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Combined Edges Systems and C-space

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Abstract

The combined edges systems, which are the work's central idea, are used to introduce and investigate the C-space in this paper. We introduce the crucial and necessary stepping stone for the concepts of c-derived graphs, c-closed graphs, and c-closure.

1. Introduction

Many people once thought that abstract topological structures only had a little role to play in the generalization of real lines and complex planes, or that they had some ties to algebra and other disciplines of mathematics. And it appears that there is a significant disconnect between the uses of these frameworks in actual life. We observed that in some circumstances, the idea of relation is employed to generate topologies for use in significant applications, such as computation topologies [12], recombination spaces [9, 10], and information granulation used in biological sciences and some other fields of applications. A subfield of mathematics called topological graph theory [1, 2, 3, 4, 13] has notions that are used in practically every other subfield of mathematics as well as several practical contexts. We think that topological graph structure will serve as a crucial foundation for bridging the topology-applications divide. For all terminology and notation in graph theory a novel idea in topology on a signed graph and topology on transitive products of a signed graph was researched by Subbaih [5] in 2007. Karunakaran [6] established topology τ_g on a graph G from a collection of spanning subgraphs of G in the same year. Thomas [7] investigated topology in 2013 and determined the topological numbers of several graphs using set indexers. By using two

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fixed vertices and determining vertex and edge incidence depending on the distance between them, Shokry [8] described a new technique for creating graph topology in 2015. A undirected graph or graph is pair $\Omega = (\mho(\Omega), \mathcal{E}(\Omega))$ where $\mho(\Omega)$ is a non-empty set whose elements are called points or vertices (called vertex set) and $\mathcal{E}(\Omega)$ is the set of unordered pairs of elements of $\mho(\Omega)$ (called edge set). An edge of a graph that joins a vertex to itself is called a loop. If two edges of a graph are joined by a vertex, then these edges are called the edges ϱ incident with the edges ϱ_1. The set of ϱ is $\{\varrho_1 \in \mathcal{E}(\Omega): \varrho_1 \text{incident with } \varrho\}$ and the edges ϱ non incident with the edges ϱ_1. The set of ϱ is $\{\varrho_1 \in \mathcal{E}(\Omega): \varrho_1 \text{nonincident with } \varrho\}$. A graph is symmetric if $(\mho_1, \mho_2) \in \mathcal{E}(\Omega)$ implies $(\mho_2, \mho_1) \in \mathcal{E}(\Omega)$, antisymmetric if $(\mho_1, \mho_2) \in \mathcal{E}(\Omega)$ and $(\mho_2, \mho_1) \in \mathcal{E}(\Omega)$ implies $\mho_2 = \mho_1$. A subgraph of a graph Ω is a graph each of whose vertices belong to $\mho(\Omega)$ and each of whose edges belong to $\mathcal{E}(\Omega)$. An empty graph if the vertices set and edge set is empty. A degree of a vertex $\mho$ in a graph Ω is the number of edges of Ω incident with $\mho$. Null graph is a graph that all of its vertices are of the same degree of degree zero. A star graph of order n (denoted by S_n) is a graph that all edges are incident to each other. A subfamily $Þ_c$ of Ω is said to supra topology on Ω if (i) $\Omega, \phi \in \mu$ (ii) if $ƕ_i \in Þ_c, \forall\, i$, then $\cup\, ƕ_i \in Þ_c$. $(\Omega, Þ_c)$ is called supra topology space. A supra topology space $(\Omega, Þ_c)$ is T_1-space if $\forall\, \varrho_1, \varrho_2 \in \Omega$ such that $\varrho_1 \neq \varrho_2$, there exists $ƕ$ and $ƙ$ are open set such that $\varrho_1 \in ƕ$ & $\varrho_1 \notin ƙ$ and $\varrho_2 \notin ƕ$ & $\varrho_2 \in ƙ$.

2. Combined Edges Systems and C-spaces

In this section, we introduce and investigate the notions of combined edges systems, C-space and c-derived of und. g. (undirected graph).

Definition 2.1. Let $\Omega = (\mho(\Omega), \mathcal{E}(\Omega))$ be an und. g. and an edge $\varrho \in \mathcal{E}(\Omega)$.

1. The edge ϱ incident with the edge ϱ_1 the set of ϱ is denoted by I_ϱ and defined by:

$$I\mathcal{E}(\varrho) = \{\varrho_1 \in \mathcal{E}(\Omega): \varrho_1 \text{incident with } \varrho\}.$$

2. The edge ϱ non incident with the edge ϱ_1 the set of ϱ is denoted by NI_ϱ and defined by:

$$NI\mathcal{E}(\varrho) = \{\varrho_1 \in \mathcal{E}(\Omega): \varrho_1 \text{non incident with } \varrho\}.$$

Definition 2.2. Let $\Omega = (\mho(\Omega), \mathcal{E}(\Omega))$ be an und. g. Then the incident with the edge system (resp. non incident with the edge system) of an edge $\varrho \in \mathcal{E}(\Omega)$ is denoted by $I\mathcal{E}S(\varrho)$ (resp. $NI\mathcal{E}S(\varrho)$) and defined by: $I\mathcal{E}S(\varrho) = \{I\mathcal{E}(\varrho)\}$, (resp. $NI\mathcal{E}S(\varrho) = \{NI\mathcal{E}(\varrho)\}$).

Example 2.3. Let $\Omega = (\mho(\Omega), \mathcal{E}(\Omega))$ be an und. g. such that $\mho(\Omega) = \{ʋ_1, ʋ_2, ʋ_3, ʋ_4, ʋ_5, ʋ_6\}$, $\mathcal{E}(\Omega) = \{\varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_5, \varrho_6, \varrho_7\}$.

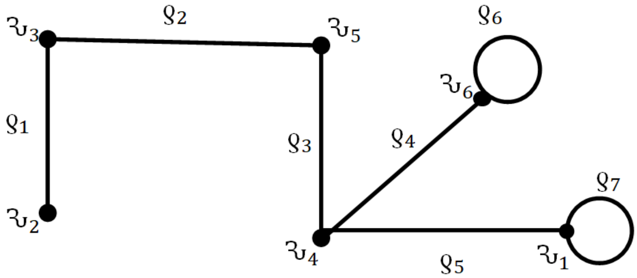

Figure 2.1. und. g. Ω given in Example (2.3).

