



Special Cases of Generalized Leonardo Numbers: Modified p -Leonardo, p -Leonardo-Lucas and p -Leonardo Numbers

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Abstract

In this paper, we define and investigate modified p -Leonardo, p -Leonardo-Lucas and p -Leonardo sequences as special cases of the generalized Leonardo sequence. We present Binet's formulas, generating functions, Simson formulas, and the summation formulas for these sequences. Moreover, we give some identities and matrices related with these sequences. Furthermore, we show that there are close relations between modified p -Leonardo, p -Leonardo-Lucas, p -Leonardo numbers and Fibonacci, Lucas numbers.

1 Introduction and Preliminaries

The sequence of Fibonacci numbers $\{F_n\}$ is defined by

$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 2, \quad F_0 = 0, \quad F_1 = 1,$$

and the sequence of Lucas numbers $\{L_n\}$ is defined by

$$L_n = L_{n-1} + L_{n-2}, \quad n \geq 2, \quad L_0 = 2, \quad L_1 = 1.$$

The generalizations of Fibonacci and Lucas sequences lead to several nice and interesting sequences.

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In [3], Catarino and Borges introduced a new sequence of numbers called Lenonardo numbers. They defined Lenonardo numbers as

$$l_n = l_{n-1} + l_{n-2} + 1, \quad n \geq 2,$$

with $l_0 = 1, l_1 = 1$. Lenonardo sequences has been studied by many authors, see for example, [1,2,4,6,7].

In Soykan [6], two sequences related to Lenonardo sequence were defined: Modified Leonardo and Leonardo-Lucas numbers are defined as

$$G_n = G_{n-1} + G_{n-2} + 1, \quad \text{with } G_0 = 0, G_1 = 1, \quad n \geq 2,$$

and

$$H_n = H_{n-1} + H_{n-2} - 1, \quad \text{with } H_0 = 3, H_1 = 2, \quad n \geq 2,$$

respectively. The sequences $\{G_n\}$, $\{H_n\}$ and $\{l_n\}$ satisfy the following third order linear recurrences:

$$\begin{aligned} G_n &= 2G_{n-1} - G_{n-3}, & G_0 = 0, G_1 = 1, G_2 = 2, \\ H_n &= 2H_{n-1} - H_{n-3}, & H_0 = 3, H_1 = 2, H_2 = 4, \\ l_n &= 2l_{n-1} - l_{n-3}, & l_0 = 1, l_1 = 1, l_2 = 3. \end{aligned}$$

In Soykan [6], generalized Leonardo sequence $\{W_n\}_{n \geq 0} = \{W_n(W_0, W_1, W_2)\}_{n \geq 0}$ is defined by the third-order recurrence relation

$$W_n = 2W_{n-1} - W_{n-3} \tag{1.1}$$

with the initial values $W_0 = c_0, W_1 = c_1, W_2 = c_2$ not all being zero. The sequence $\{W_n\}_{n \geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = 2W_{-(n-2)} - W_{-(n-3)}$$

for $n = 1, 2, 3, \dots$. Therefore, recurrence (1.1) holds for all integer n .

Note that the sequences $\{G_n\}$, $\{H_n\}$ and $\{l_n\}$ are the special cases of the generalized Leonardo sequence $\{W_n\}$. Next, we list some properties of the sequences $\{W_n\}$, $\{G_n\}$, $\{H_n\}$ and $\{l_n\}$ as follows (see Soykan [6] for details).

- Binet formula of generalized Leonardo numbers can be given as

$$\begin{aligned}
 W_n &= \frac{z_1\alpha^n}{(\alpha - \beta)(\alpha - \gamma)} + \frac{z_2\beta^n}{(\beta - \alpha)(\beta - \gamma)} + \frac{z_3\gamma^n}{(\gamma - \alpha)(\gamma - \beta)} \quad (1.2) \\
 &= \frac{z_1\alpha^{n+1} - z_2\beta^{n+1}}{(\alpha - \beta)} - z_3
 \end{aligned}$$

where

$$z_1 = W_2 - (2 - \alpha)W_1 + (1 - \alpha)W_0, \quad (1.3)$$

$$z_2 = W_2 - (2 - \beta)W_1 + (1 - \beta)W_0, \quad (1.4)$$

$$z_3 = W_2 - W_1 - W_0. \quad (1.5)$$

Here, α, β and γ are the roots of the cubic equation

$$x^3 - 2x^2 + 1 = (x^2 - x - 1)(x - 1) = 0.$$

Moreover,

$$\begin{aligned}
 \alpha &= \frac{1 + \sqrt{5}}{2}, \\
 \beta &= \frac{1 - \sqrt{5}}{2}, \\
 \gamma &= 1.
 \end{aligned}$$

Note that

$$\begin{aligned}
 \alpha + \beta + \gamma &= 2, \\
 \alpha\beta + \alpha\gamma + \beta\gamma &= 0, \\
 \alpha\beta\gamma &= -1,
 \end{aligned}$$

or

$$\alpha + \beta = 1, \quad \alpha\beta = -1.$$

- For all integers n , modified Leonardo, Leonardo-Lucas and Leonardo

numbers can be expressed using Binet's formulas as

$$\begin{aligned} G_n &= \frac{\alpha^{n+2} - \beta^{n+2}}{\alpha - \beta} - 1, \\ H_n &= \alpha^n + \beta^n + 1, \\ l_n &= \frac{2(\alpha^{n+1} - \beta^{n+1})}{\alpha - \beta} - 1, \end{aligned}$$

respectively. Note that Binet's formulas of Fibonacci and Lucas numbers, respectively, are

$$\begin{aligned} F_n &= \frac{\alpha^n}{\alpha - \beta} + \frac{\beta^n}{\beta - \alpha} = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \\ L_n &= \alpha^n + \beta^n, \end{aligned}$$

and so

$$G_n = F_{n+2} - 1, \quad (1.6)$$

$$H_n = L_n + 1, \quad (1.7)$$

$$l_n = 2F_{n+1} - 1. \quad (1.8)$$

- We have the following interrelations:

$$5G_n = 3L_{n+1} + L_n - 5, \quad (1.9)$$

$$H_n = 2F_{n+1} - F_n + 1, \quad (1.10)$$

$$5l_n = 2L_{n+1} + 4L_n - 5. \quad (1.11)$$

- The ordinary generating function $f_{W_n}(x) = \sum_{n=0}^{\infty} W_n x^n$ of the generalized Leonardo sequence is given by

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0 + (W_1 - 2W_0)x + (W_2 - 2W_1)x^2}{1 - 2x + x^3}.$$

- (Simson Formula of Generalized Leonardo Numbers) For all integers n , we

have

$$\begin{aligned} & \begin{vmatrix} W_{n+2} & W_{n+1} & W_n \\ W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \end{vmatrix} \\ &= (-1)^n(-W_2^3 + W_1^3 - W_0^3 + 2(2W_1W_2^2 + W_0W_1^2 - 2W_1^2W_2 + W_0^2W_2) \\ & \quad - 3W_0W_1W_2). \end{aligned}$$

- We now present a few special identities for the generalized Leonardo sequence $\{W_n\}$. For all integers n and m , the following identities hold:

- (Catalan’s identity of the generalized Leonardo sequence)

$$\begin{aligned} W_{n+m}W_{n-m} - W_n^2 &= ((W_1 - W_0)F_{n+m+2} + (W_2 - 2W_1 + W_0)F_{n+m+1} - \\ & (W_2 - W_1 - W_0))((W_1 - W_0)F_{n-m+2} + (W_2 - 2W_1 + W_0)F_{n-m+1} - (W_2 - \\ & W_1 - W_0)) - ((W_1 - W_0)F_{n+2} + (W_2 - 2W_1 + W_0)F_{n+1} - (W_2 - W_1 - W_0))^2. \end{aligned}$$

- (Cassini’s identity of the generalized Leonardo sequence)

$$\begin{aligned} W_{n+1}W_{n-1} - W_n^2 &= (-W_2^2 - W_1^2 + W_0^2 + 3W_1W_2 - W_0W_2 - W_0W_1) \\ & F_{n+1}^2 + (W_2^2 + W_1^2 - W_0^2 - 3W_1W_2 + W_0W_2 + W_0W_1)F_n^2 + (W_2 - 3W_1 + \\ & 2W_0)(W_2 - W_1 - W_0)F_{n+1} + (2W_2 - 5W_1 + 3W_0)(-W_2 + W_1 + W_0)F_n + \\ & (W_2^2 + W_1^2 - W_0^2 - 3W_1W_2 + W_0W_2 + W_0W_1)F_{n+1}F_n. \end{aligned}$$

- (d’Ocagne’s identity)

$$\begin{aligned} W_{m+1}W_n - W_mW_{n+1} &= (W_0 - W_1)(W_0 + W_1 - W_2)(F_{n+1} - F_{m+1}) + \\ & (W_0 - 2W_1 + W_2)(W_0 + W_1 - W_2)(F_m - F_n) + (W_0^2 - W_1^2 - W_2^2 - W_0W_1 - \\ & W_0W_2 + 3W_1W_2)(F_nF_{m+1} - F_mF_{n+1}). \end{aligned}$$

- (Melham’s identity)

$$\begin{aligned} W_{n+1}W_{n+2}W_{n+6} - W_{n+3}^3 &= (W_0^2 - W_1^2 - W_2^2 - W_0W_1 - W_0W_2 + \\ & 3W_1W_2)(-(W_1 - W_2)F_{n+1}^3 + (W_0 - W_1)F_n^3 + (W_0 - W_2)F_{n+1}F_n^2 - (W_0 - \\ & 2W_1 + W_2)F_{n+1}^2F_n) + (W_0 + W_1 - W_2)((5W_0 + 2W_1 - 7W_2)(W_0 + W_1 - \\ & 2W_2)F_{n+1}^2 + (W_0 - W_2)(2W_0 + 3W_1 - 5W_2)F_n^2 + (7W_0^2 + 3W_1^2 + 17W_2^2 + \\ & 7W_0W_1 - 21W_0W_2 - 13W_1W_2)F_{n+1}F_n) + (W_0 + W_1 - W_2)^2(-(4W_0 + \\ & 3W_1 - 7W_2)F_{n+1} - (3W_0 + W_1 - 4W_2)F_n). \end{aligned}$$

- For $n \in \mathbb{Z}$, generalized Leonardo numbers have the following identity:

$$W_{-n} = (-1)^{-n}(W_{2n} - H_n W_n + \frac{1}{2}(H_n^2 - H_{2n})W_0).$$

- If we define

$$A = \begin{pmatrix} 2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$B_n = \begin{pmatrix} G_{n+1} & -G_{n-1} & -G_n \\ G_n & -G_{n-2} & -G_{n-1} \\ G_{n-1} & -G_{n-3} & -G_{n-2} \end{pmatrix},$$

then

$$B_n = A^n$$

i.e.,

$$\begin{pmatrix} 2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} G_{n+1} & -G_{n-1} & -G_n \\ G_n & -G_{n-2} & -G_{n-1} \\ G_{n-1} & -G_{n-3} & -G_{n-2} \end{pmatrix}.$$

- For all integers m, n , we have

$$W_{n+m} = W_n G_{m+1} - W_{n-1} G_{m-1} - W_{n-2} G_m. \quad (1.12)$$

- For all integers m, n , we have

$$\begin{aligned} & (W_0 + W_1 - W_2)(W_0^2 - W_1^2 - W_2^2 - W_0 W_1 - W_0 W_2 + 3W_1 W_2)W_{n+m} \\ = & W_n(-(W_0^2 - 2W_1^2 + W_1 W_2)W_{m+3} + (W_2^2 + W_0 W_1 - 2W_1 W_2)W_{m+2} \\ & + (-W_1^2 + W_0 W_2)W_{m+1}) \\ & - W_{n-1}(-(W_0^2 - 2W_1^2 + W_1 W_2)W_{m+1} + (W_2^2 + W_0 W_1 - 2W_1 W_2)W_m \\ & + (-W_1^2 + W_0 W_2)W_{m-1}) \\ & - W_{n-2}(-(W_0^2 - 2W_1^2 + W_1 W_2)W_{m+2} + (W_2^2 + W_0 W_1 - 2W_1 W_2)W_{m+1} \\ & + (-W_1^2 + W_0 W_2)W_m). \end{aligned}$$

Note that all the properties of the sequences $\{W_n\}$, $\{G_n\}$, $\{H_n\}$ and $\{l_n\}$ listed above are given and proved in Soykan [6]. In the next sections, we present new results.

2 Special Cases of Generalized Leonardo Sequence

Now, we define new three sequences related to modified Leonardo, Leonardo-Lucas and Leonardo sequences: Modified p -Leonardo, p -Leonardo-Lucas and p -Leonardo numbers are defined, respectively, as

$$\begin{aligned} G_{p,n} &= G_{p,n-1} + G_{p,n-2} + p, & G_{p,0} &= 0, G_{p,1} = 1, & n &\geq 2, \\ H_{p,n} &= H_{p,n-1} + H_{p,n-2} - p, & H_{p,0} &= 3, H_{p,1} = p + 1, & n &\geq 2, \\ l_{p,n} &= l_{p,n-1} + l_{p,n-2} + p, & l_{p,0} &= 1, l_{p,1} = 1 & n &\geq 2, \end{aligned}$$

where p is an arbitrary non-zero complex (or real) number.

The sequences $\{G_{p,n}\}$, $\{H_{p,n}\}$ and $\{l_{p,n}\}$ satisfy the following third order linear recurrences:

$$\begin{aligned} G_{p,n} &= 2G_{p,n-1} - G_{p,n-3}, & G_{p,0} &= 0, G_{p,1} = 1, G_{p,2} = p + 1, \\ H_{p,n} &= 2H_{p,n-1} - H_{p,n-3}, & H_{p,0} &= 3, H_{p,1} = p + 1, H_{p,2} = 4, \\ l_{p,n} &= 2l_{p,n-1} - l_{p,n-3}, & l_{p,0} &= 1, l_{p,1} = 1, l_{p,2} = p + 2. \end{aligned}$$

Note that

$$\begin{aligned} G_{1,n} &= G_n, \\ H_{1,n} &= H_n, \\ l_{1,n} &= l_n, \end{aligned}$$

and note also that the sequences $\{G_{p,n}\}$, $\{H_{p,n}\}$ and $\{l_{p,n}\}$ are the special cases of the generalized Leonardo sequence $\{W_n\}$. So, we can use all the results given in [6].

