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New Topological Structure related from Digraph

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Abstract

This work introduces a novel topological constraint called the intopological digraph space imposed by a subbasis $\overrightarrow{I_{ẹ}}^{v}$ contains one vertice such that the edge ẹ is indegree of it and investigates some properties of this structure. Our objective is to provide a crucial first step in the study of some of the characteristics of digraphs by utilizing the appropriate topology.

1. Introduction

Graph theory is a useful mathematical tool in many areas and is regarded as a core concept in separate mathematics for two reasons. First, graphs are mathematically chosen from a theoretical perspective. Despite being only simple relational combinations, graphs may perform topological space, collection objects, and many other mathematical groups. The second rationale is that certain concepts are more practical in real-world settings when they are represented using graphs. Regarding the connection between graph theory and topology, topological concepts are expressed by one of the tools of the graph, such as converting a set of edges or a set of vertices to topological space and studying other topological concepts of this space. Topology is one of the most well-known and contemporary topics that has occupied a wide area of mathematicians. Several earlier studies on the subject of topological graphs are included below. Evans et al. [1] first proposed the concept of topology on digraphs in 1967. Between the collection of all topologies with n vertices, they discovered only one relationship. Bhargava and Ahlborn [2] looked at the topological space connected to digraphs in 1968. They expanded the previous finding to encompass infinite graphs. In 1983, Majumdar [3] created graph

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topology from continuous multivalued functions that was connected between a dense subset of topology. A domination set of a graph and a dense subset of topology were linked by Subramanian in 2001 [4]. A novel idea in topology on a signed graph and topology on transitive products of a signed graph was researched by Subbaih [5] in 2007. Karunakaran [6] established topology τ_g on a graph G from a collection of spanning subgraphs of G in the same year. Thomas [7] investigated topology in 2013 and determined the topological numbers of several graphs using set indexers. By using two fixed vertices and determining vertex and edge incidence depending on the distance between them, Shokry [8] described a new technique for creating graph topology in 2015. When applying topology to a digraph in 2018, Abdu and Kilicman [9] associated two topologies with the set of edges dubbed compatible and incompatible edges topologies. Furthermore, Al'Dzhabri [10] presented new topological space structures connected to digraphs in 2020 by combining new topologies with digraphs that were generated from particular open sets known as DG-topological space. A few more kinds of open sets linked to graphs were also introduced in 2020 by Al'Dzhabri [11].

2. Preliminaries

In this work, some basic notions of graph theory [12], and topology [13] are presented. A graph (resp., directed graph or digraph) $\mathrm{Ɗ} = (\mathrm{Ṿ}, \mathrm{Ẹ})$ consists of a vertex set Ṿ and an edge set Ẹ of unordered (resp., ordered) pairs of elements of Ṿ. To avoid ambiguities, we assume that the vertex and edge sets are disjoint. We say that two vertices v and w of a graph (resp., digraph Ɗ) are adjacent if there is an edge of the form vw (rsep., $\overrightarrow{\mathrm{wv}}$ or $\overrightarrow{\mathrm{vw}}$) joining them, and the vertices v and w are then incident with such an edge. A subdigraph H of a digraph Ɗ is a digraph, each of whose vertices belong to Ṿ and each of whose edges belong to Ẹ. The degree of a vertex v of Ɗ is the number of edges incident with v, and written deg(v). A vertex of degree zero is an isolated vertex. In digraph, the outdegree, of a vertex v of Ɗ is the number of edges of the form $\overrightarrow{\mathrm{vw}}$ and denoted by $\mathrm{d}^{+}(\mathrm{v})$, similarly, the indegree of a vertex v of Ɗ is the number of edges of the form $\overrightarrow{\mathrm{wv}}$, and denoted by $\mathrm{d}^{-}(\mathrm{v})$. A vertex of out-degree and in-degree are zero is an isolated vertex. A topology τ on a set X is a combination of subset of X, called open, such that the union of the member of any subset of τ is a member of τ, the intersection of the members of any finite subset of τ is a member of τ, and both empty set and X are in τ and the ordered pair (X, τ) is called topological space. The topology $\tau = P(X)$ on X is called discrete topology while the topology $\tau = \{X, \emptyset\}$ on X is called indiscrete topology. A topology in which arbitrary intersection of open set is open called Alexandroff space

3. Intopological Digraph Space

We introduce our new subbasis family to generate a topology on the set of vertices V of a digraph $D = (V, E)$.

Definition 3.1. Let $D = (V, E)$ be a digraph. We define $\overrightarrow{I_{e}}^{v}$ a set contains one vertice such that the edge e is indegree of it. Also define $\overrightarrow{S_{D}}^{v}$ as follows: $\overrightarrow{S_{D}}^{v} = V(D) \cup \{\overrightarrow{I_{e}}^{v} \mid e \in E\}$. Hence $\overrightarrow{S_{D}}^{v}$ forms a subbasis for a topology $\overrightarrow{\tau_{D}}^{v}$ on V called intopological digraph space $\overrightarrow{\tau_{D}}^{v}$ (briefly intop. digsp.) of D.

Example 3.2. Let $D = (V, E)$ be digraph in Figure 1 such that $V = \{v_1, v_2, v_3, v_4, v_5\}$, $E = \{e_1, e_2, e_3, e_4, e_5\}$.

