$\mathcal{E}(\Omega)$	$\{\varrho_2,\varrho_3,\varrho_4,\varrho_5\}$
$\{\varrho_4,\varrho_5,\varrho_1\}$	$\mathcal{E}(\Omega)$	$\mathcal{E}(\Omega)$	$\mathcal{E}(\Omega)$
$\{\varrho_4,\varrho_5,\varrho_2\}$	$\mathcal{E}(\Omega)$	$\{\varrho_1,\varrho_2,\varrho_4,\varrho_5\}$	$\{\varrho_1,\varrho_2,\varrho_4,\varrho_5\}$

$\{\varrho_1, \varrho_3, \varrho_5\}$	$\mathcal{E}(\Omega)$	$\mathcal{E}(\Omega)$	$\mathcal{E}(\Omega)$
$\{\varrho_1, \varrho_2, \varrho_3, \varrho_4\}$	$\mathcal{E}(\Omega)$	$\mathcal{E}(\Omega)$	$\mathcal{E}(\Omega)$
$\{\varrho_1, \varrho_2, \varrho_3, \varrho_5\}$	$\mathcal{E}(\Omega)$	$\mathcal{E}(\Omega)$	$\mathcal{E}(\Omega)$
$\{\varrho_2, \varrho_3, \varrho_4, \varrho_5\}$	$\mathcal{E}(\Omega)$	$\mathcal{E}(\Omega)$	$\mathcal{E}(\Omega)$
$\{\varrho_1, \varrho_3, \varrho_4, \varrho_5\}$	$\mathcal{E}(\Omega)$	$\mathcal{E}(\Omega)$	$\mathcal{E}(\Omega)$
$\{\varrho_1, \varrho_2, \varrho_4, \varrho_5\}$	$\{\varrho_1\}$	$\mathcal{E}(\Omega)$	$\mathcal{E}(\Omega)$
$\mathcal{E}(\Omega)$	$\mathcal{E}(\Omega)$	$\mathcal{E}(\Omega)$	$\mathcal{E}(\Omega)$
ϕ	ϕ	ϕ	ϕ

Table 2.3. $B_{\mathrm{i}}^{1}(\mathcal{E}(hʋ))$, $B_{\mathrm{n}}^{1}(\mathcal{E}(hʋ))$ and $B_{\mathrm{c}}^{1}(\mathcal{E}(hʋ))$ for all $hʋ \subseteq \Omega$.

$\mathcal{E}(hʋ)$	$B_{\mathrm{i}}^{1}(\mathcal{E}(hʋ))$	$B_{\mathrm{n}}^{1}(\mathcal{E}(hʋ))$	$B_{\mathrm{c}}^{1}(\mathcal{E}(hʋ))$
$\{\varrho_1\}$	$\{\varrho_1, \varrho_2\}$	$\{\varrho_1, \varrho_3, \varrho_4, \varrho_5\}$	$\{\varrho_1\}$
$\{\varrho_2\}$	$\{\varrho_1, \varrho_2, \varrho_3\}$	$\{\varrho_2, \varrho_4, \varrho_5\}$	$\{\varrho_2\}$
$\{\varrho_3\}$	$\{\varrho_2, \varrho_3, \varrho_4, \varrho_5\}$	$\{\varrho_1, \varrho_3\}$	$\{\varrho_3\}$
$\{\varrho_4\}$	$\{\varrho_3, \varrho_4, \varrho_5\}$	$\{\varrho_1, \varrho_2, \varrho_4\}$	$\{\varrho_4\}$
$\{\varrho_5\}$	$\{\varrho_3, \varrho_4, \varrho_5\}$	$\{\varrho_1, \varrho_2, \varrho_5\}$	$\{\varrho_5\}$
$\{\varrho_1, \varrho_2\}$	$\{\varrho_2, \varrho_3\}$	$\mathcal{E}(\Omega)$	$\{\varrho_2, \varrho_3\}$
$\{\varrho_1, \varrho_3\}$	$\mathcal{E}(\Omega)$	$\{\varrho_1, \varrho_4, \varrho_5\}$	$\{\varrho_1, \varrho_4, \varrho_5\}$
$\{\varrho_1, \varrho_4\}$	$\mathcal{E}(\Omega)$	$\mathcal{E}(\Omega)$	$\mathcal{E}(\Omega)$
$\{\varrho_1, \varrho_5\}$	$\mathcal{E}(\Omega)$	$\mathcal{E}(\Omega)$	$\mathcal{E}(\Omega)$
$\{\varrho_2, \varrho_3\}$	$\mathcal{E}(\Omega)$	$\mathcal{E}(\Omega)$	$\mathcal{E}(\Omega)$
$\{\varrho_2, \varrho_4\}$	$\mathcal{E}(\Omega)$	$\{\varrho_1, \varrho_2, \varrho_4, \varrho_5\}$	$\{\varrho_1, \varrho_2, \varrho_4, \varrho_5\}$
$\{\varrho_2, \varrho_5\}$	$\mathcal{E}(\Omega)$	$\{\varrho_1, \varrho_2, \varrho_4, \varrho_5\}$	$\{\varrho_1, \varrho_2, \varrho_4, \varrho_5\}$
$\{\varrho_3, \varrho_4\}$	$\{\varrho_2, \varrho_3, \varrho_4, \varrho_5\}$	$\{\varrho_1, \varrho_2, \varrho_3, \varrho_4\}$	$\{\varrho_2, \varrho_3, \varrho_4\}$
$\{\varrho_3, \varrho_5\}$	$\{\varrho_2, \varrho_3, \varrho_4, \varrho_5\}$	$\{\varrho_1, \varrho_2, \varrho_3, \varrho_5\}$	$\{\varrho_2, \varrho_3, \varrho_4\}$
$\{\varrho_4, \varrho_5\}$	$\{\varrho_3, \varrho_4, \varrho_5\}$	$\{\varrho_1, \varrho_2, \varrho_4, \varrho_5\}$	$\{\varrho_4, \varrho_5\}$
$\{\varrho_1, \varrho_2, \varrho_3\}$	$\{\varrho_3, \varrho_4, \varrho_5\}$	$\{\varrho_1, \varrho_2, \varrho_4, \varrho_5\}$	$\{\varrho_4, \varrho_5\}$
$\{\varrho_1, \varrho_2, \varrho_4\}$	$\{\varrho_2, \varrho_3, \varrho_4, \varrho_5\}$	$\{\varrho_1, \varrho_2, \varrho_3, \varrho_5\}$	$\{\varrho_2, \varrho_3, \varrho_5\}$
$\{\varrho_1, \varrho_2, \varrho_5\}$	$\{\varrho_2, \varrho_3, \varrho_4, \varrho_5\}$	$\{\varrho_1, \varrho_2, \varrho_3, \varrho_4\}$	$\{\varrho_2, \varrho_3, \varrho_4\}$
$\{\varrho_2, \varrho_3, \varrho_4\}$	$\mathcal{E}(\Omega)$	$\mathcal{E}(\Omega)$	$\mathcal{E}(\Omega)$
$\{\varrho_2, \varrho_3, \varrho_5\}$	$\mathcal{E}(\Omega)$	$\mathcal{E}(\Omega)$	$\mathcal{E}(\Omega)$