Then

$$\mathrm{I}\mathcal{E}(\varrho_1) = \{\varrho_2\}, \mathrm{I}\mathcal{E}(\varrho_2) = \{\varrho_1, \varrho_3\}, \mathrm{I}\mathcal{E}(\varrho_3) = \{\varrho_2, \varrho_4, \varrho_5\}, \mathrm{I}\mathcal{E}(\varrho_4) = \{\varrho_3, \varrho_5, \varrho_6\},$$

$$\mathrm{I}\mathcal{E}(\varrho_5) = \{\varrho_3, \varrho_4, \varrho_7\}, \mathrm{I}\mathcal{E}(\varrho_6) = \{\varrho_4, \varrho_6\}, \mathrm{I}\mathcal{E}(\varrho_7) = \{\varrho_5, \varrho_7\}.$$

And

$$\mathrm{I}\mathcal{ES}(\varrho_1) = \{\{\varrho_2\}\}, \mathrm{I}\mathcal{ES}(\varrho_2) = \{\{\varrho_1, \varrho_3\}\}, \mathrm{I}\mathcal{ES}(\varrho_3) = \{\{\varrho_2, \varrho_4, \varrho_5\}\},$$

$$\mathrm{I}\mathcal{ES}(\varrho_4) = \{\{\varrho_3, \varrho_5, \varrho_6\}\}, \mathrm{I}\mathcal{ES}(\varrho_5) = \{\{\varrho_3, \varrho_4, \varrho_7\}\}, \mathrm{I}\mathcal{ES}(\varrho_6) = \{\{\varrho_4, \varrho_6\}\},$$

$$\mathrm{I}\mathcal{ES}(\varrho_7) = \{\{\varrho_5, \varrho_7\}\}.$$

Also, we have

$$\mathrm{NI}\mathcal{E}(\varrho_1) = \{\varrho_3, \varrho_4, \varrho_5, \varrho_6, \varrho_7\}, \mathrm{NI}\mathcal{E}(\varrho_2) = \{\varrho_4, \varrho_5, \varrho_6, \varrho_7\}, \mathrm{NI}\mathcal{E}(\varrho_3) = \{\varrho_1, \varrho_6, \varrho_7\},$$

$$\mathrm{NI}\mathcal{E}(\varrho_4) = \{\varrho_1, \varrho_2, \varrho_7\}, \mathrm{NI}\mathcal{E}(\varrho_5) = \{\varrho_1, \varrho_2, \varrho_6\}, \mathrm{NI}\mathcal{E}(\varrho_6) = \{\varrho_1, \varrho_2, \varrho_3, \varrho_5, \varrho_7\},$$

$$\mathrm{NI}\mathcal{E}(\varrho_7) = \{\varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_6\}.$$

And

$$NI\mathcal{ES}(\varrho_1) = \{\{\varrho_3, \varrho_4, \varrho_5, \varrho_6, \varrho_7\}\}, NI\mathcal{ES}(\varrho_2) = \{\{\varrho_4, \varrho_5, \varrho_6, \varrho_7\}\},$$

$$NI\mathcal{ES}(\varrho_3) = \{\{\varrho_1, \varrho_6, \varrho_7\}\}, NI\mathcal{ES}(\varrho_4) = \{\{\varrho_1, \varrho_2, \varrho_7\}\}, NI\mathcal{ES}(\varrho_5) = \{\{\varrho_1, \varrho_2, \varrho_6\}\},$$

$$NI\mathcal{ES}(\varrho_6) = \{\{\varrho_1, \varrho_2, \varrho_3, \varrho_5, \varrho_7\}\}, NI\mathcal{ES}(\varrho_7) = \{\{\varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_6\}\}.$$

Definition 2.4. Let $\Omega = (\mho(\Omega), \mathcal{E}(\Omega))$ be an und. g. Then the combined edges systems of an edge $\varrho \in \mathcal{E}(\Omega)$ is denoted by $C\mathcal{ES}(\varrho)$ and defined by:

$$C\mathcal{E}S(\varrho) = \{I\mathcal{E}S(\varrho)\ , NI\mathcal{E}S(\varrho)\}.$$

Definition 2.5. Let $\Omega = (\mho(\Omega), \mathcal{E}(\Omega))$ be an und. g. Then the combined edges of an edge $\varrho \in \mathcal{E}(\Omega)$ is denoted by $C\mathcal{E}(\varrho)$ such that $C\mathcal{E}(\varrho) \in C\mathcal{E}S(\varrho)$.

Example 2.6. According to Example (2.3), the combined edges systems are given by:

$C\mathcal{E}S(\varrho_1) = \{\{\varrho_2\}, \{\varrho_3, \varrho_4, \varrho_5, \varrho_6, \varrho_7\}\}$, $C\mathcal{E}S(\varrho_2) = \{\{\varrho_1, \varrho_3\}, \{\varrho_4, \varrho_5, \varrho_6, \varrho_7\}\}$,

$\mathcal{E}S(\varrho_3) = \{\{\varrho_2, \varrho_4, \varrho_5\}, \{\varrho_1, \varrho_6, \varrho_7\}\}$, $C\mathcal{E}S(\varrho_4) = \{\{\varrho_3, \varrho_5, \varrho_6\}, \{\varrho_1, \varrho_2, \varrho_7\}\}$,

$C\mathcal{E}S(\varrho_5) = \{\{\varrho_3, \varrho_4, \varrho_7\}, \{\varrho_1, \varrho_2, \varrho_6\}\}$, $C\mathcal{E}S(\varrho_6) = \{\{\varrho_4, \varrho_6\}, \{\varrho_1, \varrho_2, \varrho_3, \varrho_5, \varrho_7\}\}$,

$C\mathcal{E}S(\varrho_7) = \{\{\varrho_5, \varrho_7\}, \{\varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_6\}\}$.

Definition 2.7. Let $\Omega = (\mho(\Omega), \mathcal{E}(\Omega))$ be an und. g. and suppose that $Þ_c: \mathcal{E}(\Omega) \to P(P(\mathcal{E}(\Omega)))$ is a mapping which assigns for each ϱ in $\mathcal{E}(\Omega)$ its combined edges system in $P(P(\mathcal{E}(\Omega)))$. Then the pair $(\Omega, Þ_c)$ is called the C-space.

Example 2.8. According to Example (2.3), the mapping $Þ_c$ is given by:

$Þ_c(\varrho_1) = \{\{\varrho_2\}, \{\varrho_3, \varrho_4, \varrho_5, \varrho_6, \varrho_7\}\}$, $Þ_c(\varrho_2) = \{\{\varrho_1, \varrho_3\}, \{\varrho_4, \varrho_5, \varrho_6, \varrho_7\}\}$,

$Þ_c(\varrho_3) = \{\{\varrho_2, \varrho_4, \varrho_5\}, \{\varrho_1, \varrho_6, \varrho_7\}\}$, $Þ_c(\varrho_4) = \{\{\varrho_3, \varrho_5, \varrho_6\}, \{\varrho_1, \varrho_2, \varrho_7\}\}$,

$Þ_c(\varrho_5) = \{\{\varrho_3, \varrho_4, \varrho_7\}, \{\varrho_1, \varrho_2, \varrho_6\}\}$, $Þ_c(\varrho_6) = \{\{\varrho_4, \varrho_6\}, \{\varrho_1, \varrho_2, \varrho_3, \varrho_5, \varrho_7\}\}$,

$Þ_c(\varrho_7) = \{\{\varrho_5, \varrho_7\}, \{\varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_6\}\}$.