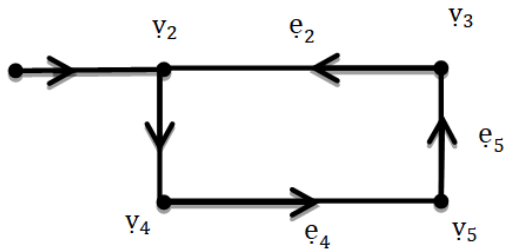

Figure 1. Simple digraph.

We have

$$\overrightarrow{I_{e_1}}^{v} = \{v_2\}, \ \overrightarrow{I_{e_2}}^{v} = \{v_2\}, \ \overrightarrow{I_{e_3}}^{v} = \{v_4\}, \ \overrightarrow{I_{e_4}}^{v} = \{v_5\}, \ \overrightarrow{I_{e_5}}^{v} = \{v_3\}$$

and

$$\overrightarrow{S_{D}}^{v} = \{V(D), \{v_2\}, \{v_3\}, \{v_4\}, \{v_5\}\}.$$

By taking finitely intersection, the basis obtained is:

$$\{V(D), \emptyset, \{v_2\}, \{v_3\}, \{v_4\}, \{v_5\}\}.$$

Then by taking all unions, the intop. digsp. can be written as:

$$\overrightarrow{\tau_{D}}^{v} = \{V(D), \emptyset, \{v_2\}, \{v_3\}, \{v_4\}, \{v_5\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_2, v_5\}, \{v_3, v_4\}, \{v_3, v_5\}, \{v_4, v_5\}, \{v_2, v_3, v_4\}, \{v_2, v_3, v_5\}, \{v_2, v_4, v_5\}, \{v_3, v_4, v_5\}, \{v_2, v_3, v_4, v_5\}\}.$$

Remark 3.3. Let C_n be cyclic digraph. If every edge is in the same direction, then we get that the intop. digsp. $\overrightarrow{\tau_{D}}^{v}$ on C_n is discrete, and if the edges are not all in the same direction, then we get that the intop. digsp. $\overrightarrow{\tau_{D}}^{v}$ on C_n is not discrete.

This remark is illustrated in the next two examples.

Example 3.4. Let C_5 be cyclic digraph such that every edge is in the same direction as shown in Figure 2.

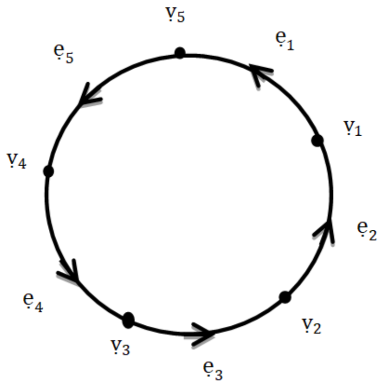

Figure 2. Cyclic digraph C_5.

We have

$$\overrightarrow{\mathrm{I}_{e_1}}^{v} = \{v_5\},\ \overrightarrow{\mathrm{I}_{e_2}}^{v}=\{v_1\},\ \overrightarrow{\mathrm{I}_{e_3}}^{v}=\{v_2\},\ \overrightarrow{\mathrm{I}_{e_4}}^{v}=\{v_3\},\ \overrightarrow{\mathrm{I}_{e_5}}^{v}=\{v_4\}$$

and

$$\overrightarrow{\mathrm{S}_{\mathrm{D}}}^{v} = \{V(\mathrm{D}), \{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_5\}\}$$

$$\overrightarrow{\tau_{\mathrm{D}}}^{v} = \{V(\mathrm{D}), \emptyset, \{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_5\}, \{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_4\}, \{v_1, v_5\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_2, v_5\}, \{v_3, v_4\}, \{v_3, v_5\}, \{v_4, v_5\}, \{v_1, v_2, v_3\}, \{v_1, v_2, v_4\}, \{v_1, v_2, v_5\}, \{v_1, v_3, v_4\}, \{v_1, v_3, v_5\}, \{v_1, v_4, v_5\}, \{v_2, v_3, v_4\}, \{v_2, v_3, v_5\}, \{v_2, v_4, v_5\}, \{v_3, v_4, v_5\}, \{v_1, v_2, v_3, v_4\}, \{v_1, v_2, v_3, v_5\}, \{v_1, v_2, v_4, v_5\}, \{v_1, v_3, v_4, v_5\}, \{v_2, v_3, v_4, v_5\}\},$$

then we get that the intop. digsp. $\overrightarrow{\tau_{\mathrm{D}}}^{v}$ of C_5 is discrete topology.

Example 3.5. Let C_6 be cyclic digraph such that edges are not all in the same direction as shown in Figure 3.

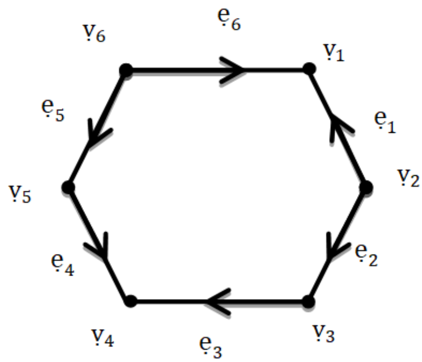

Figure 3. C_6 digraph.