$\{\varrho_3, \varrho_4, \varrho_1\}$	$\mathcal{E}(\Omega)$	$\{\varrho_1, \varrho_2, \varrho_4, \varrho_5\}$	$\{\varrho_1, \varrho_2, \varrho_4, \varrho_5\}$
$\{\varrho_3, \varrho_4, \varrho_5\}$	$\{\varrho_2, \varrho_3\}$	$\mathcal{E}(\Omega)$	$\{\varrho_2, \varrho_3\}$
$\{\varrho_4, \varrho_5, \varrho_1\}$	$\mathcal{E}(\Omega)$	$\mathcal{E}(\Omega)$	$\mathcal{E}(\Omega)$
$\{\varrho_4, \varrho_5, \varrho_2\}$	$\mathcal{E}(\Omega)$	$\{\varrho_1, \varrho_4, \varrho_5\}$	$\{\varrho_1, \varrho_4, \varrho_5\}$
$\{\varrho_1, \varrho_3, \varrho_5\}$	$\mathcal{E}(\Omega)$	$\{\varrho_1, \varrho_2, \varrho_4, \varrho_5\}$	$\{\varrho_1, \varrho_2, \varrho_4, \varrho_5\}$
$\{\varrho_1, \varrho_2, \varrho_3, \varrho_4\}$	$\{\varrho_3, \varrho_4, \varrho_5\}$	$\{\varrho_1, \varrho_2, \varrho_5\}$	$\{\varrho_5\}$
$\{\varrho_1, \varrho_2, \varrho_3, \varrho_5\}$	$\{\varrho_3, \varrho_4, \varrho_5\}$	$\{\varrho_1, \varrho_2, \varrho_5\}$	$\{\varrho_4\}$
$\{\varrho_2, \varrho_3, \varrho_4, \varrho_5\}$	$\{\varrho_1, \varrho_2\}$	$\{\varrho_1, \varrho_3, \varrho_4, \varrho_5\}$	$\{\varrho_1\}$
$\{\varrho_1, \varrho_3, \varrho_4, \varrho_5\}$	$\{\varrho_1, \varrho_2, \varrho_3\}$	$\{\varrho_2, \varrho_4, \varrho_5\}$	$\{\varrho_2\}$
$\{\varrho_1, \varrho_2, \varrho_4, \varrho_5\}$	$\{\varrho_2, \varrho_3, \varrho_4, \varrho_5\}$	$\{\varrho_1, \varrho_3\}$	$\{\varrho_3\}$
$\mathcal{E}(\Omega)$	ϕ	ϕ	ϕ
ϕ	ϕ	ϕ	ϕ

Table 2.4. $\zeta_{\mathrm{i}}^1(\mathcal{E}(\text{hʋ}))$, $\zeta_{\mathrm{n}}^1(\mathcal{E}(\text{hʋ}))$ and $\zeta_{c}^1(\mathcal{E}(\text{hʋ}))$ for all $\text{hʋ} \subseteq \Omega$.

$\mathcal{E}(\text{hʋ})$	$\zeta_{\mathrm{i}}^1(\mathcal{E}(\text{hʋ}))$	$\zeta_{\mathrm{n}}^1(\mathcal{E}(\text{hʋ}))$	$\zeta_{c}^1(\mathcal{E}(\text{hʋ}))$
$\{\varrho_1\}$	3/5	1/5	4/5
$\{\varrho_2\}$	2/5	2/5	4/5
$\{\varrho_3\}$	1/5	3/5	4/5
$\{\varrho_4\}$	2/5	2/5	4/5
$\{\varrho_5\}$	2/5	2/5	4/5
$\{\varrho_1, \varrho_2\}$	3/5	0	3/5
$\{\varrho_1, \varrho_3\}$	0	2/5	2/5
$\{\varrho_1, \varrho_4\}$	0	0	0
$\{\varrho_1, \varrho_5\}$	0	0	0
$\{\varrho_2, \varrho_3\}$	0	0	0
$\{\varrho_2, \varrho_4\}$	0	1/5	1/5
$\{\varrho_2, \varrho_5\}$	0	1/5	1/5
$\{\varrho_3, \varrho_4\}$	1/5	1/5	2/5
$\{\varrho_3, \varrho_5\}$	1/5	1/5	2/5
$\{\varrho_4, \varrho_5\}$	2/5	1/5	3/5
$\{\varrho_1, \varrho_2, \varrho_3\}$	2/5	1/5	3/5

$\{\varrho_1, \varrho_2, \varrho_4\}$	1/5	1/5	2/5
$\{\varrho_1, \varrho_2, \varrho_5\}$	1/5	1/5	2/5
$\{\varrho_2, \varrho_3, \varrho_4\}$	0	0	0
$\{\varrho_2, \varrho_3, \varrho_5\}$	0	0	0
$\{\varrho_3, \varrho_4, \varrho_1\}$	0	1/5	1/5
$\{\varrho_3, \varrho_4, \varrho_5\}$	3/5	0	3/5
$\{\varrho_4, \varrho_5, \varrho_1\}$	0	0	0
$\{\varrho_4, \varrho_5, \varrho_2\}$	0	2/5	2/5
$\{\varrho_1, \varrho_3, \varrho_5\}$	0	1/5	1/5
$\{\varrho_1, \varrho_2, \varrho_3, \varrho_4\}$	2/5	2/5	4/5
$\{\varrho_1, \varrho_2, \varrho_3, \varrho_5\}$	2/5	2/5	4/5
$\{\varrho_2, \varrho_3, \varrho_4, \varrho_5\}$	3/5	1/5	4/5
$\{\varrho_1, \varrho_3, \varrho_4, \varrho_5\}$	2/5	2/5	4/5
$\{\varrho_1, \varrho_2, \varrho_4, \varrho_5\}$	1/5	3/5	4/5
$\mathcal{E}(\Omega)$	1	1	1
ϕ	1	1	1

Theorem 2.5. *Let* $\mathfrak{Z} = (\mho(\Omega), \mathcal{E}(\Omega))$ *be a generalized approximation space and* $\text{ƕ} \subseteq \Omega$. *Then*

(a) $L^1_{\mathrm{c}}(\mathcal{E}(\text{ƕ})) = L^1_{\mathrm{i}}(\mathcal{E}(\text{ƕ})) \cup L^1_{\mathrm{n}}(\mathcal{E}(\text{ƕ}))$.

(b) $U^1_{\mathrm{c}}(\mathcal{E}(\text{ƕ})) = U^1_{\mathrm{i}}(\mathcal{E}(\text{ƕ})) \cap U^1_{\mathrm{n}}(\mathcal{E}(\text{ƕ}))$.

(c) $Bd^1_{\mathrm{c}}(\mathcal{E}(\text{ƕ})) = Bd^1_{\mathrm{i}}(\mathcal{E}(\text{ƕ})) \cap Bd^1_{\mathrm{n}}(\mathcal{E}(\text{ƕ}))$.

Proof.

(a) Let $\varrho \in (L^1_{\mathrm{i}}(\mathcal{E}(\text{ƕ})) \cup L^1_{\mathrm{n}}(\mathcal{E}(\text{ƕ}))) \Leftrightarrow \varrho \in L^1_{\mathrm{i}}(\mathcal{E}(\text{ƕ})) \vee \varrho \in L^1_{\mathrm{n}}(\mathcal{E}(\text{ƕ})) \Leftrightarrow \mathrm{I}\mathcal{E}(\varrho) \subseteq \mathcal{E}(\text{ƕ}) \vee N\mathrm{I}\mathcal{E}(\varrho) \subseteq \mathcal{E}(\text{ƕ}) \Leftrightarrow \exists\, C\mathcal{E}(\varrho)$ such that $C\mathcal{E}(\varrho) \subseteq \mathcal{E}(\text{ƕ}) \Leftrightarrow \varrho \in L^1_{\mathrm{c}}(\mathcal{E}(\text{ƕ}))$, hence $L^1_{\mathrm{c}}(\mathcal{E}(\text{ƕ})) = L^1_{\mathrm{i}}(\mathcal{E}(\text{ƕ})) \cup L^1_{\mathrm{n}}(\mathcal{E}(\text{ƕ}))$.

(b) Let $\varrho \in U^1_{\mathrm{c}}(\mathcal{E}(\text{ƕ}))$. Then there are two cases:

1) $\varrho \in \mathcal{E}(\text{ƕ}) \Rightarrow \varrho \in U^1_{\mathrm{i}}(\mathcal{E}(\text{ƕ})) \wedge \varrho \in U^1_{\mathrm{n}}(\mathcal{E}(\text{ƕ})) \Rightarrow \varrho \in U^1_{\mathrm{i}}(\mathcal{E}(\text{ƕ})) \cap U^1_{\mathrm{n}}(\mathcal{E}(\text{ƕ}))$.