Therefore $(\Omega, Þ_c)$ is a C-space.

It might see that the concept of C-spaces without additional assumptions on und. g. Ω, is two general to embrace many properties. It will be seen that with suitable definitions, a whole concept of C-spaces can be developed and certain of its results find an application in generalized rough set theory.

Definition 2.9. Let $(\Omega, Þ_c)$ be a C-space. An edge ϱ in $\mathcal{E}(\Omega)$ is called a limit edge of a graph $\hbar \subseteq \Omega$ if every combined edgesof ϱ contains at least one edge of $\hbar$ different from ϱ. The set of all limit edges of a $\hbar \subseteq \Omega$ is called the c-derived und. g. of $\hbar$ and is denoted by $[\mathcal{E}(\hbar)]_c^{`}$, and : $[\mathcal{E}(\hbar)]_c^{`} = \{\varrho \in \mathcal{E}(\Omega);\ C\mathcal{E}S(\varrho) \cap (\mathcal{E}(\hbar) - \{\varrho\}) \neq \phi\}$.

Example 2.10. In Example (2.8), if $\hbar \subseteq \Omega$, $\hbar = (\mho(\hbar), \mathcal{E}(\hbar))$ where $\mho(\hbar) = \{\Im_1, \Im_2, \Im_3, \Im_4, \Im_6\}$ and $\mathcal{E}(\hbar) = \{\varrho_1, \varrho_4, \varrho_5, \varrho_6, \varrho_7\}$.

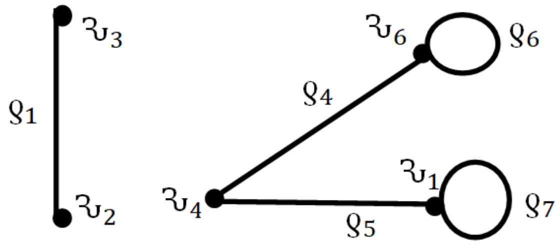

Figure 2.2. sub und. g. $\mathfrak{h}$ of Ω given in Example (2.10).

Then we note that $[\mathcal{E}(\mathfrak{h})]_c^{`} = \{\varrho_2, \varrho_3, \varrho_4, \varrho_5, \varrho_6, \varrho_7\}$.

Definition 2.11. Suppose that $\mathfrak{A}_c : P(\mathcal{E}(\Omega)) \to P(\mathcal{E}(\Omega))$ is a mapping which assigns for every $\mathfrak{h} \subseteq \Omega$ a set $\mathfrak{A}_c(\mathcal{E}(\mathfrak{h})) \subseteq \mathcal{E}(\Omega)$ such that $\mathfrak{A}_c(\mathcal{E}(\mathfrak{h})) = [\mathcal{E}(\mathfrak{h})]_c^{`}$.

Proposition 2.12. *The mapping* $\mathfrak{A}_c$ *satisfies the following properties*:

(a) $\mathfrak{A}_c(\phi) = \phi$.

(b) *If* $\mathfrak{k} \subseteq \mathfrak{h}$, *then* $\mathfrak{A}_c(\mathcal{E}(\mathfrak{k})) \subseteq \mathfrak{A}_c(\mathcal{E}(\mathfrak{h}))$ *for all* $\mathfrak{h}, \mathfrak{k} \subseteq \Omega$.

Proof. (a) Since $\mathfrak{A}_c(\mathcal{E}(\mathfrak{h})) = [\mathcal{E}(\mathfrak{h})]_c^{`}$ and $[\mathcal{E}(\mathfrak{h})]_c^{`} = \{\varrho \in \mathcal{E}(\Omega);\ C\mathcal{E}S(\varrho) \cap (\mathcal{E}(\mathfrak{h}) - \{\varrho\}) \neq \phi\}$, we note that for all $\varrho \in \mathcal{E}(\Omega)$; $C\mathcal{E}S(\varrho) \cap (\phi - \{\varrho\}) = \phi$ and hence $[\mathcal{E}(\mathfrak{h})]_c^{`} = \phi$ thus $\mathfrak{A}_c(\phi) = \phi$.

(b) Let $\Omega = (\mho(\Omega), \mathcal{E}(\Omega))$ be an und. g. and let $\mathfrak{h}, \mathfrak{k} \subseteq \Omega$ and $\mathfrak{k} \subseteq \mathfrak{h}$. To prove that $\mathfrak{A}_c(\mathcal{E}(\mathfrak{k})) \subseteq \mathfrak{A}_c(\mathcal{E}(\mathfrak{h}))$. Let $\varrho_1 \in [\mathcal{E}(\mathfrak{k})]_c^{`} \Longrightarrow C\mathcal{E}S(\varrho_1) \cap (\mathcal{E}(\mathfrak{k}) - \{\varrho_1\}) \neq \phi$.

Since $\mathcal{E}(\mathfrak{k}) \subseteq \mathcal{E}(\mathfrak{h}) \Longrightarrow C\mathcal{E}S(\varrho_1) \cap (\mathcal{E}(\mathfrak{h}) - \{\varrho_1\}) \neq \phi$ thus $\varrho_1 \in [\mathcal{E}(\mathfrak{h})]_c^{`} \Longrightarrow [\mathcal{E}(\mathfrak{k})]_c^{`} \subseteq [\mathcal{E}(\mathfrak{h})]_c^{`} \Longrightarrow \mathfrak{A}_c(\mathcal{E}(\mathfrak{k})) \subseteq \mathfrak{A}_c(\mathcal{E}(\mathfrak{h}))$.

Example 2.13. In Example (2.8), and Example (2.10), if $\mathfrak{k} \subseteq \Omega$, $\mathfrak{k} = (\mho(\mathfrak{k}), \mathcal{E}(\mathfrak{k}))$ suth that $\mho(\mathfrak{k}) = \{\mathfrak{v}_2, \mathfrak{v}_3, \mathfrak{v}_4, \mathfrak{v}_6\}, \mathcal{E}(\mathfrak{k}) = \{\varrho_1, \varrho_4, \varrho_6\}$.

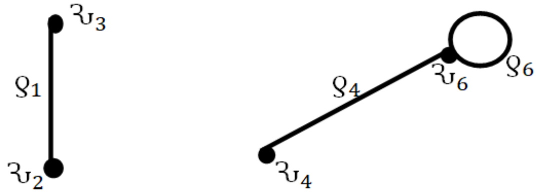

Figure 2.3. sub und. g. $\mathfrak{k}$ of Ω given in Example (2.13).