Next, we present the first few values of the modified p -Leonardo, p -Leonardo-Lucas and p -Leonardo numbers with positive and negative subscripts:

Table 1: The first few values of the special third-order numbers with positive and negative subscripts.

n	0	1	2	3	4	5	6	7	8	9
$G_{p,n}$	0	1	$p+1$	$2p+2$	$4p+3$	$7p+5$	$12p+8$	$20p+13$	$33p+21$	$54p+34$
$G_{p,-n}$		$1-p$	-1	$2-2p$	$p-3$	$5-4p$	$4p-8$	$13-9p$	$12p-21$	$34-22p$
$H_{p,n}$	3	$p+1$	4	5	$9-p$	$14-2p$	$23-4p$	$37-7p$	$60-12p$	$97-20p$
$H_{p,-n}$		$2p-2$	$5-p$	$4p-7$	$12-4p$	$9p-19$	$31-12p$	$22p-50$	$81-33p$	$56p-131$
$l_{p,n}$	1	1	$p+2$	$2p+3$	$4p+5$	$7p+8$	$12p+13$	$20p+21$	$33p+34$	$54p+55$
$l_{p,-n}$		$-p$	1	$-2p-1$	$p+2$	$-4p-3$	$4p+5$	$-9p-8$	$12p+13$	$-22p-21$

For all integers n , modified p -Leonardo, p -Leonardo-Lucas and p -Leonardo numbers can be expressed using Binet’s formulas as

$$\begin{aligned}
 G_{p,n} &= \frac{(\alpha + p - 1)}{(\alpha - \beta)(\alpha - 1)}\alpha^n + \frac{(\beta + p - 1)}{(\beta - \alpha)(\beta - 1)}\beta^n - p, \\
 H_{p,n} &= \frac{5 - 2\alpha + p(\alpha - 2)}{(\alpha - \beta)(\alpha - 1)}\alpha^n + \frac{5 - 2\beta + p(\beta - 2)}{(\beta - \alpha)(\beta - 1)}\beta^n + p, \\
 l_{p,n} &= \frac{1 + p}{(\alpha - \beta)(\alpha - 1)}\alpha^n + \frac{1 + p}{(\beta - \alpha)(\beta - 1)}\beta^n - p,
 \end{aligned}$$

respectively.

We now present generating functions.

Corollary 1. *The ordinary generating functions of modified p -Leonardo, p -Leonardo-Lucas and p -Leonardo numbers are given as*

$$\begin{aligned}
 \sum_{n=0}^{\infty} G_{p,n}x^n &= \frac{x + (p - 1)x^2}{1 - 2x + x^3}, \\
 \sum_{n=0}^{\infty} H_{p,n}x^n &= \frac{3 + (p - 5)x + (2 - 2p)x^2}{1 - 2x + x^3}, \\
 \sum_{n=0}^{\infty} l_{p,n}x^n &= \frac{1 - x + px^2}{1 - 2x + x^3},
 \end{aligned}$$

respectively.

Next, we give Simson’s formulas.

Corollary 2. For all integers n , Simson’s formulas of modified p -Leonardo, p -Leonardo-Lucas and p -Leonardo numbers are given as

$$\begin{aligned} \begin{vmatrix} G_{p,n+2} & G_{p,n+1} & G_{p,n} \\ G_{p,n+1} & G_{p,n} & G_{p,n-1} \\ G_{p,n} & G_{p,n-1} & G_{p,n-2} \end{vmatrix} &= p(-p^2 + p + 1)(-1)^n, \\ \begin{vmatrix} H_{p,n+2} & H_{p,n+1} & H_{p,n} \\ H_{p,n+1} & H_{p,n} & H_{p,n-1} \\ H_{p,n} & H_{p,n-1} & H_{p,n-2} \end{vmatrix} &= p(p^2 - 7p + 11)(-1)^n, \\ \begin{vmatrix} l_{p,n+2} & l_{p,n+1} & l_{p,n} \\ l_{p,n+1} & l_{p,n} & l_{p,n-1} \\ l_{p,n} & l_{p,n-1} & l_{p,n-2} \end{vmatrix} &= p(p + 1)^2(-1)^{n+1}, \end{aligned}$$

respectively.

3 Some Identities of Modified p -Leonardo, p -Leonardo-Lucas and p -Leonardo Numbers

In this section, we obtain some identities of modified p -Leonardo, p -Leonardo-Lucas and p -Leonardo numbers. First, we can give a few basic relations between $\{W_n\}$ and $\{G_{p,n}\}$.

Lemma 3. The following equalities are true:

- (a) $(-p^3 + p^2 + p)W_n = (-W_0p^2 + W_1p + W_0 + W_1 - W_2)G_{p,n+2} - (W_0 + W_1 - W_2 + 2pW_1 - pW_2 - 2p^2W_0 + p^2W_1)G_{p,n+1} - (W_0 + W_1 - W_2 + pW_0 - 2p^2W_1 + p^2W_2)G_{p,n}$.
- (b) $(W_0^3 - 2W_0^2W_2 - 2W_0W_1^2 + 3W_0W_1W_2 - W_1^3 + 4W_1^2W_2 - 4W_1W_2^2 + W_2^3)G_{p,n} = -(W_0^2 - W_1^2 - W_0W_2 + W_1W_2 - pW_1^2 + pW_0W_2)W_{n+2} + (W_0^2 + W_2^2 + W_0W_1 - 2W_0W_2 - W_1W_2 - pW_0^2 + 2pW_0W_2 - pW_1W_2)W_{n+1} - (W_1^2 + W_2^2 + W_0W_1 - W_0W_2 - 2W_1W_2 - pW_2^2 - pW_0W_1 + 2pW_1W_2)W_n$.

Proof. Note that all the identities hold for all integers p, n . We prove (a). To show (a), writing

$$W_n = a \times G_{p,n+2} + b \times G_{p,n+1} + c \times G_{p,n}$$

and solving the system of equations

$$W_0 = a \times G_{p,2} + b \times G_{p,1} + c \times G_{p,0}$$

$$W_1 = a \times G_{p,3} + b \times G_{p,2} + c \times G_{p,1}$$

$$W_2 = a \times G_{p,4} + b \times G_{p,3} + c \times G_{p,2}$$

we find that

$$a = \frac{1}{-p^3 + p^2 + p}(-W_0p^2 + W_1p + W_0 + W_1 - W_2),$$

$$b = -\frac{1}{-p^3 + p^2 + p}(W_0 + W_1 - W_2 + 2pW_1 - pW_2 - 2p^2W_0 + p^2W_1),$$

$$c = -\frac{1}{-p^3 + p^2 + p}(W_0 + W_1 - W_2 + pW_0 - 2p^2W_1 + p^2W_2).$$

The other equalities can be proved similarly. \square

Note that all the identities in the above Lemma can be proved by induction as well.

Next, we present a few basic relations between $\{W_n\}$ and $\{H_{p,n}\}$.