2) $\varrho \in \mathcal{E}(\Omega) - \mathcal{E}(\text{ƕ})$, since $\varrho \in U^1_{c}(\mathcal{E}(\text{ƕ})) \Rightarrow$ for all $C\mathcal{E}(\varrho)$, $C\mathcal{E}(\varrho) \cap \mathcal{E}(\text{ƕ}) \neq \emptyset \Rightarrow (\mathrm{I}\mathcal{E}(\varrho) \cap \mathcal{E}(\text{ƕ}) \neq \emptyset) \wedge (N\mathrm{I}\mathcal{E}(\varrho) \cap \mathcal{E}(\text{ƕ}) \neq \emptyset) \Rightarrow \varrho \in U^1_{\mathrm{i}}(\mathcal{E}(\text{ƕ})) \wedge \varrho \in U^1_{\mathrm{n}}(\mathcal{E}(\text{ƕ})) \Rightarrow \varrho \in U^1_{\mathrm{i}}(\mathcal{E}(\text{ƕ})) \cap U^1_{\mathrm{n}}(\mathcal{E}(\text{ƕ}))$.

Conversely, $\varrho \in (L_{\mathrm{i}}^{1}(\mathcal{E}(ƕ)) \cap U_{\mathrm{n}}^{1}(\mathcal{E}(ƕ)))$, then there are two cases:

1) $\varrho \in \mathcal{E}(ƕ) \Rightarrow \varrho \in U_{\mathrm{c}}^{1}(\mathcal{E}(ƕ))$.

2) $\varrho \in \mathcal{E}(\Omega) - \mathcal{E}(ƕ)$, since $\varrho \in (L_{\mathrm{i}}^{1}(\mathcal{E}(ƕ)) \cap U_{\mathrm{n}}^{1}(\mathcal{E}(ƕ))) \Rightarrow (\mathrm{I}\mathcal{E}(\varrho) \cap \mathcal{E}(ƕ) \neq \emptyset) \wedge (N\mathrm{I}\mathcal{E}(\varrho) \cap \mathcal{E}(ƕ) \neq \emptyset) \Rightarrow$ for all $C\mathcal{E}(\varrho)$, $C\mathcal{E}(\varrho) \cap \mathcal{E}(ƕ) \neq \emptyset \Rightarrow \varrho \in U_{\mathrm{c}}^{1}(\mathcal{E}(ƕ))$. Consequently, $U_{\mathrm{c}}^{1}(\mathcal{E}(ƕ)) = U_{\mathrm{i}}^{1}(\mathcal{E}(ƕ)) \cap U_{\mathrm{n}}^{1}(\mathcal{E}(ƕ))$.

(c) Let $\varrho \in Bd_{\mathrm{c}}^{1}(\mathcal{E}(ƕ)) \Rightarrow \varrho \in U_{\mathrm{c}}^{1}(\mathcal{E}(ƕ)) \wedge \varrho \notin L_{\mathrm{c}}^{1}(\mathcal{E}(ƕ))$ since $\varrho \in U_{\mathrm{c}}^{1}(\mathcal{E}(ƕ))$ by Theorem $(2.5(b))$ we get $\varrho \in (U_{\mathrm{i}}^{1}(\mathcal{E}(ƕ)) \cap U_{\mathrm{n}}^{1}(\mathcal{E}(ƕ))) \Rightarrow \varrho \in U_{\mathrm{i}}^{1}(\mathcal{E}(ƕ))$ and $\varrho \in U_{\mathrm{n}}^{1}(\mathcal{E}(ƕ))$. Since $\varrho \notin L_{\mathrm{c}}^{1}(\mathcal{E}(ƕ))$ by Theorem $(2.5(a))$ we get $\varrho \notin (L_{\mathrm{i}}^{1}(\mathcal{E}(ƕ)) \cup L_{\mathrm{n}}^{1}(\mathcal{E}(ƕ))) \Rightarrow \varrho \notin L_{\mathrm{i}}^{1}(\mathcal{E}(ƕ))$ and $\varrho \notin L_{\mathrm{n}}^{1}(\mathcal{E}(ƕ))$ and hence $\varrho \in Bd_{\mathrm{i}}^{1}(\mathcal{E}(ƕ))$ and $\varrho \in Bd_{\mathrm{n}}^{1}(\mathcal{E}(ƕ)) \Rightarrow \varrho \in (Bd_{\mathrm{i}}^{1}(\mathcal{E}(ƕ)) \cap Bd_{\mathrm{n}}^{1}(\mathcal{E}(ƕ)))$. Conversely, $\varrho \in (Bd_{\mathrm{i}}^{1}(\mathcal{E}(ƕ)) \cap Bd_{\mathrm{n}}^{1}(\mathcal{E}(ƕ))) \Rightarrow \varrho \in Bd_{\mathrm{i}}^{1}(\mathcal{E}(ƕ))$ and $\varrho \in Bd_{\mathrm{n}}^{1}(\mathcal{E}(ƕ))$ since $\varrho \in Bd_{\mathrm{i}}^{1}(\mathcal{E}(ƕ)) \Rightarrow \varrho \in U_{\mathrm{i}}^{1}(\mathcal{E}(ƕ))$ and $\varrho \notin L_{\mathrm{i}}^{1}(\mathcal{E}(ƕ))$ and since $\varrho \in Bd_{\mathrm{n}}^{1}(\mathcal{E}(ƕ)) \Rightarrow \varrho \in U_{\mathrm{n}}^{1}(\mathcal{E}(ƕ))$ and $\varrho \notin L_{\mathrm{n}}^{1}(\mathcal{E}(ƕ))$ and hence $\varrho \in (U_{\mathrm{i}}^{1}(\mathcal{E}(ƕ)) \cap U_{\mathrm{n}}^{1}(\mathcal{E}(ƕ)))$ by Theorem $(2.5(b))$ we get $\varrho \in U_{\mathrm{c}}^{1}(\mathcal{E}(ƕ))$ and $\varrho \notin (L_{\mathrm{i}}^{1}(\mathcal{E}(ƕ)) \cup L_{\mathrm{n}}^{1}(\mathcal{E}(ƕ)))$ by Theorem $(2.5(a))$ we get $\varrho \notin L_{\mathrm{c}}^{1}(\mathcal{E}(ƕ))$ then $\varrho \in Bd_{\mathrm{c}}^{1}(\mathcal{E}(ƕ))$.

Proposition 2.6. *Let* $\mathfrak{Z} = (\mho(\Omega), \mathcal{E}(\Omega))$ *be a generalized approximation space and* $ƕ, \mathfrak{k} \subseteq \Omega$. *Then*

(1) $L_{\mathrm{c}}^{1}(\mathcal{E}(ƕ)) \subseteq \mathcal{E}(ƕ)$.

(2) $L_{\mathrm{c}}^{1}(\mathcal{E}(\Omega)) = \mathcal{E}(\Omega)$.

(3) $L_{\mathrm{c}}^{1}(\emptyset) = \emptyset$.

(4) *If* $\mathcal{E}(ƕ) \subseteq \mathcal{E}(\mathfrak{k})$, *then* $L_{\mathrm{c}}^{1}(\mathcal{E}(ƕ)) \subseteq L_{\mathrm{c}}^{1}(\mathcal{E}(\mathfrak{k}))$.

(5) $L_{\mathrm{c}}^{1}(\mathcal{E}(ƕ) \cap \mathcal{E}(\mathfrak{k})) \subseteq L_{\mathrm{c}}^{1}(\mathcal{E}(ƕ)) \cap L_{\mathrm{c}}^{1}(\mathcal{E}(\mathfrak{k}))$.

(6) $L_{\mathrm{c}}^{1}(\mathcal{E}(ƕ)) \cup L_{\mathrm{c}}^{1}(\mathcal{E}(\mathfrak{k})) \subseteq L_{\mathrm{c}}^{1}(\mathcal{E}(ƕ) \cup \mathcal{E}(\mathfrak{k}))$.

(7) $L_{\mathrm{c}}^{1}(\mathcal{E}(ƕ)) = \mathcal{E}(\Omega) - [U_{\mathrm{c}}^{1}(\mathcal{E}(\Omega) - \mathcal{E}(ƕ))]$.

Proof.

The proof $(1), (2)$ and (3) by Definition $(2.1(c))$.