We have

$$\overrightarrow{I_{e_1}}^{v} = \{v_1\},\ \overrightarrow{I_{e_2}}^{v}=\{v_3\},\ \overrightarrow{I_{e_3}}^{v}=\{v_4\},\ \overrightarrow{I_{e_4}}^{v}=\{v_4\},\ \overrightarrow{I_{e_5}}^{v}=\{v_5\},\ \overrightarrow{I_{e_6}}^{v}=\{v_1\}$$

and

$$\overrightarrow{S_D}^{v} = \{V(D),\{v_1\},\{v_3\},\{v_4\},\{v_5\}\}$$

$$\overrightarrow{\tau_D}^{v} = \{\ V(D),\emptyset,\{v_1\},\{v_3\},\{v_4\},\{v_5\},\{v_1,v_3\},\{v_1,v_4\},\{v_1,v_5\},\{v_3,v_4\},\{v_3,v_5\},$$
$$\{v_4,v_5\},\{v_1,v_3,v_4\},\{v_1,v_3,v_5\},\{v_1,v_4,v_5\},\{v_3,v_4,v_5\},\{v_1,v_3,v_4,v_5\},$$

then we get that the intop. digsp. $\overrightarrow{\tau_D}^{v}$ of C_6 is not discrete .

Remark 3.6. Let P_n be a path digraph. Then the intop. digsp. $\overrightarrow{\tau_D}^{v}$ on P_n is not discrete topology.

This remark is illustrated in the next example.

Example 3.7. Let P_5 be a path digraph as shown in Figure 4.

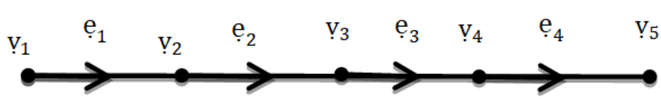

Figure 4. Path digraph P_5.

We have

$$\overrightarrow{I_{e_1}}^{v} = \{v_2\},\ \overrightarrow{I_{e_2}}^{v}=\{v_3\},\ \overrightarrow{I_{e_3}}^{v}=\{v_4\},\ \overrightarrow{I_{e_4}}^{v}=\{v_5\}$$

and

$$\overrightarrow{S_D}^{v} = \{V(D), \{v_2\}, \{v_3\}, \{v_4\}, \{v_5\}\}$$

$$\overrightarrow{\tau_D}^{v} = \{V(D), \emptyset, \{v_2\}, \{v_3\}, \{v_4\}, \{v_5\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_2, v_5\}, \{v_3, v_4\}\}, \{v_3, v_5\},$$
$$\{v_4, v_5\}, \{v_2, v_3, v_4\}, \{v_2, v_3, v_5\}, \{v_2, v_4, v_5\}, \{v_3, v_4, v_5\}, \{v_2, v_3, v_4, v_5\}.$$

We note the intop. digsp. $\overrightarrow{\tau_D}^{v}$ is not discrete.

Example 3.8. Let P_5 be a path digraph as shown in Figure 5. Then we get that the intop. digsp. $\overrightarrow{\tau_D}^{v}$ of P_5 is not discrete.

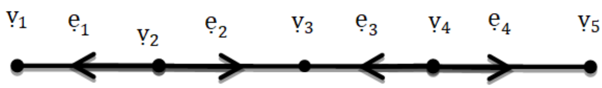

Figure 5. Path digraph P_5.

We have

$$\overrightarrow{I_{e_1}}^{v} = \{v_1\}, \ \overrightarrow{I_{e_2}}^{v} = \{v_3\}, \ \overrightarrow{I_{e_3}}^{v} = \{v_3\}, \ \overrightarrow{I_{e_4}}^{v} = \{v_5\}$$

and

$$\overrightarrow{S_D}^{v} = \{V(D), \{v_1\}, \{v_3\}, \{v_5\}\}$$

$$\overrightarrow{\tau_D}^{v} = \{V(D), \emptyset, \{v_1\}, \{v_3\}, \{v_5\}, \{v_1, v_3\}, \{v_1, v_5\}, \{v_3, v_5\}, \{v_1, v_3, v_5\}\}.$$

We note the intop. digsp. $\overrightarrow{\tau_D}^{v}$ is not discrete.

Definition 3.9. Let $D = (V, E)$ be a digraph. Then $\overrightarrow{E_v}$ is the set of all edges that indgree to the vertice v.

Example 3.10. According to Example 3.2, we get

$$\overrightarrow{E_{v_1}} = \{\emptyset\}, \ \overrightarrow{E_{v_2}} = \{e_1, e_2\}, \ \overrightarrow{E_{v_3}} = \{e_5\}, \ \overrightarrow{E_{v_4}} = \{e_3\}, \ \overrightarrow{E_{v_5}} = \{e_4\}.$$

Proposition 3.11. *Let* $\overrightarrow{\tau_D}^{v}$ *be intop. digsp. of the digraph* $D = (V, E)$. *If* $\overrightarrow{E_v} \neq \emptyset$, *then* $\{v\} \in \overrightarrow{\tau_D}^{v}$ *for every* $v \in V$.

Proof. Let D be digraph. Since $\overrightarrow{E_v} \neq \emptyset$, we get $\bigcap_{e \in \overrightarrow{E_v}} \overrightarrow{I_e}^{v} = \{v\}$ [because $\overrightarrow{I_e}^{v} = \{v\}$, $\forall\, e \in \overrightarrow{E_v}$].