Lemma 4. *The following equalities are true:*

$$(a) \quad p(p^2 - 7p + 11)W_n = -(11W_0 + 11W_1 - 11W_2 - 5pW_0 - 4pW_1 + 2pW_2 + p^2W_2)H_{p,n+2} + (11W_0 + 11W_1 - 11W_2 - 8pW_0 - 3pW_1 + 4pW_2 + p^2W_0)H_{p,n+1} + (11W_0 + 11W_1 - 11W_2 - 4pW_0 - 8pW_1 + 5pW_2 + p^2W_1)H_{p,n}.$$

$$(b) \quad (-W_0^3 + 2W_0^2W_2 + 2W_0W_1^2 - 3W_0W_1W_2 + W_1^3 - 4W_1^2W_2 + 4W_1W_2^2 - W_2^3)H_{p,n} = (W_0^2 - 4W_1^2 - 3W_2^2 - 3W_0W_1 + 2W_0W_2 + 7W_1W_2 + pW_0^2 - 2pW_0W_2 + pW_1W_2)W_{n+2} + (2W_0^2 + 3W_1^2 + 5W_2^2 + 5W_0W_1 - 7W_0W_2 - 8W_1W_2 - 2pW_0^2 - pW_2^2 - pW_0W_1 + 4pW_0W_2)W_{n+1} + (W_1^2 - 3W_0^2 - 2W_2^2 - 2W_0W_1 + 5W_0W_2 + W_1W_2 + pW_1^2 + 2pW_2^2 + 2pW_0W_1 - pW_0W_2 - 4pW_1W_2)W_n$$

Now, we give a few basic relations between $\{W_n\}$ and $\{l_{p,n}\}$.

Lemma 5. *The following equalities are true:*

- (a) $p(p + 1)W_n = (-W_2 + W_1 + (1 + p)W_0)l_{p,n+2} + (W_2 + (p - 1)W_1 - (2p + 1)W_0)l_{p,n+1} + ((p + 1)W_2 - (2p + 1)W_1 - W_0)l_{p,n}$.
- (b) $(W_0^3 - 2W_0^2W_2 - 2W_0W_1^2 + 3W_0W_1W_2 - W_1^3 + 4W_1^2W_2 - 4W_1W_2^2 + W_2^3)l_{p,n} = (2W_1^2 - W_0^2 + W_2^2 + W_0W_1 - 3W_1W_2 + pW_1^2 - pW_0W_2)W_{n+2} - (W_1^2 + W_2^2 + W_0W_1 - W_0W_2 - 2W_1W_2 + pW_0^2 - 2pW_0W_2 + pW_1W_2)W_{n+1} + (W_0^2 - W_1^2 - W_0W_2 + W_1W_2 + pW_2^2 + pW_0W_1 - 2pW_1W_2)W_n$.

Next, we present a few basic relations between $\{G_{p,n}\}$ and $\{H_{p,n}\}$.

Lemma 6. *The following equalities are true*

- $(p^2 - 7p + 11)G_{p,n} = -2(2p - 5)H_{p,n+4} + 2(p - 3)H_{p,n+3} - (p^2 - 9p + 15)H_{p,n+2}$,
- $(p^2 - 7p + 11)G_{p,n} = -2(3p - 7)H_{p,n+3} - (p^2 - 9p + 15)H_{p,n+2} + (4p - 10)H_{p,n+1}$,
- $(p^2 - 7p + 11)G_{p,n} = -(p^2 + 3p - 13)H_{p,n+2} + (4p - 10)H_{p,n+1} + (6p - 14)H_{p,n}$,
- $(p^2 - 7p + 11)G_{p,n} = -2(p^2 + p - 8)H_{p,n+1} + (6p - 14)H_{p,n} + (p^2 + 3p - 13)H_{p,n-1}$,
- $(p^2 - 7p + 11)G_{p,n} = 2(-2p^2 + p + 9)H_{p,n} + (p^2 + 3p - 13)H_{p,n-1} + 2(p^2 + p - 8)H_{p,n-2}$,

and

- $(-p^2 + p + 1)H_{p,n} = -(-p^2 + 3p + 1)G_{p,n+4} + (-4p^2 + 8p + 6)G_{p,n+3} - 2(-2p^2 + 3p + 3)G_{p,n+2}$,
- $(-p^2 + p + 1)H_{p,n} = 2(-p^2 + p + 2)G_{p,n+3} - 2(-2p^2 + 3p + 3)G_{p,n+2} + (-p^2 + 3p + 1)G_{p,n+1}$,
- $(-p^2 + p + 1)H_{p,n} = -2(p - 1)G_{p,n+2} + (-p^2 + 3p + 1)G_{p,n+1} - 2(-p^2 + p + 2)G_{p,n}$,
- $(-p^2 + p + 1)H_{p,n} = -(p^2 + p - 5)G_{p,n+1} - 2(-p^2 + p + 2)G_{p,n} + 2(p - 1)G_{p,n-1}$,
- $(-p^2 + p + 1)H_{p,n} = -2(2p - 3)G_{p,n} + 2(p - 1)G_{p,n-1} + (p^2 + p - 5)G_{p,n-2}$.

Next, we give a few basic relations between $\{G_{p,n}\}$ and $\{l_{p,n}\}$.

Lemma 7. *The following equalities are true:*

$$\begin{aligned}(p+1)G_{p,n} &= -2l_{p,n+4} - (p-4)l_{p,n+3} + (2p-1)l_{p,n+2}, \\(p+1)G_{p,n} &= -pl_{p,n+3} + (2p-1)l_{p,n+2} + 2l_{p,n+1}, \\(p+1)G_{p,n} &= -l_{p,n+2} + 2l_{p,n+1} + pl_{p,n}, \\(p+1)G_{p,n} &= pl_{p,n} + l_{p,n-1},\end{aligned}$$

and

$$\begin{aligned}(-p^2 + p + 1)l_{p,n} &= -(2p+1)G_{p,n+4} + (p^2 + 4p + 2)G_{p,n+3} - p(2p+1)G_{p,n+2}, \\(-p^2 + p + 1)l_{p,n} &= p^2G_{p,n+3} - p(2p+1)G_{p,n+2} + (2p+1)G_{p,n+1}, \\(-p^2 + p + 1)l_{p,n} &= -pG_{p,n+2} + (2p+1)G_{p,n+1} - p^2G_{p,n}, \\(-p^2 + p + 1)l_{p,n} &= G_{p,n+1} - p^2G_{p,n} + pG_{p,n-1}, \\(-p^2 + p + 1)l_{p,n} &= -(p^2 - 2)G_{p,n} + pG_{p,n-1} - G_{p,n-2}.\end{aligned}$$

Now, we present a few basic relations between $\{H_{p,n}\}$ and $\{l_{p,n}\}$.