(4) Let $\mathcal{E}(ƕ) \subseteq \mathcal{E}(\mathfrak{k})$ and $\varrho \in L_{\mathrm{c}}^{1}(\mathcal{E}(ƕ))$, then $\exists\ C\mathcal{E}(\varrho)$ such that $C\mathcal{E}(\varrho) \subseteq \mathcal{E}(ƕ)$ so $\varrho \in L_{\mathrm{c}}^{1}(\mathcal{E}(ƕ)) \subseteq \mathcal{E}(ƕ) \subseteq \mathcal{E}(\mathfrak{k})$. Thus we have $\varrho \in \mathcal{E}(\mathfrak{k})$ and there exist $C\mathcal{E}(\varrho)$ such that $C\mathcal{E}(\varrho) \subseteq \mathcal{E}(ƕ) \subseteq \mathcal{E}(\mathfrak{k})$. Hence, $\varrho \in L_{\mathrm{c}}^{1}(\mathcal{E}(\mathfrak{k}))$ and so $L_{\mathrm{c}}^{1}(\mathcal{E}(ƕ)) \subseteq L_{\mathrm{c}}^{1}(\mathcal{E}(\mathfrak{k}))$.

(5) Since $(\mathcal{E}(\mathfrak{h}) \cap \mathcal{E}(\mathfrak{k})) \subseteq \mathcal{E}(\mathfrak{h})$ by Proposition (2.6(4)) we get $L_c^1(\mathcal{E}(\mathfrak{h}) \cap \mathcal{E}(\mathfrak{k})) \subseteq L_c^1(\mathcal{E}(\mathfrak{h})) - - - (1)$. And since $(\mathcal{E}(\mathfrak{h}) \cap \mathcal{E}(\mathfrak{k})) \subseteq \mathcal{E}(\mathfrak{k})$ by Proposition (2.6(4)) we get $L_c^1(\mathcal{E}(\mathfrak{h}) \cap \mathcal{E}(\mathfrak{k})) \subseteq L_c^1(\mathcal{E}(\mathfrak{k})) - - - (2)$. From (1) and (2) we get $L_c^1(\mathcal{E}(\mathfrak{h}) \cap \mathcal{E}(\mathfrak{k})) \subseteq L_c^1(\mathcal{E}(\mathfrak{h})) \cap L_c^1(\mathcal{E}(\mathfrak{k}))$.

(6) Since $\mathcal{E}(\mathfrak{h}) \subseteq (\mathcal{E}(\mathfrak{h}) \cup \mathcal{E}(\mathfrak{k}))$ by Proposition (2.6(4)) we get $L_c^1(\mathcal{E}(\mathfrak{h})) \subseteq L_c^1(\mathcal{E}(\mathfrak{h}) \cup \mathcal{E}(\mathfrak{k})) - - - (1)$. And since $\mathcal{E}(\mathfrak{k}) \subseteq (\mathcal{E}(\mathfrak{h}) \cup \mathcal{E}(\mathfrak{k}))$ by Proposition (2.6(4)) we get $L_c^1(\mathcal{E}(\mathfrak{k})) \subseteq L_c^1(\mathcal{E}(\mathfrak{h}) \cup \mathcal{E}(\mathfrak{k})) - - - (2)$. From (1) and (2) we get $L_c^1(\mathcal{E}(\mathfrak{h}) \cup \mathcal{E}(\mathfrak{k})) \subseteq L_c^1(\mathcal{E}(\mathfrak{h}) \cup \mathcal{E}(\mathfrak{k}))$.

(7) Let $\varrho \in L_c^1(\mathcal{E}(\mathfrak{h})) \Rightarrow \varrho \in \mathcal{E}(\mathfrak{h})$ and $\exists\, C\mathcal{E}(\varrho) \subseteq \mathcal{E}(\mathfrak{h}) \Rightarrow \varrho \in \mathcal{E}(\Omega) - [\mathcal{E}(\Omega) - \mathcal{E}(\mathfrak{h})]$ and $\exists\, C\mathcal{E}(\varrho): C\mathcal{E}(\varrho) \cap [\mathcal{E}(\Omega) - \mathcal{E}(\mathfrak{h})] = \emptyset \Rightarrow \varrho \notin U_c^1\big(\mathcal{E}(\Omega) - \mathcal{E}(\mathfrak{h})\big) \Rightarrow \varrho \in \mathcal{E}(\Omega) - U_c^1(\mathcal{E}(\Omega) - \mathcal{E}(\mathfrak{h})) \Rightarrow L_c^1(\mathcal{E}(\mathfrak{h})) \subseteq \mathcal{E}(\Omega) - U_c^1(\mathcal{E}(\Omega) - \mathcal{E}(\mathfrak{h})) - - - - (1)$. On other side let $\varrho \in \mathcal{E}(\Omega) - U_c^1\big(\mathcal{E}(\Omega) - \mathcal{E}(\mathfrak{h})\big) \Rightarrow \varrho \in \mathcal{E}(\Omega)$ and $\varrho \notin U_c^1(\mathcal{E}(\Omega) - \mathcal{E}(\mathfrak{h})) \Rightarrow \exists\, C\mathcal{E}(\varrho): C\mathcal{E}(\varrho) \cap [\mathcal{E}(\Omega) - \mathcal{E}(\mathfrak{h})] = \emptyset$ and $\varrho \in \mathcal{E}(\Omega) - [\mathcal{E}(\Omega) - \mathcal{E}(\mathfrak{h})]$ and $\exists\, C\mathcal{E}(\varrho) \subseteq \mathcal{E}(\mathfrak{h}) \Rightarrow \varrho \in L_c^1(\mathcal{E}(\mathfrak{h})) \Rightarrow \mathcal{E}(\Omega) - U_c^1(\mathcal{E}(\Omega) - \mathcal{E}(\mathfrak{h})) \subseteq L_c^1(\mathcal{E}(\mathfrak{h})) - - - (2)$. From (1) and (2) we get $L_c^1(\mathcal{E}(\mathfrak{h})) = \mathcal{E}(\Omega) - [U_c^1(\mathcal{E}(\Omega) - \mathcal{E}(\mathfrak{h}))]$.

Proposition 2.7. *Let* $\mathfrak{Z} = (\mho(\Omega), \mathcal{E}(\Omega))$ *be a generalized approximation space and* $\mathfrak{h}, \mathfrak{k} \subseteq \Omega$. *Then*

(1) $\mathcal{E}(\mathfrak{h}) \subseteq U_c^1(\mathcal{E}(\mathfrak{h}))$.

(2) $U_c^1(\mathcal{E}(\Omega)) = \mathcal{E}(\Omega)$.

(3) $U_c^1(\emptyset) = \emptyset$.

(4) *If* $\mathcal{E}(\mathfrak{h}) \subseteq \mathcal{E}(\mathfrak{k})$, *then* $U_c^1(\mathcal{E}(\mathfrak{h})) \subseteq U_c^1(\mathcal{E}(\mathfrak{k}))$.

(5) $U_c^1(\mathcal{E}(\mathfrak{h}) \cap \mathcal{E}(\mathfrak{k})) \subseteq U_c^1(\mathcal{E}(\mathfrak{h})) \cap U_c^1(\mathcal{E}(\mathfrak{k}))$.

(6) $U_c^1(\mathcal{E}(\mathfrak{h})) \cup U_c^1(\mathcal{E}(\mathfrak{k})) \subseteq U_c^1(\mathcal{E}(\mathfrak{h}) \cup \mathcal{E}(\mathfrak{k}))$.

(7) $U_c^1(\mathcal{E}(\mathfrak{h})) = \mathcal{E}(\Omega) - [L_c^1(\mathcal{E}(\Omega) - \mathcal{E}(\mathfrak{h}))]$.

(8) $L_c^1(\mathcal{E}(\mathfrak{h})) \subseteq U_c^1(\mathcal{E}(\mathfrak{h}))$.

Proof.

The proof (1), (2) and (3) by Definition (2.1(c)).