Now by the definition of intop. digsp. $\overrightarrow{\tau_D}^{v}$, $\{v\}$ is element in the basis of intop. digsp. $\overrightarrow{\tau_D}^{v}$. Hence $\{v\} \in \overrightarrow{\tau_D}^{v}$.

Remark 3.12. Let $D = (V, E)$ discrete to be a digraph. Then the intop. digsp. $\overrightarrow{\tau_D}^{v}$ is not necessary to be discrete topology in general.

The Example 3.2 illustrates Remark 3.12.

Corollary 3.13. *Let* $D = (V, E)$ *be a digraph. Then*

(i) *If* $\overrightarrow{E_v} \neq \emptyset$ *for all* $v \in V$, *then* $\overrightarrow{\tau_D}^{v}$ *is discrete topology.*

(ii) *If* $D = (V, E)$ *is reflexive, then* $\overrightarrow{\tau_D}^{v}$ *is discrete topology.*

(iii) *If* $D = (V, E)$ *is equivalent, then* $\overrightarrow{\tau_D}^{v}$ *is discrete topology.*

(iv) *If* $D = (V, E)$ *is null digraph, then* $\overrightarrow{\tau_D}^{v}$ *is indiscrete topology.*

Proof.

(i) It follows from Proposition 3.11.

(ii) Since $D = (V, E)$ is reflexive, $\overrightarrow{E_v} \neq \emptyset$ for all $v \in V$. By (i) the proof is complete.

(iii) Clear.

(iv) Clear.

Example 3.14. Let C_5 be cyclic digraph such that every edge is in the same direction as shown in Figure 2. Then we get that the intop. digsp. $\overrightarrow{\tau_D}^{v}$ is discrete.

Remark 3.15. Let $D = (V, E)$ be a symmetric digraph. Then intop. digsp. $\overrightarrow{\tau_D}^{v}$ is not necessary to be discrete intopology in general.

The following example shows remark.

Example 3.16. Let $D = (V, E)$ be a digraph in Figure 6 such that $V=\{v_1,v_2,v_3,v_4\}$, $E = \{e_1, e_2, e_3, e_4\}$

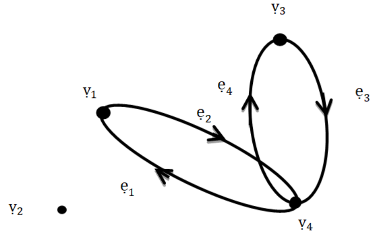

Figure 6. Digraph with loop.

We have that $\overrightarrow{\tau_{D}}^{v} = \{V(D), \emptyset, \{v_1\}, \{v_3\}, \{v_4\}, \{v_1, v_3\}, \{v_1, v_4\}, \{v_3, v_4\}, \{v_1, v_3, v_4\}\}$ is not discrete topology.

Proposition 3.17. *If* $D = (V, E)$ *is a symmetric and serial digraph, then the intop. digsp.* $\overrightarrow{\tau_{D}}^{v}$ *is discrete topology.*

Proof. Since $D = (V, E)$ is serial digraph, then $D = (V, E)$ has not isolated vertex and since the digraph $D = (V, E)$ is symmetric, we get $\overrightarrow{E_v} \neq \emptyset$ for every $v \in V$ and by Corollary 3.13(i), then $\overrightarrow{\tau_{D}}^{v}$ is discrete topology.

Proposition 3.18. *The intop. digsp.* $(V, \overrightarrow{\tau_{D}}^{v})$ *of digraph* $D = (V, E)$ *satisfies the property of Alexandroff.*

Proof. It is adequate to show that arbitrary intersection of elements of $\overrightarrow{S_{D}}^{v}$ is open. Let $A \subseteq E$ then either: $\bigcap_{e \in A} \overrightarrow{I_e}^{v} = \emptyset$ is open or $\bigcap_{e \in A} \overrightarrow{I_e}^{v} = \{v\}$ such that $e \in \overrightarrow{E_v}$ for all $e \in A$. This means $\overrightarrow{E_v} \neq \emptyset$ then by Proposition 3.11, then $\{v\} \in \overrightarrow{\tau_{D}}^{v}$. Hence $\bigcap_{e \in A} \overrightarrow{I_e}^{v}$ is open.

Then the intop. digsp. $\overrightarrow{\tau_{D}}^{v}$ satisfies the property of Alexandroff .

Definition 3.19. In any digraph $D = (V, E)$ since $(V, \overrightarrow{\tau_{D}}^{v})$ is Alexandroff space, for each $v \in V$, the intersection of all open sets containing v is the smallest open set containing v and denoted by U_v.

Also the family $\overrightarrow{M_{D}}^{v} = \{U_v | v \in V\}$ is the minimal basis for the intop. digsp. $(V, \overrightarrow{\tau_{D}}^{v})$.

Proposition 3.20. *In any digraph* $D = (V, E)$, $U_v = \bigcap_{e \in \overrightarrow{E_v}} \overrightarrow{I_e}^{v}$ *for every* $v \in V$.