Lemma 8. *The following equalities are true:*

$$\begin{aligned}(p+1)H_{p,n} &= -(p-6)l_{p,n+4} + (4p-13)l_{p,n+3} - 2(2p-3)l_{p,n+2}, \\(p+1)H_{p,n} &= (2p-1)l_{p,n+3} - 2(2p-3)l_{p,n+2} + (p-6)l_{p,n+1}, \\(p+1)H_{p,n} &= 4l_{p,n+2} + (p-6)l_{p,n+1} - (2p-1)l_{p,n}, \\(p+1)H_{p,n} &= (p+2)l_{p,n+1} - (2p-1)l_{p,n} - 4l_{p,n-1}, \\(p+1)H_{p,n} &= 5l_{p,n} - 4l_{p,n-1} - (p+2)l_{p,n-2},\end{aligned}$$

and

$$\begin{aligned}(p^2 - 7p + 11)l_{p,n} &= -(5p-14)H_{p,n+4} + (4p-15)H_{p,n+3} - (p^2 - 8p + 10)H_{p,n+2}, \\(p^2 - 7p + 11)l_{p,n} &= -(6p-13)H_{p,n+3} - (p^2 - 8p + 10)H_{p,n+2} + (5p-14)H_{p,n+1}, \\(p^2 - 7p + 11)l_{p,n} &= -(p^2 + 4p - 16)H_{p,n+2} + (5p-14)H_{p,n+1} + (6p-13)H_{p,n}, \\(p^2 - 7p + 11)l_{p,n} &= -(2p^2 + 3p - 18)H_{p,n+1} + (6p-13)H_{p,n} + (p^2 + 4p - 16)H_{p,n-1}, \\(p^2 - 7p + 11)l_{p,n} &= -(4p^2 - 23)H_{p,n} + (p^2 + 4p - 16)H_{p,n-1} + (2p^2 + 3p - 18)H_{p,n-2}.\end{aligned}$$

We can give a few basic identities by using Lemma 3.

Corollary 9. *The following identities hold:*

$$(a) \quad (-p^3 + p^2 + p)G_n = (p - 1)G_{p,n+2} - (p^2 - 1)G_{p,n+1} + G_{p,n}.$$

$$(b) \quad G_{p,n} = -(p - 1)G_{n+2} + (2p - 2)G_{n+1} + G_n.$$

$$(c) \quad (-p^3 + p^2 + p)H_n = (-3p^2 + 2p + 1)G_{p,n+2} + (4p^2 - 1)G_{p,n+1} - (3p + 1)G_{p,n}.$$

$$(d) \quad 5G_{p,n} = (8p + 1)H_{n+2} - (7p - 1)H_{n+1} - (6p + 2)H_n.$$

$$(e) \quad (-p^3 + p^2 + p)l_n = -(p^2 - p + 1)G_{p,n+2} + (p^2 + p + 1)G_{p,n+1} - (p^2 + p - 1)G_{p,n}.$$

$$(f) \quad 4G_{p,n} = -2pl_{n+2} + (2p + 2)l_{n+1} + (4p - 2)l_n.$$

Proof.

(a) and (b) Take $W_n = G_{1,n} = G_n$ in Lemma 3.

(c) and (d) Take $W_n = H_{1,n} = H_n$ in Lemma 3.

(e) and (f) Take $W_n = l_{1,n} = l_n$ in Lemma 3.

□

We now present a few basic identities as a result of Lemma 4.

Corollary 10. *The following identities hold:*

$$(a) \quad p(p^2 - 7p + 11)G_n = -(2p^2 - 11)H_{p,n+2} + (5p - 11)H_{p,n+1} + (p^2 + 2p - 11)H_{p,n}.$$

$$(b) \quad H_{p,n} = (2p - 2)G_{n+2} - (4p - 7)G_{n+1} + (p - 5)G_n.$$

$$(c) \quad p(p^2 - 7p + 11)H_n = -(4p^2 - 15p + 11)H_{p,n+2} + (3p^2 - 14p + 11)H_{p,n+1} + (2p^2 - 8p + 11)H_{p,n}.$$

$$(d) \quad 5H_{p,n} = -(7p - 7)H_{n+2} + (8p - 8)H_{n+1} + (4p + 1)H_n.$$

$$(e) \quad p(p^2 - 7p + 11)l_n = (-3p^2 + 3p + 11)H_{p,n+2} + (p^2 + p - 11)H_{p,n+1} + (p^2 + 3p - 11)H_{p,n}.$$

$$(f) \quad 2H_{p,n} = (p+3)l_{n+2} - 5l_{n+1} - (3p-2)l_n.$$

Proof.

(a) and (b) Take $W_n = G_{1,n} = G_n$ in Lemma 4.

(c) and (d) Take $W_n = H_{1,n} = H_n$ in Lemma 4.

(e) and (f) Take $W_n = l_{1,n} = l_n$ in Lemma 4.

□

We next give a few basic identities by using Lemma 5.

Corollary 11. *The following identities hold:*

$$(a) \quad p(p+1)G_n = -l_{p,n+2} + (p+1)l_{p,n+1} + l_{p,n}.$$

$$(b) \quad l_{p,n} = -pG_{n+2} + (2p+1)G_{n+1} - G_n.$$

$$(c) \quad p(p+1)H_n = (3p+1)l_{p,n+2} - (4p+1)l_{p,n+1} - l_{p,n}.$$

$$(d) \quad 5l_{p,n} = (8p+3)H_{n+2} - (7p+2)H_{n+1} - (6p+1)H_n.$$

$$(e) \quad p(p+1)l_n = (p-1)l_{p,n+2} - (p-1)l_{p,n+1} + (p+1)l_{p,n}.$$

$$(f) \quad 4l_{p,n} = -(2p-2)l_{n+2} + (2p-2)l_{n+1} + 4pl_n.$$

Proof.

(a) and (b) Take $W_n = G_{1,n} = G_n$ in Lemma 5.

(c) and (d) Take $W_n = H_{1,n} = H_n$ in Lemma 5.

(e) and (f) Take $W_n = l_{1,n} = l_n$ in Lemma 5

□

4 Relations Between Special Numbers

In this section, we present identities on p-modified Leonardo, p-Leonardo-Lucas and p-Leonardo numbers and Fibonacci and Lucas numbers.

Lemma 12. *For all integers n, we have the following identities:*

(a)

$$G_{p,n} = pF_{n+1} + F_n - p, \tag{4.1}$$

$$5G_{p,n} = (p + 2)L_{n+1} + (2p - 1)L_n - 5p. \tag{4.2}$$

(b)

$$H_{p,n} = (3 - p)F_{n+1} + (p - 2)F_n + p, \tag{4.3}$$

$$5H_{p,n} = (p - 1)L_{n+1} + (8 - 3p)L_n + 5p. \tag{4.4}$$

(c)

$$l_{p,n} = pF_{n+1} + F_{n+1} - p, \tag{4.5}$$

$$5l_{p,n} = (p + 1)L_{n+1} + 2(p + 1)L_n - 5p. \tag{4.6}$$

Proof. Note that

$$F_{n+2} = F_{n+1} + F_n,$$

$$F_{n+3} = 2F_{n+1} + F_n,$$

$$F_{n+4} = 3F_{n+1} + 2F_n,$$

$$L_{n+2} = L_{n+1} + L_n,$$

$$L_{n+3} = 2L_{n+1} + L_n.$$

We know that

$$G_n = F_{n+2} - 1,$$

$$H_n = L_n + 1,$$

$$l_n = 2F_{n+1} - 1.$$

We also know from Corollaries 9, 10 and 11 that

$$\begin{aligned} G_{p,n} &= -(p-1)G_{n+2} + (2p-2)G_{n+1} + G_n, \\ 5G_{p,n} &= (8p+1)H_{n+2} - (7p-1)H_{n+1} - (6p+2)H_n, \\ 4G_{p,n} &= -2pl_{n+2} + (2p+2)l_{n+1} + (4p-2)l_n, \end{aligned}$$

and

$$\begin{aligned} H_{p,n} &= (2p-2)G_{n+2} - (4p-7)G_{n+1} + (p-5)G_n, \\ 5H_{p,n} &= -(7p-7)H_{n+2} + (8p-8)H_{n+1} + (4p+1)H_n, \\ 2H_{p,n} &= (p+3)l_{n+2} - 5l_{n+1} - (3p-2)l_n, \end{aligned}$$

and

$$\begin{aligned} l_{p,n} &= -pG_{n+2} + (2p+1)G_{n+1} - G_n, \\ 5l_{p,n} &= (8p+3)H_{n+2} - (7p+2)H_{n+1} - (6p+1)H_n, \\ 4l_{p,n} &= -(2p-2)l_{n+2} + (2p-2)l_{n+1} + 4pl_n, \end{aligned}$$

Now, use the above identities. □

5 Special Identities

As special cases of Catalan's identity of the generalized Leonardo sequence, we have the following corollary.