(4) Let $\mathcal{E}(\mathfrak{h}) \subseteq \mathcal{E}(\mathfrak{k})$ and $\varrho \in U_c^1(\mathcal{E}(\mathfrak{h}))$, we have:

If $\varrho \in \mathcal{E}(\text{ƕ}) \Rightarrow \varrho \in \mathcal{E}(\text{ƕ}) \subseteq \mathcal{E}(\text{ƙ}) \Rightarrow \varrho \in \mathcal{E}(\text{ƙ}) \Rightarrow \varrho \in U_c^1(\mathcal{E}(\text{ƙ}))$. If $\varrho \in \mathcal{E}(\Omega) - \mathcal{E}(\text{ƕ})$ since $\varrho \in U_c^1(\mathcal{E}(\text{ƕ})) \Rightarrow \forall\, C\mathcal{E}(\varrho): C\mathcal{E}(\varrho) \cap \mathcal{E}(\text{ƕ}) \neq \emptyset$ and since $\mathcal{E}(\text{ƕ}) \subseteq \mathcal{E}(\text{ƙ}) \Rightarrow \forall\, C\mathcal{E}(\varrho): C\mathcal{E}(\varrho) \cap \mathcal{E}(\text{ƙ}) \neq \emptyset$ and hence if $\varrho \in \mathcal{E}(\text{ƙ}) - \mathcal{E}(\text{ƕ}) \Rightarrow \varrho \in \mathcal{E}(\text{ƙ}) \Rightarrow \varrho \in U_c^1(\mathcal{E}(\text{ƙ}))$ if $\varrho \in \mathcal{E}(\Omega) - \mathcal{E}(\text{ƙ}) \Rightarrow \forall\, C\mathcal{E}(\varrho): C\mathcal{E}(\varrho) \cap \mathcal{E}(\text{ƙ}) \neq \emptyset \Rightarrow \varrho \in U_c^1(\mathcal{E}(\text{ƙ}))$ thus $U_c^1(\mathcal{E}(\text{ƕ})) \subseteq U_c^1(\mathcal{E}(\text{ƙ}))$.

(5) Since $(\mathcal{E}(\text{ƕ}) \cap \mathcal{E}(\text{ƙ})) \subseteq \mathcal{E}(\text{ƕ})$ by Proposition $(2.7(4))$ we get $U_c^1(\mathcal{E}(\text{ƕ}) \cap \mathcal{E}(\text{ƙ})) \subseteq U_c^1(\mathcal{E}(\text{ƕ})) - - - (1)$. And since $(\mathcal{E}(\text{ƕ}) \cap \mathcal{E}(\text{ƙ})) \subseteq \mathcal{E}(\text{ƙ})$ by Proposition $(2.7(4))$ we get $U_c^1(\mathcal{E}(\text{ƕ}) \cap \mathcal{E}(\text{ƙ})) \subseteq U_c^1(\mathcal{E}(\text{ƙ})) - - - (2)$. From (1) and (2) we get $U_c^1(\mathcal{E}(\text{ƕ}) \cap \mathcal{E}(\text{ƙ})) \subseteq U_c^1(\mathcal{E}(\text{ƕ})) \cap U_c^1(\mathcal{E}(\text{ƙ}))$.

(6) Since $\mathcal{E}(\text{ƕ}) \subseteq (\mathcal{E}(\text{ƕ}) \cup \mathcal{E}(\text{ƙ}))$ by Proposition $(2.7(4))$ we get $U_c^1(\mathcal{E}(\text{ƕ})) \subseteq U_c^1(\mathcal{E}(\text{ƕ}) \cup \mathcal{E}(\text{ƙ})) - - - (1)$. And since $\mathcal{E}(\text{ƙ}) \subseteq (\mathcal{E}(\text{ƕ}) \cup \mathcal{E}(\text{ƙ}))$ by Proposition $(2.7(4))$ we get $U_c^1(\mathcal{E}(\text{ƙ})) \subseteq U_c^1(\mathcal{E}(\text{ƕ}) \cup \mathcal{E}(\text{ƙ})) - - - (2)$. From (1) and (2) we get $U_c^1(\mathcal{E}(\text{ƕ})) \cup U_c^1(\mathcal{E}(\text{ƙ})) \subseteq U_c^1(\mathcal{E}(\text{ƕ}) \cup \mathcal{E}(\text{ƙ}))$.

(7) By Proposition $(2.6(7))$ $L_c^1(\mathcal{E}(\text{ƕ})) = \mathcal{E}(\Omega) - [U_c^1(\mathcal{E}(\Omega) - \mathcal{E}(\text{ƕ}))] \Rightarrow \mathcal{E}(\Omega) - L_c^1(\mathcal{E}(\text{ƕ})) = \mathcal{E}(\Omega) - (\mathcal{E}(\Omega) - [U_c^1(\mathcal{E}(\Omega) - \mathcal{E}(\text{ƕ}))]) \Rightarrow U_c^1(\mathcal{E}(\Omega) - \mathcal{E}(\text{ƕ})) = \mathcal{E}(\Omega) - L_c^1(\mathcal{E}(\text{ƕ}))$. Now we replace $\mathcal{E}(\Omega) - \mathcal{E}(\text{ƕ})$ for $\mathcal{E}(\text{ƕ})$ we get $U_c^1(\mathcal{E}(\text{ƕ})) = \mathcal{E}(\Omega) - L_c^1(\mathcal{E}(\Omega) - \mathcal{E}(\text{ƕ}))$.

(8) By Proposition $(2.6.(1))$ we get $L_c^1(\mathcal{E}(\text{ƕ})) \subseteq \mathcal{E}(\text{ƕ})$ and by Proposition $(2.7(1))$ we get $\mathcal{E}(\text{ƕ}) \subseteq U_c^1(\mathcal{E}(\text{ƕ}))$ thus $L_c^1(\mathcal{E}(\text{ƕ})) \subseteq U_c^1(\mathcal{E}(\text{ƕ}))$.

Remark 2.8. Let $\mathfrak{Z} = (\text{Ʊ}(\Omega), \mathcal{E}(\Omega))$ be a generalized approximation space and $\text{ƕ}, \text{ƙ} \subseteq \Omega$. Then the following statements are not necessarily true:

(1) $L_c^1(\mathcal{E}(\text{ƕ})) = L_c^1(L_c^1(\mathcal{E}(\text{ƕ})))$.

(2) $L_c^1(\mathcal{E}(\text{ƕ})) = U_c^1(L_c^1(\mathcal{E}(\text{ƕ})))$.

(3) $\mathcal{E}(\text{ƕ}) \subseteq L_c^1(U_c^1(\mathcal{E}(\text{ƕ})))$.

(4) $L_c^1(\mathcal{E}(\text{ƕ})) \subseteq L_c^1(L_c^1(\mathcal{E}(\text{ƕ})))$.

(5) $L_c^1(\mathcal{E}(\text{ƕ}) \cup \mathcal{E}(\text{ƙ})) = L_c^1(\mathcal{E}(\text{ƕ})) \cup L_c^1(\mathcal{E}(\text{ƙ}))$.

(6) $U_c^1(\mathcal{E}(\text{ƕ})) = U_c^1(U_c^1(\mathcal{E}(\text{ƕ})))$.

(7) $U_c^1(\mathcal{E}(\text{ƕ})) = L_c^1(U_c^1(\mathcal{E}(\text{ƕ})))$.

(8) $U_c^1(L_c^1(\mathcal{E}(\text{ƕ}))) \subseteq \mathcal{E}(\text{ƕ})$.

(9) $U_c^1(U_c^1(\mathcal{E}(\text{ƕ}))) \subseteq U_c^1(\mathcal{E}(\text{ƕ}))$.

(10) $U_c^1(\mathcal{E}(\text{ƕ}) \cup \mathcal{E}(\text{ƙ})) = U_c^1(\mathcal{E}(\text{ƕ})) \cup U_c^1(\mathcal{E}(\text{ƙ}))$.

The following example is used to demonstrate this notion.

Example 2.9. In Example (2.4) we get

(1) Let $\text{ƕ} = (\mho(\text{ƕ}), \mathcal{E}(\text{ƕ}))$ such that $\mho(\text{ƕ}) = \{\mathfrak{z}_3, \mathfrak{z}_4\}$ and $\mathcal{E}(\text{ƕ}) = \{\varrho_1, \varrho_2\}$. Then $L_c^1(\mathcal{E}(\text{ƕ})) = \{\varrho_1\}$, $L_c^1(L_c^1(\mathcal{E}(\text{ƕ})) = \emptyset$. Therefore, $L_c^1(\mathcal{E}(\text{ƕ})) \neq L_c^1(L_c^1(\mathcal{E}(\text{ƕ})))$.