Proof. Since $\overrightarrow{S_{D}}^{v}$ is the subbasis of $\overrightarrow{\tau_{D}}^{v}$ and U_v the intersection of all open sets containing v, we have $U_v = \bigcap_{e \in A} \overrightarrow{I_e}^{v}$ for some subset A of E, by definition of U_v then $v \in U_v$ and since $U_v = \bigcap_{e \in A} \overrightarrow{I_e}^{v}$ implies $v \in \bigcap_{e \in A} \overrightarrow{I_e}^{v}$ then $v \in \overrightarrow{I_e}^{v}$ for all $e \in A$, since $\overrightarrow{I_e}^{v}$ contain one vertex then $\overrightarrow{I_e}^{v} = \{v\}$ for all $e \in A$. This leads to $e \in \overrightarrow{E_v}$ for each $e \in A$. Hence $A \subseteq \overrightarrow{E_v}$ and so $v \in \bigcap_{e \in \overrightarrow{E_v}} \overrightarrow{I_e}^{v} \subseteq \bigcap_{e \in A} \overrightarrow{I_e}^{v}$ and hence $v \in \bigcap_{e \in \overrightarrow{E_v}} \overrightarrow{I_e}^{v} \subseteq U_v$. From the definition of U_v the proof is complete.

Remark 3.21. Let $\mathfrak{D} = (V, E)$ be a digraph. For any $v \in V$

(i) If $\overrightarrow{E_v} \neq \emptyset$, then by Proposition 3.20 $U_v = \bigcap_{e \in \overrightarrow{E_v}} \overrightarrow{I_e}^{v} = \{v\}$ [since $\overrightarrow{I_e}^{v} = \{v\}, \forall e \in \overrightarrow{E_v}$]. Then $U_v = \{v\}$.

(ii) If $\overrightarrow{E_v} = \emptyset$, then by Proposition 3.20 $U_v = \bigcap_{e \in \overrightarrow{E_v}} \overrightarrow{I_e}^{v} = V$ [since V is the only open set in the subbasis contain v].

The following example is applied to show this remark.

Example 3.22. Let $\mathfrak{D} = (V, E)$ be a digraph in Figure 3 such that $V = \{v_1, v_2, v_3, v_4\}$, $E = \{e_1, e_2, e_3, e_4\}$.

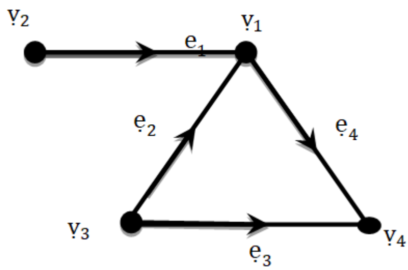

Figure 7. Dragon digraph.

$\overrightarrow{I_{e_1}}^{v} = \{v_1\}$, $\overrightarrow{I_{e_2}}^{v} = \{v_1\}$, $\overrightarrow{I_{e_3}}^{v} = \{v_4\}$, $\overrightarrow{I_{e_4}}^{v} = \{v_4\}$ $\overrightarrow{S_{\mathfrak{D}}}^{v} = \{V(\mathfrak{D}), \{v_1\}, \{v_4\}\}$, $\overrightarrow{\tau_{\mathfrak{D}}}^{v} = \{V(\mathfrak{D}), \emptyset, \{v_1\}, \{v_4\}, \{v_1, v_4\}\}$.

We get $\overrightarrow{E_{v_1}} = \{e_1, e_2\} \neq \emptyset$ then $U_{v_1} = \{v_1\}$ and $\overrightarrow{E_{v_2}} = \emptyset$ then $U_{v_2} = V$.

Theorem 3.23. *For any $u, v \in V$ in a digraph $\mathfrak{D} = (V, E)$, we have $u \in U_v$ if and only if $\overrightarrow{E_v} = \emptyset$, i.e., $U_v = \{u \in V | \overrightarrow{E_v} = \emptyset\}$.*

Proof. $\Longrightarrow$ Let $u \in U_v$ to prove $\overrightarrow{E_v} = \emptyset$.

If $\overrightarrow{E_v} \neq \emptyset$ by Remark 3.21(i) implies $U_v = \{v\} \Rightarrow u \notin U_v$ is contradiction with hypothesis, then $\overrightarrow{E_v} = \emptyset$.

$\Longleftarrow$ If $\overrightarrow{E_v} = \emptyset$ and by Remark 3.21(ii) we get $U_v = V$ and hence $u \in U_v$.

Corollary 3.24. *For any $u, v \in V$ in a digraph $\mathfrak{D} = (V, E)$, we have $u \in U_v$ if and only if $\overrightarrow{E_v} \subseteq \overrightarrow{E_u}$, i.e. $U_v = \{u \in V | \overrightarrow{E_v} \subseteq \overrightarrow{E_u}\}$.*

Proof. By Theorem 3.23 since this inequality $\overrightarrow{E_{v}} \subseteq \overrightarrow{E_{u}}$ is imposable and it is correct if $\overrightarrow{E_{v}} = \emptyset$.

Remark 3.25. The intop.digsp. $(V, \overrightarrow{\tau_D}^{v})$ in any digraph $D = (V, E)$ is discrete if and only if $\overrightarrow{E_{v}} \not\subseteq \overrightarrow{E_{u}}$ and $\overrightarrow{E_{u}} \not\subseteq \overrightarrow{E_{v}}$ for every distinct pair of vertices $u, v \in V$.

Proof. $\Longrightarrow$ Suppose that $(V, \overrightarrow{\tau_D}^{v})$ is discrete intop. digsp. $\overrightarrow{\tau_D}^{v}$, then we get $U_v = \{v\}$ for all $v \in V$ and hence $u \notin U_v$ and $v \notin U_u$ for every distinct pair of vertices $u, v \in V$, therefore by Corollary 3.24 we get $\overrightarrow{E_{v}} \not\subseteq \overrightarrow{E_{u}}$ and $\overrightarrow{E_{u}} \not\subseteq \overrightarrow{E_{v}}$ for every distinct pair of vertices $u, v \in V$.