Corollary 13. *For all integers n and m , the following identities hold:*

- (a) $G_{p,n+m}G_{p,n-m} - G_{p,n}^2 = (F_{n+m+2} + (p-1)F_{n+m+1} - p)(F_{n-m+2} + (p-1)F_{n-m+1} - p) - (F_{n+2} + (p-1)F_{n+1} - p)^2$
- (b) $H_{p,n+m}H_{p,n-m} - H_{p,n}^2 = ((p-2)F_{n+m+2} + (5-2p)F_{n+m+1} + p)((p-2)F_{n-m+2} + (5-2p)F_{n-m+1} + p) - ((p-2)F_{n+2} + (5-2p)F_{n+1} + p)^2$
- (c) $l_{p,n+m}l_{p,n-m} - l_{p,n}^2 = -(p+1)(pF_{m+n+1} + pF_{n-m+1} - (p+1)F_{-m+n+1}F_{m+n+1} + (p+1)F_{n+1}^2 - 2pF_{n+1})$

As special cases of Cassini’s identity of the generalized Leonardo sequence, we have the following corollary.

Corollary 14. *For all integers n , the following identities hold:*

- (a) $G_{p,n+1}G_{p,n-1} - G_{p,n}^2 = (-p^2 + p + 1)F_{n+1}^2 + (p^2 - p - 1)F_n^2 + (p^2 - p - 1)F_nF_{n+1} + p(p - 2)F_{n+1} - p(2p - 3)F_n.$
- (b) $H_{p,n+1}H_{p,n-1} - H_{p,n}^2 = (p^2 - 7p + 11)F_n^2 + (p^2 - 7p + 11)F_nF_{n+1} - p(5p - 12)F_n + ((-p^2 + 7p - 11)F_{n+1}^2 + p(3p - 7)F_{n+1}).$
- (c) $l_{p,n+1}l_{p,n-1} - l_{p,n}^2 = -(p + 1)^2F_{n+1}^2 + (p + 1)^2F_n^2 + (p + 1)^2F_nF_{n+1} + p(p + 1)F_{n+1} - p(2p + 2)F_n.$

As special cases of the d’Ocagne’s and Melham’s identities, we have the following three corollaries. First one presents d’Ocagne’s and Melham’s identities of modified p-Leonardo sequence $\{G_{p,n}\}$.

Corollary 15. *Let n and m be any integers. Then the following identities are true:*

(a) *(d’Ocagne’s identity)*

$$G_{p,m+1}G_{p,n} - G_{p,m}G_{p,n+1} = (-p^2 + p + 1)(F_nF_{m+1} - F_mF_{n+1}) - p(F_{m+1} - F_{n+1}) - p(p - 1)(F_m - F_n).$$

(b) *(Melham’s identity)*

$$G_{p,n+1}G_{p,n+2}G_{p,n+6} - G_{p,n+3}^3 = -p(p^2 - p - 1)F_{n+1}^3 + (p^2 - p - 1)F_n^3 - p(7p + 5)(2p + 1)F_{n+1}^2 - p(p + 1)(5p + 2)F_n^2 + (p - 1)(p^2 - p - 1)F_{n+1}^2F_n + (p + 1)(p^2 - p - 1)F_{n+1}F_n^2 - p(17p^2 + 21p + 7)F_{n+1}F_n + p^2(7p + 4)F_{n+1} + p^2(4p + 3)F_n.$$

Second one presents d’Ocagne’s and Melham’s identities of p-Leonardo-Lucas sequence $\{H_{p,n}\}$.

Corollary 16. *Let n and m be any integers. Then the following identities are true:*

(a) (*d'Ocagne's identity*)

$$H_{p,m+1}H_{p,n} - H_{p,m}H_{p,n+1} = -p(p-2)(F_{n+1} - F_{m+1}) - p(2p-5)(F_m - F_n) + (-p^2 + 7p - 11)(F_n F_{m+1} - F_m F_{n+1}).$$

(b) (*Melham's identity*)

$$H_{p,n+1}H_{p,n+2}H_{p,n+6} - H_{p,n+3}^3 = (p-3)(p^2 - 7p + 11)F_{n+1}^3 + (p-2)(p^2 - 7p + 11)F_n^3 + p(p-4)(2p-11)F_{n+1}^2 - p(3p-11)F_n^2 - (2p-5)(p^2 - 7p + 11)F_{n+1}^2 F_n + (p^2 - 7p + 11)F_{n+1} F_n^2 + p(3p^2 - 25p + 55)F_{n+1} F_n - p^2(3p-13)F_{n+1} - p^2(p-6)F_n.$$

Third one presents d'Ocagne's and Melham's identities of p-Leonardo sequence $\{l_{p,n}\}$.

Corollary 17. *Let n and m be any integers. Then the following identities are true:*

(a) (*d'Ocagne's identity*)

$$l_{m+1}l_{p,n} - l_m l_{p,n+1} = -(p+1)^2(F_n F_{m+1} - F_m F_{n+1}) - p(p+1)(F_m - F_n).$$

(b) (*Melham's identity*)

$$l_{p,n+1}l_{p,n+2}l_{p,n+6} - l_{p,n+3}^3 = -(p+1)((p+1)^2 F_{n+1}^3 + 14p(p+1)F_{n+1}^2 + 5p(p+1)F_n^2 - (p+1)^2 F_{n+1}^2 F_n - (p+1)^2 F_{n+1} F_n^2 + 17p(p+1)F_{n+1} F_n - 7p^2 F_{n+1} - 4p^2 F_n).$$

6 On the Recurrence Properties of Special Cases of Generalized Leonardo Sequence

We present modified p-Leonardo, p-Leonardo-Lucas and p-Leonardo numbers at the negative index. Here, $H_n = H_{1,n}$.

Corollary 18. *For $n \in \mathbb{Z}$, we have the following recurrence relations:*

(a) *modified p-Leonardo sequence:*

$$G_{p,-n} = (-1)^{-n}(G_{p,2n} - H_n G_{p,n}).$$

(b) *p*-Leonardo-Lucas sequence:

$$H_{p,-n} = (-1)^{-n}(H_{p,2n} - H_n H_{p,n} + \frac{3}{2}(H_n^2 - H_{2n})).$$

(c) *p*-Leonardo sequence:

$$l_{p,-n} = \frac{1}{2}(-1)^{-n} (H_n^2 - H_{2n} + 2l_{p,2n} - 2H_n l_{p,n}).$$

Note that since

$$\begin{aligned} G_{p,n} &= pF_{n+1} + F_n - p, \\ H_{p,n} &= (3 - p)F_{n+1} + (p - 2)F_n + p, \\ l_{p,n} &= pF_{n+1} + F_{n+1} - p, \end{aligned}$$

and

$$F_{-n} = (-1)^{n+1} F_n,$$

we get

$$\begin{aligned} G_{p,-n} &= (-1)^n(pF_{n-1} - F_n) - p, \\ H_{p,-n} &= (-1)^n((3 - p)F_{n-1} - (p - 2)F_n) + p, \\ l_{p,-n} &= (p + 1)(-1)^n F_{n-1} - p. \end{aligned}$$

7 Sum Formulas

The following Corollary gives sum formulas of Fibonacci and Lucas numbers.