(2) Let $\text{ƕ} = (\mho(\text{ƕ}), \mathcal{E}(\text{ƕ}))$ such that $\mho(\text{ƕ}) = \{\mathfrak{z}_2, \mathfrak{z}_3, \mathfrak{z}_4\}$ and $\mathcal{E}(\text{ƕ}) = \{\varrho_1, \varrho_2, \varrho_3\}$. Then $L_c^1(\mathcal{E}(\text{ƕ})) = \{\varrho_1, \varrho_2, \varrho_3\}$, $U_c^1(L_c^1(\mathcal{E}(\text{ƕ})) = \mathcal{E}(\Omega)$. Therefore, $L_c^1(\mathcal{E}(\text{ƕ})) \neq U_c^1(L_c^1(\mathcal{E}(\text{ƕ}))$.

(3) Let $\text{ƕ} = (\mho(\text{ƕ}), \mathcal{E}(\text{ƕ}))$ such that $\mho(\text{ƕ}) = \{\mathfrak{z}_1, \mathfrak{z}_2\}$ and $\mathcal{E}(\text{ƕ}) = \{\varrho_4, \varrho_5\}$. Then $U_c^1(\mathcal{E}(\text{ƕ})) = \{\varrho_4, \varrho_5\}$, $L_c^1(U_c^1(\mathcal{E}(\text{ƕ})) = \emptyset$. Therefore, $\mathcal{E}(\text{ƕ}) \nsubseteq L_c^1(U_c^1(\mathcal{E}(\text{ƕ}))$.

(4) Let $\text{ƕ} = (\mho(\text{ƕ}), \mathcal{E}(\text{ƕ}))$ such that $\mho(\text{ƕ}) = \{\mathfrak{z}_3, \mathfrak{z}_4\}$ and $\mathcal{E}(\text{ƕ}) = \{\varrho_1, \varrho_2\}$. Then $L_c^1(\mathcal{E}(\text{ƕ})) = \{\varrho_1\}$, $L_c^1(L_c^1(\mathcal{E}(\text{ƕ})) = \emptyset$. Therefore, $L_c^1(\mathcal{E}(\text{ƕ})) \nsubseteq L_c^1(L_c^1(\mathcal{E}(\text{ƕ}))$.

(5) Let $\text{ƕ} = (\mho(\text{ƕ}), \mathcal{E}(\text{ƕ}))$ such that $\mho(\text{ƕ}) = \{\mathfrak{z}_1, \mathfrak{z}_2\}$ and $\mathcal{E}(\text{ƕ}) = \{\varrho_5\}$. And $\text{ƙ} = (\mho(\text{ƙ}), \mathcal{E}(\text{ƙ}))$ such that $\mho(\text{ƙ}) = \{\mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_4\}$ and $\mathcal{E}(\text{ƙ}) = \{\varrho_1, \varrho_2\}$. Then $L_c^1(\mathcal{E}(\text{ƕ})) = \emptyset$, $L_c^1(\mathcal{E}(\text{ƙ})) = \{\varrho_1\}$, $L_c^1(\mathcal{E}(\text{ƕ})) \cup L_c^1(\mathcal{E}(\text{ƙ})) = \{\varrho_1\}$, $L_c^1(\mathcal{E}(\text{ƕ}) \cup \mathcal{E}(\text{ƙ})) = \{\varrho_1, \varrho_5\}$. Therefore, $L_c^1(\mathcal{E}(\text{ƕ}) \cup \mathcal{E}(\text{ƙ})) \neq L_c^1(\mathcal{E}(\text{ƕ})) \cup L_c^1(\mathcal{E}(\text{ƙ}))$.

(6) Let $\text{ƕ} = (\mho(\text{ƕ}), \mathcal{E}(\text{ƕ}))$ such that $\mho(\text{ƕ}) = \{\mathfrak{z}_3, \mathfrak{z}_4\}$ and $\mathcal{E}(\text{ƕ}) = \{\varrho_1, \varrho_2\}$. Then $U_c^1(\mathcal{E}(\text{ƕ})) = \{\varrho_1, \varrho_2, \varrho_3\}$, $U_c^1(U_c^1(\mathcal{E}(\text{ƕ})) = \mathcal{E}(\Omega)$. Therefore, $U_c^1(\mathcal{E}(\text{ƕ})) \neq U_c^1(U_c^1(\mathcal{E}(\text{ƕ}))$.

(7) Let $\text{ƕ} = (\mho(\text{ƕ}), \mathcal{E}(\text{ƕ}))$ such that $\mho(\text{ƕ}) = \{\mathfrak{z}_4\}$ and $\mathcal{E}(\text{ƕ}) = \{\varrho_1\}$. Then $U_c^1(\mathcal{E}(\text{ƕ})) = \{\varrho_1\}$, $L_c^1(U_c^1(\mathcal{E}(\text{ƕ})) = \emptyset$. Therefore, $U_c^1(\mathcal{E}(\text{ƕ})) \neq L_c^1(U_c^1(\mathcal{E}(\text{ƕ}))$.

(8) Let $\text{ƕ} = (\mho(\text{ƕ}), \mathcal{E}(\text{ƕ}))$ such that $\mho(\text{ƕ}) = \{\mathfrak{z}_2, \mathfrak{z}_3, \mathfrak{z}_4\}$ and $\mathcal{E}(\text{ƕ}) = \{\varrho_1, \varrho_2, \varrho_4\}$. Then $L_c^1(\mathcal{E}(\text{ƕ})) = \{\varrho_1, \varrho_4\}$, $U_c^1(L_c^1(\mathcal{E}(\text{ƕ})) = \mathcal{E}(\Omega)$. Therefore, $U_c^1(L_c^1(\mathcal{E}(\text{ƕ})) \nsubseteq \mathcal{E}(\text{ƕ})$.

(9) Let $\text{ƕ} = (\mho(\text{ƕ}), \mathcal{E}(\text{ƕ}))$ such that $\mho(\text{ƕ}) = \{\mathfrak{z}_3, \mathfrak{z}_4\}$ and $\mathcal{E}(\text{ƕ}) = \{\varrho_1, \varrho_2\}$. Then $U_c^1(\mathcal{E}(\text{ƕ})) = \{\varrho_1, \varrho_2, \varrho_3\}$, $U_c^1(U_c^1(\mathcal{E}(\text{ƕ})) = \mathcal{E}(\Omega)$. Therefore, $U_c^1(U_c^1(\mathcal{E}(\text{ƕ})) \nsubseteq U_c^1(\mathcal{E}(\text{ƕ}))$.

(10) Let $\text{ƕ} = (\mho(\text{ƕ}), \mathcal{E}(\text{ƕ}))$ such that $\mho(\text{ƕ}) = \{\mathfrak{z}_4\}$ and $\mathcal{E}(\text{ƕ}) = \{\varrho_1\}$. And $\text{ƙ} = (\mho(\text{ƙ}), \mathcal{E}(\text{ƙ}))$ such that $\mho(\text{ƙ}) = \{\mathfrak{z}_3, \mathfrak{z}_4\}$ and $\mathcal{E}(\text{ƙ}) = \{\varrho_2\}$. Then $U_c^1(\mathcal{E}(\text{ƕ})) = \{\varrho_1\}$, $U_c^1(\mathcal{E}(\text{ƙ})) = \{\varrho_2\}$, $U_c^1(\mathcal{E}(\text{ƕ})) \cup U_c^1(\mathcal{E}(\text{ƙ})) = \{\varrho_1, \varrho_2\}$, $U_c^1(\mathcal{E}(\text{ƕ}) \cup \mathcal{E}(\text{ƙ})) = \{\varrho_1, \varrho_2, \varrho_3\}$. Therefore, $U_c^1(\mathcal{E}(\text{ƕ}) \cup \mathcal{E}(\text{ƙ})) \neq U_c^1(\mathcal{E}(\text{ƕ})) \cup U_c^1(\mathcal{E}(\text{ƙ}))$.