Then $[\mathcal{E}(\text{ƙ})]^{`}_{c} = \{ϱ_2, ϱ_3, ϱ_4, ϱ_5, ϱ_6\}$. And by Example (2.10), $[\mathcal{E}(\text{ƕ})]^{`}_{c} = \{ϱ_2, ϱ_3, ϱ_4, ϱ_5, ϱ_6, ϱ_7\}$ we get that $\text{ƙ} \subseteq \text{ƕ}$ and $[\mathcal{E}(\text{ƙ})]^{`}_{c} \subseteq [\mathcal{E}(\text{ƕ})]^{`}_{c}$.

Proposition 2.14. *Let* $(\Omega, \text{Þ}_c)$ *be a C-space for all* $\text{ƕ}, \text{ƙ} \subseteq \Omega$. *Then,*

(a) $[\mathcal{E}(\text{ƕ})]^{`}_{c} \cup [\mathcal{E}(\text{ƙ})]^{`}_{c} \subseteq [\mathcal{E}(\text{ƕ} \cup \text{ƙ})]^{`}_{c}$.

(b) $[\mathcal{E}(\text{ƕ} \cap \text{ƙ})]^{`}_{c} \subseteq [\mathcal{E}(\text{ƕ})]^{`}_{c} \cap [\mathcal{E}(\text{ƙ})]^{`}_{c}$.

Proof.

(a) Let $\Omega = (\text{Ʊ}(\Omega), \mathcal{E}(\Omega))$ be an und. g. and let $\text{ƕ}, \text{ƙ} \subseteq \Omega$. To prove that $[\mathcal{E}(\text{ƕ})]^{`}_{c} \cup [\mathcal{E}(\text{ƙ})]^{`}_{c} \subseteq [\mathcal{E}(\text{ƕ} \cup \text{ƙ})]^{`}_{c}$.

Since $\text{ƕ} \subseteq (\text{ƕ} \cup \text{ƙ})$ by Proposition (2.12.(b)) we get

$$[\mathcal{E}(\text{ƕ})]^{`}_{c} \subseteq [\mathcal{E}(\text{ƕ} \cup \text{ƙ})]^{`}_{c} --- (1).$$

And $\text{ƙ} \subseteq (\text{ƕ} \cup \text{ƙ})$ by Proposition (2.12.(b)) we get

$$[\mathcal{E}(\text{ƙ})]^{`}_{c} \subseteq [\mathcal{E}(\text{ƕ} \cup \text{ƙ})]^{`}_{c} --- (2).$$

From(1) and (2) we get $[\mathcal{E}(\text{ƕ})]^{`}_{c} \cup [\mathcal{E}(\text{ƙ})]^{`}_{c} \subseteq [\mathcal{E}(\text{ƕ} \cup \text{ƙ})]^{`}_{c}$.

(b) Let $\Omega = (\text{Ʊ}(\Omega), \mathcal{E}(\Omega))$ be an und. g. and let $\text{ƕ}, \text{ƙ} \subseteq \Omega$. To prove that $[\mathcal{E}(\text{ƕ} \cap \text{ƙ})]^{`}_{c} \subseteq [\mathcal{E}(\text{ƕ})]^{`}_{c} \cap [\mathcal{E}(\text{ƙ})]^{`}_{c}$.

Since $(\text{ƕ} \cap \text{ƙ}) \subseteq \text{ƕ}$ by Proposition (2.12.(b)) we get

$$[\mathcal{E}(\text{ƕ} \cap \text{ƙ})]^{`}_{c} \subseteq [\mathcal{E}(\text{ƕ})]^{`}_{c} --- (1).$$

And $(\text{ƕ} \cap \text{ƙ}) \subseteq (\text{ƙ})$ by Proposition (2.12.(b)) we get

$$[\mathcal{E}(\text{ƕ} \cap \text{ƙ})]^{`}_{c} \subseteq [\mathcal{E}(\text{ƙ})]^{`}_{c} --- (2).$$

From (1) and (2) we get $[\mathcal{E}(\text{ƕ} \cap \text{ƙ})]^{`}_{c} \subseteq [\mathcal{E}(\text{ƕ})]^{`}_{c} \cap [\mathcal{E}(\text{ƙ})]^{`}_{c}$.

Note 2.15. Let $(\Omega, \text{Þ}_c)$ be a C-space for all $\text{ƕ}, \text{ƙ} \subseteq \Omega$. Then,

(a) $[\mathcal{E}(\text{ƕ} \cup \text{ƙ})]^{`}_{c} \subseteq [\mathcal{E}(\text{ƕ})]^{`}_{c} \cup [\mathcal{E}(\text{ƙ})]^{`}_{c}$.

(b) $[\mathcal{E}(\text{ƕ})]^{`}_{c} \cap [\mathcal{E}(\text{ƙ})]^{`}_{c} \subseteq [\mathcal{E}(\text{ƕ} \cap \text{ƙ})]^{`}_{c}$.

It is not necessary to achieve the follwing example illustrates this.

Example 2.16. In Example (2.8), if $\text{ƕ}, \text{ƙ} \subseteq \Omega$, $\text{ƕ} = (\text{Ʊ}(\text{ƕ}), \mathcal{E}(\text{ƕ}))$ such that $\text{Ʊ}(\text{ƕ}) = \{\text{Ʋ}_3, \text{Ʋ}_4, \text{Ʋ}_5\}, \mathcal{E}(\text{ƕ}) = \{ϱ_2, ϱ_3\}$, $\text{ƙ} = (\text{Ʊ}(\text{ƙ}), \mathcal{E}(\text{ƙ}))$ such that $\text{Ʊ}(\text{ƙ}) = \{\text{Ʋ}_1, \text{Ʋ}_4, \text{Ʋ}_6\}, \mathcal{E}(\text{ƙ}) = \{ϱ_4, ϱ_6, ϱ_7\}$.

Figure 2.4. sub und. g. $\mathfrak{h}, \mathfrak{k}$ of Ω given in Example (2.16).

Then we note that $[\mathcal{E}(\mathfrak{h})]'_c = \{\varrho_1, \varrho_4, \varrho_5\}, [\mathcal{E}(\mathfrak{k})]'_c = \{\varrho_3, \varrho_4, \varrho_5, \varrho_6\}, [\mathcal{E}(\mathfrak{h} \cup \mathfrak{k})]'_c = \{\varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_5, \varrho_6\}$,

$$[\mathcal{E}(\mathfrak{h} \cap \mathfrak{k})]'_c = \phi.$$

Proposition 2.17. *Let* $(\Omega, \text{Þ}_c)$ *be a C-space for all* $\mathfrak{h} \subseteq \Omega$. *If* $\varrho \in [\mathcal{E}(\mathfrak{h})]'_c$, *then* $\varrho \in [\mathcal{E}(\mathfrak{h} - \varrho)]'_c$.