Corollary 19. *For $n \geq 0$, Fibonacci and Lucas numbers have the following properties:*

1.

(a) $\sum_{k=0}^n F_k = 2F_n + F_{n-1} - 1.$

(b) $\sum_{k=0}^n F_{2k} = 2F_{2n} - F_{2n-2} - 1.$

$$(c) \sum_{k=0}^n F_{2k+1} = 2F_{2n+1} - F_{2n-1}.$$

2.

$$(a) \sum_{k=0}^n L_k = 2L_n + L_{n-1} - 1.$$

$$(b) \sum_{k=0}^n L_{2k} = 2L_{2n} - L_{2n-2} + 1.$$

$$(c) \sum_{k=0}^n L_{2k+1} = 2L_{2n+1} - L_{2n-1} - 2.$$

Proof. It is given in Soykan [5, Corollary 4.5]. □

The following Corollary presents sum formulas of modified p -Leonardo, p -Leonardo-Lucas and p -Leonardo numbers.

Corollary 20. *For $n \geq 0$, modified p -Leonardo, p -Leonardo-Lucas and p -Leonardo numbers have the following properties:*

1.

$$(a) \sum_{k=0}^n G_{p,k} = (2p+1)F_{n+1} + (p+1)F_n - (n+2)p - 1.$$

$$(b) \sum_{k=0}^n G_{p,2k} = (p+1)F_{2n+1} + pF_{2n} - (n+1)p - 1.$$

$$(c) \sum_{k=0}^n G_{p,2k+1} = (2p+1)F_{2n+1} + (p+1)F_{2n} - (n+2)p.$$

2.

$$(a) 5 \sum_{k=0}^n H_{p,k} = (6-p)L_{n+1} + (7-2p)L_n + 5(n+1)p - 5.$$

$$(b) 5 \sum_{k=0}^n H_{p,2k} = (7-2p)L_{2n+1} + (p-1)L_{2n} + 5np + 10.$$

$$(c) 5 \sum_{k=0}^n H_{p,2k+1} = (6-p)L_{2n+1} + (7-2p)L_{2n} + 5(n+2)p - 15.$$

3.

$$(a) \sum_{k=0}^n l_{p,k} = 2(p+1)F_{n+1} + (p+1)F_n - (n+2)p - 1.$$

$$(b) \sum_{k=0}^n l_{p,2k} = (p+1)F_{2n+1} + (p+1)F_{2n} - (n+1)p.$$

$$(c) \sum_{k=0}^n l_{p,2k+1} = 2(p+1)F_{2n+1} + (p+1)F_{2n} - (n+2)p - 1.$$

Proof. The proof follows from Corollary 19 and the identities (4.2), (4.4) and (4.6), i.e.,

$$\begin{aligned} G_{p,n} &= pF_{n+1} + F_n - p, \\ 5H_{p,n} &= (p - 1)L_{n+1} + (8 - 3p)L_n + 5p, \\ l_{p,n} &= pF_{n+1} + F_{n+1} - p. \end{aligned}$$

□

8 Matrices and Some Identities Associated With Generalized p-Leonardo Numbers

Next, we present formulas for the p-modified Leonardo, p-Leonardo-Lucas and p-Leonardo numbers.

Corollary 21. *For all integers n, we have the following formulas for the p-modified Leonardo, p-Leonardo-Lucas and p-Leonardo numbers.*

(a) *Modified p-Leonardo Numbers.*

$$\begin{pmatrix} 2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \frac{1}{-p^3 + p^2 + p} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

where

$$\begin{aligned} a_{11} &= (p - 1) G_{p,n+3} - (p^2 - 1) G_{p,n+2} + G_{p,n+1} \\ a_{21} &= (p - 1) G_{p,n+2} - (p^2 - 1) G_{p,n+1} + G_{p,n} \\ a_{31} &= (p - 1) G_{p,n+1} - (p^2 - 1) G_{p,n} + G_{p,n-1} \end{aligned}$$

$$\begin{aligned} a_{12} &= -((p - 1) G_{p,n+1} - (p^2 - 1) G_{p,n} + G_{p,n-1}) \\ a_{22} &= -((p - 1) G_{p,n} - (p^2 - 1) G_{p,n-1} + G_{p,n-2}) \\ a_{32} &= -((p - 1) G_{p,n-1} - (p^2 - 1) G_{p,n-2} + G_{p,n-3}) \end{aligned}$$

$$\begin{aligned}
 a_{13} &= -((p-1)G_{p,n+2} - (p^2-1)G_{p,n+1} + G_{p,n}) \\
 a_{23} &= -((p-1)G_{p,n+1} - (p^2-1)G_{p,n} + G_{p,n-1}) \\
 a_{33} &= -((p-1)G_{p,n} - (p^2-1)G_{p,n-1} + G_{p,n-2})
 \end{aligned}$$

(b) *p*-Leonardo-Lucas Numbers.

$$\begin{pmatrix} 2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \frac{1}{p(p^2-7p+11)} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

where

$$\begin{aligned}
 b_{11} &= -(2p^2-11)H_{p,n+3} + (5p-11)H_{p,n+2} + (p^2+2p-11)H_{p,n+1} \\
 b_{21} &= -(2p^2-11)H_{p,n+2} + (5p-11)H_{p,n+1} + (p^2+2p-11)H_{p,n} \\
 b_{31} &= -(2p^2-11)H_{p,n+1} + (5p-11)H_{p,n} + (p^2+2p-11)H_{p,n-1} \\
 b_{12} &= -(-(2p^2-11)H_{p,n+1} + (5p-11)H_{p,n} + (p^2+2p-11)H_{p,n-1}) \\
 b_{22} &= -(-(2p^2-11)H_{p,n} + (5p-11)H_{p,n-1} + (p^2+2p-11)H_{p,n-2}) \\
 b_{32} &= -(-(2p^2-11)H_{p,n-1} + (5p-11)H_{p,n-2} + (p^2+2p-11)H_{p,n-3}) \\
 b_{13} &= -(-(2p^2-11)H_{p,n+2} + (5p-11)H_{p,n+1} + (p^2+2p-11)H_{p,n}) \\
 b_{23} &= -(-(2p^2-11)H_{p,n+1} + (5p-11)H_{p,n} + (p^2+2p-11)H_{p,n-1}) \\
 b_{33} &= -(-(2p^2-11)H_{p,n} + (5p-11)H_{p,n-1} + (p^2+2p-11)H_{p,n-2})
 \end{aligned}$$

(c) *p*-Leonardo Numbers.