Corollary 2.10. *Let* $\mathfrak{Z} = (\text{Ʊ}(\Omega), \mathcal{E}(\Omega))$ *be a generalized approximation space and* Ω *be antisymmetric und. g. and* $\text{hʋ}, \text{ƙ} \subseteq \Omega$. *Then*

(1) $L_c^1(\mathcal{E}(\text{hʋ})) = \mathcal{E}(\text{hʋ})$.

(2) $L_c^1(\mathcal{E}(\text{hʋ}) \cap \mathcal{E}(\text{ƙ})) = L_c^1(\mathcal{E}(\text{hʋ})) \cap L_c^1(\mathcal{E}(\text{ƙ}))$.

(3) $L_c^1(\mathcal{E}(\text{hʋ})) \cup L_c^1\big(\mathcal{E}(\text{ƙ})\big) = L_c^1(\mathcal{E}(\text{hʋ}) \cup \mathcal{E}(\text{ƙ}))$.

(4) $\mathcal{E}(\text{hʋ}) = U_c^1(\mathcal{E}(\text{hʋ}))$.

(5) $U_c^1(\mathcal{E}(\text{hʋ}) \cap \mathcal{E}(\text{ƙ})) = U_c^1(\mathcal{E}(\text{hʋ})) \cap U_c^1(\mathcal{E}(\text{ƙ}))$.

(6) $U_c^1(\mathcal{E}(\text{hʋ})) \bigcup U_c^1(\mathcal{E}(\text{ƙ})) = U_c^1(\mathcal{E}(\text{hʋ}) \cup \mathcal{E}(\text{ƙ}))$.

(7) $L_c^1(\mathcal{E}(\text{hʋ})) = U_c^1(\mathcal{E}(\text{hʋ}))$.

Proof.

(1) Let Ω be an antisymmetric und. g. and $\text{hʋ} \subseteq \Omega$ by Proposition (2.6(1)) we get $L_{\mathrm{c}}^1(\mathcal{E}(\text{hʋ})) \subseteq \mathcal{E}(\text{hʋ}) - - - (1)$. Let $\varrho \in \mathcal{E}(\text{hʋ})$ and $\varrho \notin L_{\mathrm{c}}^1(\mathcal{E}(\text{hʋ})) \Rightarrow \forall\, C\mathcal{E}(\varrho) \nsubseteq \mathcal{E}(\text{hʋ})$ and this contradiction with Ω is antisymmetric since $\forall\, \varrho \in \mathcal{E}(\text{hʋ}) \Rightarrow I\mathcal{E}(\varrho) \subseteq \mathcal{E}(\text{hʋ})$ and hence $\varrho \in L_{\mathrm{c}}^1(\mathcal{E}(\text{hʋ})) \Rightarrow \mathcal{E}(\text{hʋ}) \subseteq L_{\mathrm{c}}^1(\mathcal{E}(\text{hʋ})) - - - (2)$. From (1) and (2) we get $L_{\mathrm{c}}^1(\mathcal{E}(\text{hʋ})) = \mathcal{E}(\text{hʋ})$.

(2) Let Ω be an antisymmetric und. g. and $\text{hʋ}, \text{ƙ} \subseteq \Omega$ by Proposition (2.6(5)) we get $L_{\mathrm{c}}^1(\mathcal{E}(\text{hʋ}) \cap \mathcal{E}(\text{ƙ})) \subseteq L_{\mathrm{c}}^1(\mathcal{E}(\text{hʋ})) \cap L_{\mathrm{c}}^1(\mathcal{E}(\text{ƙ})) - - - (1)$. Let $\varrho \in [L_{\mathrm{c}}^1(\mathcal{E}(\text{hʋ})) \cap L_{\mathrm{c}}^1(\mathcal{E}(\text{ƙ}))] \Rightarrow \varrho \in L_{\mathrm{c}}^1(\mathcal{E}(\text{hʋ})) \wedge \varrho \in L_{\mathrm{c}}^1(\mathcal{E}(\text{ƙ}))$ by Corollary (2.10(1)) we get $\varrho \in \mathcal{E}(\text{hʋ}) \wedge \varrho \in \mathcal{E}(\text{ƙ}) \Rightarrow \varrho \in \big(\mathcal{E}(\text{hʋ}) \cap \mathcal{E}(\text{ƙ})\big)$ by Corollary (2.10(1)) we get $\varrho \in L_{\mathrm{c}}^1(\mathcal{E}(\text{hʋ}) \cap \mathcal{E}(\text{ƙ})) \Rightarrow L_{\mathrm{c}}^1(\mathcal{E}(\text{hʋ})) \cap L_{\mathrm{c}}^1(\mathcal{E}(\text{ƙ})) \subseteq L_{\mathrm{c}}^1(\mathcal{E}(\text{hʋ}) \cap \mathcal{E}(\text{ƙ})) - - - (2)$. From (1) and (2) we get $L_{\mathrm{c}}^1(\mathcal{E}(\text{hʋ}) \cap \mathcal{E}(\text{ƙ})) = L_{\mathrm{c}}^1(\mathcal{E}(\text{hʋ})) \cap L_{\mathrm{c}}^1(\mathcal{E}(\text{ƙ}))$.

(3) Let Ω be an antisymmetric und. g. and $\text{hʋ}, \text{ƙ} \subseteq \Omega$ by Proposition (2.6(6)) we get $L_{\mathrm{c}}^1(\mathcal{E}(\text{hʋ})) \bigcup L_{\mathrm{c}}^1(\mathcal{E}(\text{ƙ})) \subseteq L_{\mathrm{c}}^1(\mathcal{E}(\text{hʋ}) \cup \mathcal{E}(\text{ƙ})) - - - (1)$. Let $\varrho \in L_{\mathrm{c}}^1\big(\mathcal{E}(\text{hʋ}) \cup \mathcal{E}(\text{ƙ})\big)$ by Corollary (2.10(1)) we get $\varrho \in (\mathcal{E}(\text{hʋ}) \cup \mathcal{E}(\text{ƙ})) \Rightarrow \varrho \in \mathcal{E}(\text{hʋ}) \vee \varrho \in \mathcal{E}(\text{ƙ})$ by Corollary (2.10(1)) we get $\varrho \in L_{\mathrm{c}}^1(\mathcal{E}(\text{hʋ})) \vee \varrho \in L_{\mathrm{c}}^1(\mathcal{E}(\text{ƙ})) \Rightarrow \varrho \in [L_{\mathrm{c}}^1(\mathcal{E}(\text{hʋ})) \bigcup L_{\mathrm{c}}^1(\mathcal{E}(\text{ƙ}))] \Rightarrow L_{\mathrm{c}}^1\big(\mathcal{E}(\text{hʋ}) \cup \mathcal{E}(\text{ƙ})\big) \subseteq L_{\mathrm{c}}^1(\mathcal{E}(\text{hʋ})) \bigcup L_{\mathrm{c}}^1(\mathcal{E}(\text{ƙ})) - - - (2)$. From (1) and (2) we get $L_{\mathrm{c}}^1(\mathcal{E}(\text{hʋ})) \bigcup L_{\mathrm{c}}^1(\mathcal{E}(\text{ƙ})) = L_{\mathrm{c}}^1(\mathcal{E}(\text{hʋ}) \cup \mathcal{E}(\text{ƙ}))$.

(4) Let Ω be an antisymmetric und. g. and $\text{hʋ} \subseteq \Omega$ by Proposition (2.7(1)) we get

$\mathcal{E}(ƕ) \subseteq U_c^1(\mathcal{E}(ƕ)) - - - (1)$. Let $\varrho \in U_c^1(\mathcal{E}(ƕ)) \Rightarrow \varrho \in \mathcal{E}(ƕ)$ or $\varrho \in \mathcal{E}(\Omega) - \mathcal{E}(ƕ)$; $\forall\, C\mathcal{E}(\varrho) \cap \mathcal{E}(ƕ) \neq \emptyset$. If $\varrho \in \mathcal{E}(ƕ) \Rightarrow U_c^1(\mathcal{E}(ƕ)) \subseteq \mathcal{E}(ƕ)$. If $\varrho \in \mathcal{E}(\Omega) - \mathcal{E}(ƕ)$; $\forall\, C\mathcal{E}(\varrho) \cap \mathcal{E}(ƕ) \neq \emptyset \Rightarrow \exists\, \varrho_1 \in C\mathcal{E}(\varrho)$ and $\varrho_1 \in \mathcal{E}(ƕ) \Rightarrow \varrho_1 \in I\mathcal{E}(\varrho)$ and this contradiction with Ω is antisymmetric and hence $\varrho \notin \mathcal{E}(\Omega) - \mathcal{E}(ƕ)$; $\forall\, C\mathcal{E}(\varrho) \cap \mathcal{E}(ƕ) \neq \emptyset$. Thus $\varrho \in \mathcal{E}(ƕ) \Rightarrow U_c^1(\mathcal{E}(ƕ)) \subseteq \mathcal{E}(ƕ) - - - (2)$. From (1) and (2) we get $\mathcal{E}(ƕ) = U_c^1(\mathcal{E}(ƕ))$.