Proof. Let $\Omega = (\mho(\Omega), \mathcal{E}(\Omega))$ be an und. g. and let $\mathfrak{h} \subseteq \Omega$. Suppose that $\varrho \in [\mathcal{E}(\mathfrak{h})]'_c$

$$\Rightarrow C\mathcal{E}S(\varrho) \cap (\mathcal{E}(\mathfrak{h}) - \{\varrho\}) \neq \phi$$

$$\Rightarrow C\mathcal{E}S(\varrho) \cap (\mathcal{E}(\mathfrak{h}) \cap \{\varrho\}^c) \neq \phi$$

$$\Rightarrow C\mathcal{E}S(\varrho) \cap (\mathcal{E}(\mathfrak{h} - \varrho) \cap \{\varrho\}^c) \neq \phi$$

$$\Rightarrow C\mathcal{E}S(\varrho) \cap (\mathcal{E}(\mathfrak{h} - \varrho) - \{\varrho\}) \neq \phi.$$

Thus $\varrho \in [\mathcal{E}(\mathfrak{h} - \varrho)]'_c$.

Example 2.18. In Example (2.8), if $\mathfrak{h} \subseteq \Omega, \mathfrak{h} = (\mho(\mathfrak{h}), \mathcal{E}(\mathfrak{h}))$ such that $V(\mathfrak{h}) = \{\mathfrak{v}_3, \mathfrak{v}_4, \mathfrak{v}_5\}, \mathcal{E}(\mathfrak{h}) = \{\varrho_2, \varrho_3\}$.

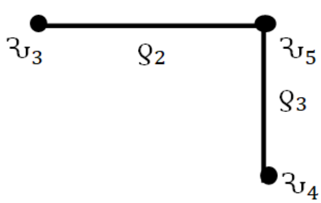

Figure 2.5. sub und. g. $\mathfrak{h}$ of Ω given in Example (2.18).

Then we note that $[\mathcal{E}(\mathfrak{h})]'_c = \{\varrho_1, \varrho_4, \varrho_5\}$ and $\varrho_1 \in [\mathcal{E}(\mathfrak{h})]'_c$, $\varrho_1 \in [\mathcal{E}(\mathfrak{h} - \varrho_1)]'_c$, $\varrho_4 \in [\mathcal{E}(\mathfrak{h})]'_c$, $\varrho_4 \in [\mathcal{E}(\mathfrak{h} - \varrho_4)]'_c$, $\varrho_5 \in [\mathcal{E}(\mathfrak{h})]'_c$, $\varrho_5 \in [\mathcal{E}(\mathfrak{h} - \varrho_5)]'_c$.

Theorem 2.19. *Every sub und. g.* ƕ *of* Ω *contains only one edge, then the c-derived und. g. of* ƕ *is empty set, i.e.,* $[\mathcal{E}(\text{ƕ})]_c^{`} = \phi$.

Proof. Let $\Omega = (\text{Ʊ}(\Omega), \mathcal{E}(\Omega))$ be an und. g. and let $\text{ƕ} \subseteq \Omega$. Suppose that $\mathcal{E}(\text{ƕ}) = \{\varrho\}$. Now to prove that $[\mathcal{E}(\text{ƕ})]_c^{`} = \phi$. We will prove that by contradiction. Suppose that $[\{\varrho\}]_c^{`} \neq \phi \Longrightarrow \exists\, \varrho_1 \in [\{\varrho\}]_c^{`} \Longrightarrow C\mathcal{E}S(\varrho_1) \cap (\{\varrho\} - \{\varrho_1\}) \neq \phi \Longrightarrow \varrho \neq \varrho_1$ and $\varrho \in I\mathcal{E}S(\varrho_1)$ and $\varrho \in NI\mathcal{E}S(\varrho_1)$ and this a contradiction, then $\varrho_1 \notin [\{\varrho\}]_c^{`} \Longrightarrow [\mathcal{E}(\text{ƕ})]_c^{`} = \phi$. The proof is complete.

Theorem 2.20. *If* $\Omega = (\text{Ʊ}(\Omega), \mathcal{E}(\Omega))$ *is star und. g., then for every sub und. g.* ƕ *of* Ω *the c-derived und. g. of* ƕ *is empty set, i.e.,* $[\mathcal{E}(\text{ƕ})]_c^{`} = \phi$.

Proof. Let $\Omega = (\text{Ʊ}(\Omega), \mathcal{E}(\Omega))$ be a star und. g. and $\text{ƕ} \subseteq \Omega$. To prove that $[\mathcal{E}(\text{ƕ})]_c^{`} = \phi$. We will prove that by contradiction. Suppose that $[\mathcal{E}(\text{ƕ})]_c^{`} \neq \phi \Longrightarrow \exists\, \varrho \in [\mathcal{E}(\text{ƕ})]_c^{`} \Longrightarrow C\mathcal{E}S(\varrho) \cap (\mathcal{E}(\text{ƕ}) - \{\varrho\}) \neq \phi \Longrightarrow C\mathcal{E}S(\varrho) \neq \phi \Longrightarrow \exists\, \varrho_1 \in C\mathcal{E}S(\varrho)$ and $\varrho_1 \in \mathcal{E}(\text{ƕ})$ and hence $\exists\, \varrho_2$ incident with ϱ_1 and ϱ_2 non incident with ϱ_1 and this a contradiction with Ω is star graph. And hence $[\mathcal{E}(\text{ƕ})]_c^{`} = \phi$.

Theorem 2.21. *If* $\Omega = (\text{Ʊ}(\Omega), \mathcal{E}(\Omega))$ *is null und. g., then for every sub und. g.* ƕ *of* Ω *the c-derived und. g. of* ƕ *is empty set, i.e.* $[\mathcal{E}(\text{ƕ})]_c^{`} = \phi$.

Proof. Clear.

Definition 2.22. Two C-spaces $(\Omega_1, \text{Þ}_{c_1})$ and $(\Omega_2, \text{Þ}_{c_2})$ are said to be c-equivalent if the c-derived und. g. of each sub und. g. in $(\Omega_1, \text{Þ}_{c_1})$ equal to the c-derived und. g. of the same sub und. g. in $(\Omega_2, \text{Þ}_{c_2})$.

In other words, the two C-spaces $(\Omega_1, \text{Þ}_{c_1})$ and $(\Omega_2, \text{Þ}_{c_2})$ are c-equivalent if and only if $[\mathcal{E}(\text{ƕ})]_{c_1}^{`} = [\mathcal{E}(\text{ƕ})]_{c_2}^{`}$, for all $\mathcal{E}(\text{ƕ}) \subseteq \mathcal{E}(\Omega_1), \mathcal{E}(\text{ƕ}) \subseteq \mathcal{E}(\Omega_2)$.