$$\begin{pmatrix} 2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \frac{1}{p(p+1)} \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}$$

where

$$\begin{aligned}
 c_{11} &= -l_{p,n+3} + (p+1)l_{p,n+2} + l_{p,n+1} \\
 c_{21} &= -l_{p,n+2} + (p+1)l_{p,n+1} + l_{p,n} \\
 c_{31} &= -l_{p,n+1} + (p+1)l_{p,n} + l_{p,n-1}
 \end{aligned}$$

$$\begin{aligned} c_{12} &= -(-l_{p,n+1} + (p + 1) l_{p,n} + l_{p,n-1}) \\ c_{22} &= -(-l_{p,n} + (p + 1) l_{p,n-1} + l_{p,n-2}) \\ c_{32} &= -(-l_{p,n-1} + (p + 1) l_{p,n-2} + l_{p,n-3}) \end{aligned}$$

$$\begin{aligned} c_{13} &= -(-l_{p,n+2} + (p + 1) l_{p,n+1} + l_{p,n}) \\ c_{23} &= -(-l_{p,n+1} + (p + 1) l_{p,n} + l_{p,n-1}) \\ c_{33} &= -(-l_{p,n} + (p + 1) l_{p,n-1} + l_{p,n-2}) \end{aligned}$$

Proof. We know that

$$A^n = \begin{pmatrix} 2 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} G_{n+1} & -G_{n-1} & -G_n \\ G_n & -G_{n-2} & -G_{n-1} \\ G_{n-1} & -G_{n-3} & -G_{n-2} \end{pmatrix}.$$

Now, use the following identities respectively:

$$\begin{aligned} (-p^3 + p^2 + p)G_n &= (p - 1) G_{p,n+2} - (p^2 - 1) G_{p,n+1} + G_{p,n} \quad (\text{Corollary 9}), \\ p(p^2 - 7p + 11)G_n &= -(2p^2 - 11)H_{p,n+2} + (5p - 11)H_{p,n+1} \\ &\quad + (p^2 + 2p - 11)H_{p,n} \quad (\text{Corollary 10}), \\ p(p + 1)G_n &= -l_{p,n+2} + (p + 1) l_{p,n+1} + l_{p,n} \quad (\text{Corollary 11}). \end{aligned}$$

□

In the following corollary, we give identities for the p-modified Leonardo, p-Leonardo-Lucas and p-Leonardo numbers.

Corollary 22. *For all integers m, n, we have*

$$\begin{aligned} G_{p,n+m} &= G_{p,n}G_{m+1} - G_{p,n-1}G_{m-1} - G_{p,n-2}G_m, \\ H_{p,n+m} &= H_{p,n}G_{m+1} - H_{p,n-1}G_{m-1} - H_{p,n-2}G_m, \\ l_{p,n+m} &= l_{p,n}G_{m+1} - l_{p,n-1}G_{m-1} - l_{p,n-2}G_m, \end{aligned}$$

where

$$\begin{aligned}G_{m-1} &= G_{1,m-1}, \\G_m &= G_{1,m}, \\G_{m+1} &= G_{1,m+1},\end{aligned}$$

and

$$\begin{aligned}(p^3 - p^2 - p)G_{p,n+m} &= -((p-1)G_{p,m+3} + (1-p^2)G_{p,m+2} + G_{p,m+1})G_{p,n} \\&+ ((p-1)G_{p,m+1} + (1-p^2)G_{p,m} + G_{p,m-1})G_{p,n-1} \\&+ ((p-1)G_{p,m+2} + (1-p^2)G_{p,m+1} + G_{p,m})G_{p,n-2},\end{aligned}$$

and

$$\begin{aligned}-p(p^2 - 7p + 11)H_{p,n+m} &= -((11 - 2p^2)H_{p,m+3} + (5p - 11)H_{p,m+2} \\&+ (p^2 + 2p - 11)H_{p,m+1})H_{p,n} \\&+ ((11 - 2p^2)H_{p,m+1} + (5p - 11)H_{p,m} \\&+ (p^2 + 2p - 11)H_{p,m-1})H_{p,n-1} \\&+ ((11 - 2p^2)H_{p,m+2} + (5p - 11)H_{p,m+1} \\&+ (p^2 + 2p - 11)H_{p,m})H_{p,n-2},\end{aligned}$$

and

$$\begin{aligned}p(p+1)^2 l_{p,n+m} &= -(p+1)(l_{p,m+3} - (p+1)l_{p,m+2} - l_{p,m+1})l_{p,n} \\&+ (p+1)(l_{p,m+1} - (p+1)l_{p,m} - l_{p,m-1})l_{p,n-1} \\&+ (p+1)(l_{p,m+2} - (p+1)l_{p,m+1} - l_{p,m})l_{p,n-2}.\end{aligned}$$

9 Conclusion

Recently, there have been so many studies of the sequences of numbers in the literature and the sequences of numbers were widely used in many research areas, such as architecture, nature, art, physics and engineering. In this work, we study on special third order linear recurrence relations.

We can summarize the sections as follows:

- In Section 1, we present some background about generalized Leonardo numbers.
- In Section 2, we define new three sequences related to modified Leonardo, Leonardo-Lucas and Leonardo sequences, namely, modified p-Leonardo, p-Leonardo-Lucas and p-Leonardo numbers.
- In Section 3, we present some identities of modified p-Leonardo, p-Leonardo-Lucas and p-Leonardo numbers.
- In Section 4, we give identities on p-modified Leonardo, p-Leonardo-Lucas and p-Leonardo numbers and Fibonacci and Lucas numbers.
- In Section 5, we present special cases of Catalan's, Cassini's, d'Ocagne's and Melham's identities of the generalized Leonardo sequence.
- In Section 6, we present expression of modified p-Leonardo, p-Leonardo-Lucas and p-Leonardo numbers at the negative index.
- In Section 7, the sum formulas of modified p-Leonardo, p-Leonardo-Lucas and p-Leonardo numbers have been given.
- In Section 8, we present matrices and some identities associated with generalized p-Leonardo numbers.

References

- [1] Y. Alp and G. Koçer, Some properties of Leonardo numbers, *Konuralp Journal of Mathematics* 9(1) (2021), 183-189.
- [2] P. Catarino and A. Borges, A note on incomplete Leonardo numbers, *Integers* 20 (2020).
- [3] P. Catarino and A. Borges, On Leonardo numbers, *Acta Mathematica Universitatis Comenianae* 89(1) (2019), 75-86.

- [4] A.G. Shannon, A note on generalized Leonardo numbers, *Notes on Number Theory and Discrete Mathematics* 25(3) (2019), 97-101.
<https://doi.org/10.7546/nntdm.2019.25.3.97-101>
- [5] Y. Soykan, Some properties of generalized Fibonacci numbers: Identities, recurrence properties and closed forms of the sum formulas $\sum_{k=0}^n x^k W_{mk+j}$, *Archives of Current Research International* 21(3) (2021), 11-38.
<https://doi.org/10.9734/acri/2021/v21i330235>
- [6] Y. Soykan, Generalized Leonardo numbers, *Journal of Progressive Research in Mathematics* 18(4) (2021), 58-84.
- [7] R.P.M. Vieira, F.R.V. Alves and P.M.M.C. Catarino, Relações bidimensionais e identidades da sequência de Leonardo, *Revista Sergipana de Matemática e Educação Matemática* 4(2) (2019), 156-173.
<https://doi.org/10.34179/revistem.v4i2.11863>

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