(5) Let Ω be an antisymmetric und. g. and $ƕ, ƙ \subseteq \Omega$ by Proposition (2.7(5)) we get $U_c^1(\mathcal{E}(ƕ) \cap \mathcal{E}(ƙ)) \subseteq U_c^1(\mathcal{E}(ƕ)) \cap U_c^1(\mathcal{E}(ƙ)) - - - (1)$. Let $\varrho \in [U_c^1(\mathcal{E}(ƕ)) \cap U_c^1(\mathcal{E}(ƙ))] \Rightarrow \varrho \in U_c^1(\mathcal{E}(ƕ)) \wedge \varrho \in U_c^1(\mathcal{E}(ƙ))$ by Corollary (2.10(4)) we get $\varrho \in \mathcal{E}(ƕ) \wedge \varrho \in \mathcal{E}(ƙ) \Rightarrow \varrho \in (\mathcal{E}(ƕ) \cap \mathcal{E}(ƙ))$ by Corollary (2.10(4)) we get $\varrho \in U_c^1(\mathcal{E}(ƕ) \cap \mathcal{E}(ƙ)) \Rightarrow U_c^1(\mathcal{E}(ƕ)) \cap U_c^1(\mathcal{E}(ƙ)) \subseteq U_c^1(\mathcal{E}(ƕ) \cap \mathcal{E}(ƙ)) - - - (2)$. From (1) and (2) we get $U_c^1(\mathcal{E}(ƕ) \cap \mathcal{E}(ƙ)) = U_c^1(\mathcal{E}(ƕ)) \cap U_c^1(\mathcal{E}(ƙ))$.

(6) Let Ω be an antisymmetric und. g. and $ƕ, ƙ \subseteq \Omega$ by Proposition (2.7(6)) we get $U_c^1(\mathcal{E}(ƕ)) \cup U_c^1(\mathcal{E}(ƙ)) \subseteq U_c^1(\mathcal{E}(ƕ) \cup \mathcal{E}(ƙ)) - - - (1)$. Let $\varrho \in U_c^1(\mathcal{E}(ƕ) \cup \mathcal{E}(ƙ))$ by Corollary (2.10(4)) we get $\varrho \in (\mathcal{E}(ƕ) \cup \mathcal{E}(ƙ)) \Rightarrow \varrho \in \mathcal{E}(ƕ) \vee \varrho \in \mathcal{E}(ƙ)$ by Corollary (2.10(4)) we get $\varrho \in U_c^1(\mathcal{E}(ƕ)) \vee \varrho \in U_c^1(\mathcal{E}(ƙ)) \Rightarrow \varrho \in [U_c^1(\mathcal{E}(ƕ)) \cup U_c^1(\mathcal{E}(ƙ))] \Rightarrow U_c^1(\mathcal{E}(ƕ) \cup \mathcal{E}(ƙ)) \subseteq U_c^1(\mathcal{E}(ƕ)) \cup U_c^1(\mathcal{E}(ƙ)) - - - (2)$. From (1) and (2) we get $U_c^1(\mathcal{E}(ƕ)) \cup U_c^1(\mathcal{E}(ƙ)) = U_c^1(\mathcal{E}(ƕ) \cup \mathcal{E}(ƙ))$.

(7) Let Ω be an antisymmetric und. g. and $ƕ \subseteq \Omega$ by Corollary (2.10(1)) we get $L_c^1(\mathcal{E}(ƕ)) = \mathcal{E}(ƕ) - - - (1)$. And by Corollary (2.10(4)) we get $U_c^1(\mathcal{E}(ƕ)) = \mathcal{E}(ƕ) - - - (2)$. From (1) and (2) we get $L_c^1(\mathcal{E}(ƕ)) = U_c^1(\mathcal{E}(ƕ))$.

Example 2.11. Let $\Omega = (Ʊ(\Omega), \mathcal{E}(\Omega))$ such that $Ʊ(\Omega) = \{ꝩ_1, ꝩ_2, ꝩ_3, ꝩ_4\}$ and $\mathcal{E}(\Omega) = \{\varrho_1, \varrho_2, \varrho_3, \varrho_4\}$.

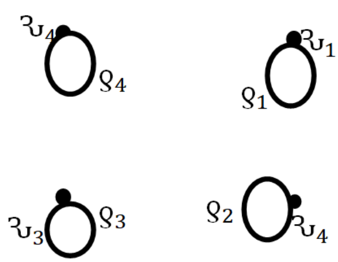

Figure 2.2. Antisymmetric und. g. Ω given in Example (2.11).

Hence $Þ_c$ is defined by

$$Þ_c(\wp_1) = \{\{\wp_1\}, \{\wp_2, \wp_3, \wp_4\}\},\ Þ_c(\wp_2) = \{\{\wp_2\}, \{\wp_1, \wp_3, \wp_4\}\},$$

$$Þ_c(\wp_3) = \{\{\wp_3\}, \{\wp_1, \wp_2, \wp_4\}\},\ Þ_c(\wp_4) = \{\{\wp_4\}, \{\wp_1, \wp_2, \wp_3\}\}.$$

Table 2.5. $L_c^1(\mathcal{E}(ƕ))$, and $U_c^1(\mathcal{E}(ƕ))$ for all $ƕ \subseteq \Omega$.

$\mathcal{E}(ƕ)$	$L_c^1(\mathcal{E}(ƕ))$	$U_c^1(\mathcal{E}(ƕ))$
$\{\wp_1\}$	$\{\wp_1\}$	$\{\wp_1\}$
$\{\wp_2\}$	$\{\wp_2\}$	$\{\wp_2\}$
$\{\wp_3\}$	$\{\wp_3\}$	$\{\wp_3\}$
$\{\wp_4\}$	$\{\wp_4\}$	$\{\wp_4\}$
$\{\wp_1, \wp_2\}$	$\{\wp_1, \wp_2\}$	$\{\wp_1, \wp_2\}$
$\{\wp_1, \wp_3\}$	$\{\wp_1, \wp_3\}$	$\{\wp_1, \wp_3\}$
$\{\wp_1, \wp_4\}$	$\{\wp_1, \wp_4\}$	$\{\wp_1, \wp_4\}$
$\{\wp_2, \wp_3\}$	$\{\wp_2, \wp_3\}$	$\{\wp_2, \wp_3\}$
$\{\wp_2, \wp_4\}$	$\{\wp_2, \wp_4\}$	$\{\wp_2, \wp_4\}$
$\{\wp_3, \wp_4\}$	$\{\wp_3, \wp_4\}$	$\{\wp_3, \wp_4\}$
$\{\wp_1, \wp_2, \wp_3\}$	$\{\wp_1, \wp_2, \wp_3\}$	$\{\wp_1, \wp_2, \wp_3\}$
$\{\wp_1, \wp_2, \wp_4\}$	$\{\wp_1, \wp_2, \wp_4\}$	$\{\wp_1, \wp_2, \wp_4\}$
$\{\wp_1, \wp_3, \wp_4\}$	$\{\wp_1, \wp_3, \wp_4\}$	$\{\wp_1, \wp_3, \wp_4\}$
$\{\wp_2, \wp_3, \wp_4\}$	$\{\wp_2, \wp_3, \wp_4\}$	$\{\wp_2, \wp_3, \wp_4\}$
$\mathcal{E}(\Omega)$	$\mathcal{E}(\Omega)$	$\mathcal{E}(\Omega)$
$\emptyset$	$\emptyset$	$\emptyset$

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