$\Longleftarrow$ Let $\overrightarrow{E_{v}} \not\subseteq \overrightarrow{E_{u}}$ and $\overrightarrow{E_{u}} \not\subseteq \overrightarrow{E_{v}}$ for every distinct pair of vertices $u, v \in V$, then by Corollary 3.24 we get $U_v = \{v\}$ for all $v \in V$ and hence, $\overrightarrow{\tau_D}^{v}$ is discrete Intopology .

Remark 3.26. The intop. digsp. $(V, \overrightarrow{\tau_D}^{v})$ in any digraph $D = (V, E)$ is not necessary to be T_0 in general .

Example 3.27. According to example 3.22 we get that the intop. digsp. $\overrightarrow{\tau_D}^{v} = \{V(D), \emptyset, \{v_1\}, \{v_4\}, \{v_1, v_4\}\}$ is not T_0 because $v_2, v_3 \in V(D)$ but $\nexists$ open set A such that $v_2 \in A$ and $v_3 \notin A$ or $v_2 \notin A$ and $v_3 \in A$.

Remark 3.28. Let C_n be a cyclic such that every edge is in the same direction. Then we get that the intop. digsp. $\overrightarrow{\tau_D}^{v}$ is T_0 and if the edges are not all in the same direction we get that the intop.digsp. is not necessary T_0.

The next example illustrates Remark 3.28.

Example 3.29.

(i) According to example 3.4 we note that the C_5 digraph all edges in the same direction and hence we note the intop. disp. $\overrightarrow{\tau_D}^{v}$ on C_5 is T_0.

(ii) According to example 3.5 we note that the C_6 all edges are not in the same direction and hence we note the intop. digsp. $\overrightarrow{\tau_D}^{v}$ is not T_0, because $v_2, v_6 \in V$ but $\nexists\, u \in \overrightarrow{\tau_D}^{v}$ such that $v_2 \in u$ and $v_6 \notin u$ or $v_2 \notin u$ and $v_6 \in u$.

Remark 3.30. Let P_n be a path such that every edge is in the same direction. Then we get that the intop. digsp. $\overrightarrow{\tau_D}^{v}$ is T_0 and if the edges are not all in the same direction we get that the intop.digsp. is not necessary T_0.

The next two examples illustrate Remark 3.30.

Example 3.31.

(i) According to example 3.7 we note that the P_5 digraph all edges in the same direction and hence we note the intop. disp. $\overrightarrow{\tau_D}^{v}$ on P_5 is T_0.

(ii) According to example 3.8 we note that the P_5 all edges are not in the same direction and hence we note the intop. digsp. $\overrightarrow{\tau_{\mathrm{D}}}^{v}$ is not T_0, because $ṿ_2, ṿ_4 \in Ṿ$ but $\nexists\, u \in \overrightarrow{\tau_D}^{v}$ such that $ṿ_2 \in u$ and $ṿ_4 \notin u$ or $ṿ_2 \notin u$ and $ṿ_4 \in u$.

Proposition 3.32. *The intop. digsp.* $(Ṿ, \overrightarrow{\tau_D}^{v})$ *in any digraph* $Ɗ = (Ṿ, Ẹ)$ *is* T_0 *if and only if* $\overrightarrow{Ẹ_{ṿ}} \neq \overrightarrow{Ẹ_u}$ *for every distinct pair of vertices* $u, ṿ \in Ṿ$.

Proof. $\Longrightarrow$ Suppose $\overrightarrow{\tau_D}^{v}$ is T_0 and $u, ṿ \in Ṿ$ is distinct pair of vertices to prove $\overrightarrow{Ẹ_{ṿ}} \neq \overrightarrow{Ẹ_u}$ i.e. [to prove $\overrightarrow{Ẹ_{ṿ}} \nsubseteq \overrightarrow{Ẹ_u}$ or $\overrightarrow{Ẹ_u} \nsubseteq \overrightarrow{Ẹ_{ṿ}}$]

If $\overrightarrow{Ẹ_{ṿ}} = \overrightarrow{Ẹ_u}$, then $\overrightarrow{Ẹ_{ṿ}} \subseteq \overrightarrow{Ẹ_u}$ and $\overrightarrow{Ẹ_u} \subseteq \overrightarrow{Ẹ_{ṿ}}$ and we get by Corollary 3.23 $u \in \mathrm{U}_{ṿ}$ and $ṿ \in \mathrm{U}_u$ this contradiction [since $\overrightarrow{\tau_D}^{v}$ is T_0], so that $\overrightarrow{Ẹ_{ṿ}} \nsubseteq \overrightarrow{Ẹ_u}$ or $\overrightarrow{Ẹ_u} \nsubseteq \overrightarrow{Ẹ_{ṿ}}$ and hence $\overrightarrow{Ẹ_{ṿ}} \neq \overrightarrow{Ẹ_u}$.