Example 2.23. Let $\Omega_1 = (\text{Ʊ}(\Omega_1), \mathcal{E}(\Omega_1))$, $\Omega_2 = (\text{Ʊ}(\Omega_2), \mathcal{E}(\Omega_2))$ where $\text{Ʊ}(\Omega_1) = \text{Ʊ}(\Omega_2) = \{\text{Ʋ}_1, \text{Ʋ}_2, \text{Ʋ}_3\}$, $\mathcal{E}(\Omega_1) = \{\varrho_1, \varrho_2, \varrho_3\}$ and $\mathcal{E}(\Omega_2) = \{\varrho_1\}$.

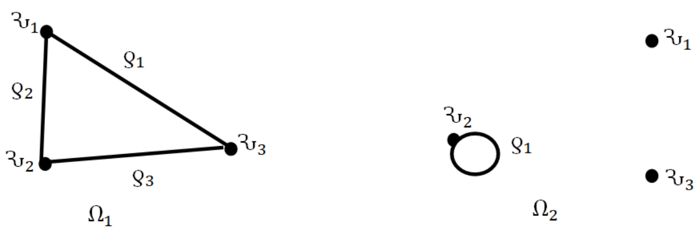

Figure 2.6. und. g. Ω_1 and Ω_2 given in Example (2.23).

Then Þ_{c_1} induced by Ω_1 is given by:

$\text{Þ}_{c_1}(\varrho_1) = \{\{\varrho_2, \varrho_3\}, \phi\}, \text{Þ}_{c_1}(\varrho_2) = \{\{\varrho_1, \varrho_3\}, \phi\}, \text{Þ}_{c_1}(\varrho_3) = \{\{\varrho_1, \varrho_2\}, \phi\}.$

Also Þ_{c_2} induced by Ω_2 is given by: $\text{Þ}_{c_2}(\varrho_1) = \{\{\varrho_1\}, \phi\}$. We note that the two C-spaces $(\Omega_1, \text{Þ}_{c_1})$ and $(\Omega_2, \text{Þ}_{c_2})$ are c-equivalent, since $[\mathcal{E}(\text{ƕ})]^{`}_{c_1} = [\mathcal{E}(\text{ƕ})]^{`}_{c_2} = \phi$, $\forall\, \mathcal{E}(\text{ƕ}) \subseteq \mathcal{E}(\Omega_1), \mathcal{E}(\Omega_2)$.

Example 2.24. Let $\Omega_1 = (\text{Ʊ}(\Omega_1), \mathcal{E}(\Omega_1))$, $\Omega_2 = (\text{Ʊ}(\Omega_2), \mathcal{E}(\Omega_2))$ where $\text{Ʊ}(\Omega_1) = \text{Ʊ}(\Omega_2) = \{\text{Ʋ}_1, \text{Ʋ}_2, \text{Ʋ}_3\}, \mathcal{E}(\Omega_1) = \{\varrho_1, \varrho_2, \varrho_3, \varrho_4\}$ and $\mathcal{E}(\Omega_2) = \{\varrho_1, \varrho_2, \varrho_3, \varrho_4\}$.

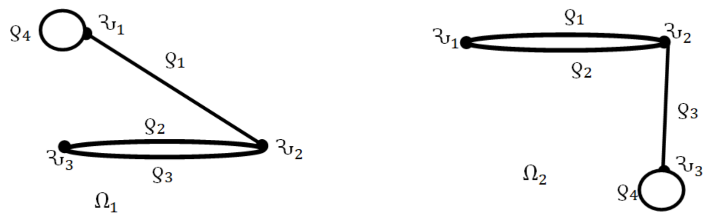

Figure 2.7. und. g. Ω_1 and Ω_2 given in Example (2.24).

Then Þ_{c_1} induced by Ω_1 is given by:

$$\text{Þ}_{c_1}(\varrho_1) = \{\{\varrho_2, \varrho_3, \varrho_4\}, \phi\}, \text{Þ}_{c_1}(\varrho_2) = \{\{\varrho_1, \varrho_3\}, \{\varrho_4\}\},$$

$$\text{Þ}_{c_1}(\varrho_3) = \{\{\varrho_1, \varrho_2\}, \{\varrho_4\}\}, \text{Þ}_{c_1}(\varrho_4) = \{\{\varrho_1, \varrho_4\}, \{\varrho_2, \varrho_3\}\}.$$

Also Þ_{c_2} induced by Ω_2 is given by:

$$\text{Þ}_{c_2}(\varrho_1) = \{\{\varrho_2, \varrho_3\}, \{\varrho_4\}\}, \text{Þ}_{c_2}(\varrho_2) = \{\{\varrho_1, \varrho_3\}, \{\varrho_4\}\},$$

$$\text{Þ}_{c_2}(\varrho_3) = \{\{\varrho_1, \varrho_2, \varrho_4\}, \phi\}, \text{Þ}_{c_2}(\varrho_4) = \{\{\varrho_3, \varrho_4\}, \{\varrho_1, \varrho_2\}\}.$$

Accordingly, there exists $\text{ƕ} \subseteq \Omega_1, \Omega_2$, namely $\mathcal{E}(\text{ƕ}) = \{\varrho_1, \varrho_2\}$ such that $[\mathcal{E}(\text{ƕ})]^{`}_{c_1} \neq [\mathcal{E}(\text{ƕ})]^{`}_{c_2}$, hence the two C-spaces $(\Omega_1, \text{Þ}_{c_1})$ and $(\Omega_2, \text{Þ}_{c_2})$ are not c-equivalent.

Corollary 2.25. *If* $\Omega_1 = (\text{Ʊ}(\Omega_1), \mathcal{E}(\Omega_1))$ *is null und. g. and* $\Omega_2 = (\text{Ʊ}(\Omega_2), \mathcal{E}(\Omega_2))$ *is star und.g., then two C-spaces* $(\Omega_1, \text{Þ}_{c_1})$ *and* $(\Omega_2, \text{Þ}_{c_2})$ *are c-equivalent.*

Proof. Let $\Omega_1 = (\text{Ʊ}(\Omega_1), \mathcal{E}(\Omega_1))$ be a null und. g. and $\Omega_2 = (\text{Ʊ}(\Omega_2), \mathcal{E}(\Omega_2))$ is star und. g.

To prove that $(\Omega_1, Þ_{c_1})$ and $(\Omega_2, Þ_{c_2})$ are c-equivalent. By Theorem (2.20), and Theorem (2.22), we get $[\mathcal{E}(\hbar)]^{`}_{c_1} = \phi, \forall \hbar \subseteq \Omega_1$ and $[\mathcal{E}(\hbar)]^{`}_{c_2} = \phi\ \forall \hbar \subseteq \Omega_2 \implies [\mathcal{E}(\hbar)]^{`}_{c_1} = [\mathcal{E}(\hbar)]^{`}_{c_2}\ \forall \hbar \subseteq \Omega_1, \Omega_1$ and hence $(\Omega_1, Þ_{c_1})$ and $(\Omega_2, Þ_{c_2})$ are c-equivalent.

Definition 2.26. A C-space $(\Omega, Þ_c)$ is called symmetric (resp. antisymmetric) if $Þ_c$ is induced by a symmetric (resp. antisymmetric) und. graph.

Example 2.27. Let $\Omega = \big(\mho(\Omega), \mathcal{E}(\Omega)\big)$ be an und. g. such that $\mho(\Omega) = \{\mathfrak{r}_1, \mathfrak{r}_2, \mathfrak{r}_3\}$, $\mathcal{E}(\Omega) = \{\varrho_1, \varrho_2, \varrho_3\}$.

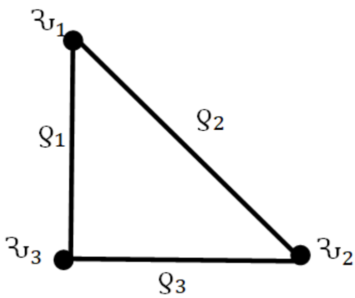

Figure 2.8. und. g. Ω given in Example (2.27).

Hence $Þ_c$ is defined by $Þ_c(\varrho_1) = \big\{\{\varrho_2, \varrho_3\}, \phi\big\}, Þ_c(\varrho_2) = \big\{\{\varrho_1, \varrho_3\}, \phi\big\}, Þ_c(\varrho_3) = \big\{\{\varrho_1, \varrho_2\}, \phi\big\}$. Thus if $(\mathfrak{r}_1, \mathfrak{r}_2) \in \mathcal{E}(\Omega)$ implies $(\mathfrak{r}_2, \mathfrak{r}_1) \in \mathcal{E}(\Omega)$ since [Ω is und. g. $\implies (\mathfrak{r}_2, \mathfrak{r}_1) = (\mathfrak{r}_1, \mathfrak{r}_2)$]. Then Ω is symmetric und. g. and hence $Þ_c$ is induced by symmetric und. g. Ω. Thus $(\Omega, Þ_c)$ is symmetric C-space.

The following und. g. shown in Figure (2.9) is antisymmetric und. g.

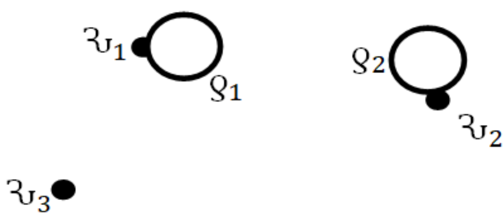

Figure 2.9. und. g. Ω.

Theorem 2.28. *If* $\Omega = (\mho(\Omega), \mathcal{E}(\Omega))$ *is antisymmetric und. g., then for every sub und. g.* $\hbar$ *of* Ω *the c-derived und. g. of* $\hbar$ *is empty set, i.e.* $[\mathcal{E}(\hbar)]^{`}_{c} = \phi$.

Proof. Let $\Omega = (\mho(\Omega), \mathcal{E}(\Omega))$ be an antisymmetric und. g. and $\hbar \subseteq \Omega$. To prove that $[\mathcal{E}(\hbar)]_c^{`} = \phi$. We will prove that by contradiction. Suppose that $[\mathcal{E}(\hbar)]_c^{`} \neq \phi \Longrightarrow \exists\, \varrho \in [\mathcal{E}(\hbar)]_c^{`} \Longrightarrow CES(\varrho) \cap (\mathcal{E}(\hbar) - \{\varrho\}) \neq \phi \Longrightarrow \exists\, \varrho_1 \in CES(\varrho)$ and $\varrho_1 \in (\mathcal{E}(\hbar) - \{\varrho\})$ $\Longrightarrow \varrho_1 \neq \varrho \Longrightarrow \varrho_1 \in I\mathcal{E}(\varrho)$ and this a contradiction with $I\mathcal{E}(\varrho) = \{\varrho\}$ since Ω is antisymmetric und. g. Thus $[\mathcal{E}(\hbar)]_c^{`} = \phi$.

3. c-closed and c-closure

In this section, we introduce the notions of c-closed and c-closure and we study some of their properties.

Definition 3.1. A C-space $(\Omega, Þ_c)$, which contains all its limit edges is called c-closed. The family $\mathfrak{T}_{Þ_c}$ of all c-closed of a C-space is defined by:

$$\mathfrak{T}_{Þ_c} = \left\{\mathcal{E}(\hbar) \subseteq \mathcal{E}(\Omega);\ [\mathcal{E}(\hbar)]_c^{`} \subseteq \mathcal{E}(\hbar)\right\}.$$

Theorem 3.2. *Let $(\Omega, Þ_c)$ be a C-space. Then ϕ and Ω are c-closed.*

Proof. Let $(\Omega, Þ_c)$ be a C-space. Since $[\phi]_c^{`} = \phi \subseteq \phi$. Thus $\phi$ is c-closed and since $[\Omega]_c^{`} \subseteq \Omega$ and hence Ω is c-closed.

Theorem 3.3. *A C-space, the intersection of any family of c-closed is c-closed.*

Proof. Let $(\Omega, Þ_c)$ be a C-space such that $\Bbbk \subseteq \Omega$ and $\mathcal{E}(\Bbbk) = \cap_i (\hbar_i); i \in I$, the intersection of the c-close $\hbar_i \subseteq \Omega, i \in I$. Hence $\Bbbk \subseteq \hbar_i$ for all $i \in I$ which implies $[\mathcal{E}(\Bbbk)]_c^{`} \subseteq [\mathcal{E}(\hbar_i)]_c^{`}$ for all $i \in I$. But $[\mathcal{E}(\hbar_i)]_c^{`} \subseteq \mathcal{E}(\hbar_i)$ for all $i \in I$ since $\hbar_i$ is c-closed and so $[\mathcal{E}(\Bbbk)]_c^{`} \subseteq \mathcal{E}(\hbar_i)$ for all $i \in I$ thus, $[\mathcal{E}(\Bbbk)]_c^{`} \subseteq \cap_i \left(\mathcal{E}(\hbar_i)\right) = \mathcal{E}(\Bbbk)$, hence $\Bbbk$ is c-closed. From definition of a c-closed.

If $\cap_i (\hbar_i) = \phi$ by Theorem (3.2), we get $\cap_i (\hbar_i)$ is c-closed.