$\Leftarrow$ Let $\overrightarrow{Ẹ_{ṿ}} \neq \overrightarrow{Ẹ_u}$ for every distinct pair of vertices $u, ṿ \in Ṿ$ then either $\overrightarrow{Ẹ_{ṿ}} \nsubseteq \overrightarrow{Ẹ_u}$ and hence $\overrightarrow{Ẹ_{ṿ}} \neq \emptyset$ implies that by Remark 3.21(i) $\mathrm{U}_{ṿ} = \{ṿ\}$ then there exists $\mathrm{U}_{ṿ}$is open set such that $ṿ \in \mathrm{U}_{ṿ}$ and $u \notin \mathrm{U}_{ṿ}$ (because $\mathrm{U}_{ṿ} = \{ṿ\}$ and $u, ṿ \in Ṿ$ is distinct pair of vertices), or $\overrightarrow{Ẹ_u} \nsubseteq \overrightarrow{Ẹ_{ṿ}}$ and hence $\overrightarrow{Ẹ_u} \neq \emptyset$ implies that by Remark 3.21(i) $\mathrm{U}_u = \{u\}$ then there exists U_u is open set such that $u \in \mathrm{U}_u$ and $ṿ \notin \mathrm{U}_u$ (because $\mathrm{U}_u = \{u\}$ and $u, ṿ \in Ṿ$ is distinct pair of vertices), therefore $\overrightarrow{\tau_{\mathrm{D}}}^{v}$ is T_0.

The next example illustrates Proposition 3.32.

Example 3.33. According to example 3.2, we note that $\overrightarrow{Ẹ_{ṿ}} \neq \overrightarrow{Ẹ_u}$ for every distinct pair of vertices $u, ṿ \in Ṿ$ [since $\overrightarrow{Ẹ_{ṿ_1}} = \emptyset$, $\overrightarrow{Ẹ_{ṿ_2}} = \{ẹ_1, ẹ_2\}$] and hence we get $\overrightarrow{\tau_{\mathrm{D}}}^{v}$ is T_0 and we note that in the example 3.27 there exist two vertices $ṿ_2, ṿ_3$ such that $\overrightarrow{Ẹ_{ṿ_2}} = \overrightarrow{Ẹ_{ṿ_3}}$ and hence the $\overrightarrow{\tau_{\mathrm{D}}}^{v}$ not T_0 .

Remark 3.34. The intop. digsp. $(Ṿ, \overrightarrow{\tau_D}^{v})$ in any digraph $Ɗ = (Ṿ, Ẹ)$ is not necessary to be T_1 in general.

Example 3.35. According to example 3.22 we get the intop. digsp. $\overrightarrow{\tau_D}^v = \{V(D), \emptyset, \{v_1\}, \{v_4\}, \{v_1, v_4\}\}$ is not T_1 because $v_2, v_3 \in V(D)$ but $\nexists$ open set A such that $v_2 \in A$ and $v_3 \notin A$.

Remark 3.36. Let C_n be a cyclic such that every edge is in the same direction. Then we get that the intop. digsp. $\overrightarrow{\tau_D}^v$ is T_1 and if the edges are not all in the same direction we get that the intop. digsp. is not T_1.

The next example illustrates Remark 3.36.

Example 3.37.

(i) According to example 3.4 we note that the C_5 digraph all edges in the same direction and hence we note the intop. disp. $\overrightarrow{\tau_D}^v$ on C_5 is T_1.

(ii) According to example 3.5 we note that the C_6 all edges are not in the same direction and hence we note the intop. digsp. $\overrightarrow{\tau_D}^v$ is not T_1, because $v_2, v_6 \in V$ but $\nexists\, u \in \overrightarrow{\tau_D}^v$ such that $v_2 \in u$ and $v_6 \notin u$.

Remark 3.38. Let P_n be a path digraph. Then the intop. digsp. on P_n is not T_1.

This remark is illustrated in the next example.

Example 3.39.

(i) According to example 3.7 we note that the P_5 digraph all edges in the same direction and hence we note the intop. disp. $\overrightarrow{\tau_D}^v$ on P_5 is not T_1.

(ii) According to example 3.8 we note that the P_5 all edges are not in the same direction and hence we note the intop. digsp. $\overrightarrow{\tau_D}^v$ is not T_1.

Proposition 3.40. *The intop. digsp.* $(V, \overrightarrow{\tau_D}^v)$ *in any digraph* $D = (V, E)$ *is* T_1 *if and only if* $\overrightarrow{\tau_D}^v$ *is discrete.*

Proof. $\Rightarrow$ Let intopological space $(V, \overrightarrow{\tau_D}^v)$ be T_1. Suppose that $\overrightarrow{\tau_D}^v$ is not discrete then by Corollary 3.13(i) we get there exist $v \in V$ such that $\overrightarrow{E_v} = \emptyset$ and by Remark 3.2(ii) $U_v = V$ implies that $\overrightarrow{\tau_D}^v$ is not T_1 this is a contradiction with hypothesis, thus $(V, \overrightarrow{\tau_D}^v)$ is discrete.

$\Leftarrow$ If $(V, \overrightarrow{\tau_D}^v)$ is discrete intop. digsp., then $(V, \overrightarrow{\tau_D}^v)$ is T_1.

Corollary 3.41. *The intop. digsp.* $(\text{V}, \overrightarrow{\tau_D}^{V})$ *in any digraph* $D = (V, E)$ *is* T_1 *if and only if* $\overrightarrow{E_v} \nsubseteq \overrightarrow{E_u}$ *and* $\overrightarrow{E_u} \nsubseteq \overrightarrow{E_v}$ *for every distinct pair of vertices* $u, v \in V$.

Proof. The proof is easy by properties 3.33 and Remark 3.25.

Corollary 3.42. *Let* $D = (V, E)$ *be a digraph. For every* $v \in V$ *we have* $U_v \subseteq \overrightarrow{I_e}^{v}$ *for all* $e \in \overrightarrow{E_v}$ *and so* $\overline{U_v} \subseteq \overline{\overrightarrow{I_e}^{v}}$ *for all* $e \in \overrightarrow{E_v}$.

Proof. By Proposition 3.20, $U_v = \bigcap_{e \in \overrightarrow{E_v}} \overrightarrow{I_e}^{v}$ for every $v \in V$. Therefore $U_v \subseteq \overrightarrow{I_e}^{v}$ for all $e \in \overrightarrow{E_v}$. To prove $\overline{U_v} \subseteq \overline{\overrightarrow{I_e}^{v}}$ for all $e \in \overrightarrow{E_v}$, let $u \in \overline{U_v}$ this implies $U \cap U_v \neq \emptyset$ for all open sets U containing u. Since $U_v \subseteq \overrightarrow{I_e}^{v}$ this implies $U \cap \overrightarrow{I_e}^{v} \neq \emptyset$ for all open sets U containing u. Hence $u \in \overline{\overrightarrow{I_e}^{v}}$ and so $\overline{U_v} \subseteq \overline{\overrightarrow{I_e}^{v}}$ for all $e \in \overrightarrow{E_v}$.

Corollary 3.43. *Given a digraph* $D = (V, E)$ *for every* $v \in V$, $\overline{\{v\}} \subseteq \overline{U_v} \subseteq \overline{\overrightarrow{I_e}^{v}}$ *for all* $e \in \overrightarrow{E_v}$.

Proof. Let $u \in \overline{\{v\}}$ this implies $U \cap \{v\} \neq \emptyset$ for all open sets U containing u. Since $\{v\} \subseteq U_v$ this implies $U \cap U_v \neq \emptyset$ for all open sets U containing u. Hence $u \in \overline{U_v}$ and so, $\overline{\{v\}} \subseteq \overline{U_v}$. Also by Corollary 3.42, $\overline{\{v\}} \subseteq \overline{U_v} \subseteq \overline{\overrightarrow{I_e}^{v}}$, for all $e \in \overrightarrow{E_v}$.

Corollary 3.44. *For any* $u, v \in V$ *in a digraph* $D = (V, E)$, *we have* $u \in \overline{\{v\}}$ *if and only if* $\overrightarrow{E_u} \subseteq \overrightarrow{E_v}$.

Proof. $u \in \overline{\{v\}} \Leftrightarrow U \cap \{v\} \neq \emptyset$ for all open sets U containing $u \Leftrightarrow v \in U_u \Leftrightarrow \overrightarrow{E_u} \subseteq \overrightarrow{E_v}$, by Corollary 3.42.

References

[1] J.W. Evans, F. Harary and M.S. Lynn, On the computer enumeration of finite topologies, *Communications of the ACM* 10(5) (1967), 295-297. https://doi.org/10.1145/363282.363311

[2] T.N. Bhargava and T.J. Ahlborn, On topological spaces associated with digraphs, *Acta Mathematica Academiae Scientiarum Hungaricae* 19 (1968), 47-52. https://doi.org/10.1007/BF01894678

[3] T. K. Majumdar, A uniform topological structure on a graph and its properties, Ph.D thesis, India, 1983

[4] A Subramaninan, Studies in graph theory - topological graphs and related topics, Ph.D. thesis, Manonmaniam Sundaranar University, India, 2001

[5] S. P. Subbaih, A study of graph theory: topology, steiner domination and semigraph concepts, Ph.D. thesis, Madurai Kamaraj University, India, 2007.

[6] K. Karunakaran, Topics in graph theory-topological approach, Ph.D. thesis, University of Kerala, India, 2007.

[7] U. Thomas, A study on topological set-indexers of graphs, Ph.D. thesis, Mahatma Gandhi University, India, 2013.

[8] M. Shokry, Generating topology on graphs operations on graphs, *Applied Mathematical Science* 9(54) (2015), 2843-2857. https://doi.org/10.12988/ams.2015.5154.

[9] K.A. Abdu and A. Kilicman, Topologies on the edges sets of directed graphs, *International Journal of Mathematical Analysis* 12(2) (2018), 71-84. https://doi.org/10.12988/ijma.2018.814

[10] Kh. Sh. Al'Dzhabri, A. M. Hamza and Y. S. Eissa, On DG-topological spaces associated with directed graphs, *Journal of Discrete Mathematical Sciences & Cryptography* 23(5) (2020), 1039-1046. https://doi.org/10.1080/09720529.2020.1714886

[11] Kh. Sh. Al'Dzhabri and M. F. H. Al'murshidy, On certain types of topological spaces associated with digraphs, *Journal of Physics: Conference Series* 1591 (2020), 012055. https://doi.org/10.1088/1742-6596/1591/1/012055

[12] S.S. Ray, Graph Theory with Algorithms and its Applications: In Applied Sciences and Technology, Springer, New Delhi, 2013.

[13] J.M. Møller, General Topology, Authors' Notes, 2007.