If $\cap_i (\hbar_i) = \Omega$ by Theorem (3.2), we get $\cap_i (\hbar_i)$ is c-closed.

Remark 3.4. The union of two c-closed contained in a C-space do not need to be c-closed as shown in the following example.

Example 3.5. Let $\Omega = \left(\mho(\Omega), \mathcal{E}(\Omega)\right)$ be an und. g. and $\mho(\Omega) = \{\mathfrak{V}_1, \mathfrak{V}_2, \mathfrak{V}_3, \mathfrak{V}_4\}$, $\mathcal{E}(\Omega) = \{\varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_5\}$.

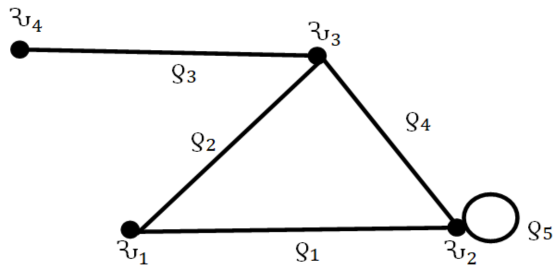

Figure 3.1. und. g. Ω given in Example (3.5).

$$Þ_c(\varrho_1) = \{\{\varrho_2, \varrho_4, \varrho_5\}, \{\varrho_3\}\}, Þ_c(\varrho_2) = \{\{\varrho_1, \varrho_3, \varrho_4\}, \{\varrho_5\}\},$$

$$Þ_c(\varrho_3) = \{\{\varrho_2, \varrho_4\}, \{\varrho_1, \varrho_5\}\}, Þ_c(\varrho_4) = \{\{\varrho_1, \varrho_2, \varrho_3, \varrho_5\}, \phi\},$$

$$Þ_c(\varrho_5) = \{\{\varrho_1, \varrho_4, \varrho_5\}, \{\varrho_2, \varrho_3\}\}.$$

Accordingly, the family $\mathfrak{T}_{Þ_c}$ of all c-closed of this C-space is given by:

$$\mathfrak{T}_{Þ_c} = \{\mathcal{E}(\Omega), \phi, \{\varrho_1\}, \{\varrho_2\}, \{\varrho_3\}, \{\varrho_4\}, \{\varrho_5\}, \{\varrho_1, \varrho_2, \varrho_3, \varrho_5\}\}.$$

Obviously, the sub und. g. $ƕ = (Ʊ(ƕ), \mathcal{E}(ƕ))$ such that $Ʊ(ƕ) = \{\mathfrak{v}_3, \mathfrak{v}_4\}, \mathcal{E}(ƕ) = \{\varrho_3\}$ and $ƙ = (Ʊ(ƙ), \mathcal{E}(ƙ))$ such that $Ʊ(ƙ) = \{\mathfrak{v}_1, \mathfrak{v}_3\}, \mathcal{E}(ƙ) = \{\varrho_2\}$. Then we note that $\{\varrho_3\}, \{\varrho_2\}$ are c-closed, but their union $ƕ \cup ƙ = (Ʊ(ƕ \cup ƙ), \mathcal{E}(ƕ \cup ƙ))$ such that $Ʊ(ƕ \cup ƙ) = \{\mathfrak{v}_1, \mathfrak{v}_3, \mathfrak{v}_4\}, \mathcal{E}(ƕ \cup ƙ) = \{\varrho_2, \varrho_3\}$ is not c-closed, [since $[\mathcal{E}(ƕ \cup ƙ)]_c^{`} = \{\varrho_1\} \Longrightarrow [\mathcal{E}(ƕ \cup ƙ)]_c^{`} \nsubseteq \mathcal{E}(ƕ \cup ƙ)$].

Theorem 3.6. *Let* $(\Omega, Þ_c)$ *be a C-space. Then*

(a) *Every sub und. g.* $ƕ$ *of* Ω *contains only one edge is c-closed.*

(b) *If* Ω *is antisymmetric und. g., then every sub und. g.* $ƕ$ *of* Ω *is c-closed.*

Proof.

(a) Let $ƕ$ is sub und. g. of Ω contains only one edge $\Longrightarrow \mathcal{E}(ƕ) = \{\varrho\}$.

By Theorem (2.19), we get $[\mathcal{E}(ƕ)]_c^{`} = \phi$. And hence $[\mathcal{E}(ƕ)]_c^{`} \subseteq \mathcal{E}(ƕ)$. Therefore $ƕ$ is c-closed.

(b) Let Ω be an antisymmetric und. g. and let $ƕ \subseteq \Omega$ be any sub und. g. by Theorem (2.28), we get $[\mathcal{E}(ƕ)]_c^{`} = \phi$. And hence $[\mathcal{E}(ƕ)]_c^{`} \subseteq \mathcal{E}(ƕ)$. Therefore $ƕ$ is c-closed.

Corollary 3.7. *Let* $(\Omega, Þ_c)$ *be a C-space, and* Ω *is antisymmetric und. g. Then* $\mathfrak{T}_{Þ_c} = P(\mathcal{E}(\Omega))$.

Proof. It is clear by Theorem (3.6(b)).

Theorem 3.8. *If* $(\Omega, Þ_c)$ *is a C-space and* $ƕ \subseteq \Omega$ *is c-closed, then every und. g. contained in* $ƕ$ *and containing* $[\mathcal{E}(ƕ)]^{\grave{}}_c$ *is c-closed.*

Proof. Let $(\Omega, Þ_c)$ be a C-space and $ƕ, ƙ \subseteq \Omega$ such that $ƕ$ is c-closed and $[\mathcal{E}(ƕ)]^{\grave{}}_c \subseteq \mathcal{E}(ƙ) \subseteq \mathcal{E}(ƕ)$. Since $\mathcal{E}(ƙ) \subseteq \mathcal{E}(ƕ)$, then $[\mathcal{E}(ƙ)]^{\grave{}}_c \subseteq [\mathcal{E}(ƕ)]^{\grave{}}_c$ and so $[\mathcal{E}(ƙ)]^{\grave{}}_c \subseteq \mathcal{E}(ƙ)$, therefore $ƙ$ is c-closed.

Corollary 3.9. *If* $(\Omega, Þ_c)$ *is a C-space and* $ƕ \subseteq \Omega$ *is c-closed, then* $[\mathcal{E}(ƕ)]^{\grave{}}_c$ *is c-closed.*

Proof. Let $ƕ \subseteq \Omega$ and c-closed. By Proposition (2.12(b)) we get $[[\mathcal{E}(ƕ)]^{\grave{}}_c]^{\grave{}}_c \subseteq [\mathcal{E}(ƕ)]^{\grave{}}_c$ and hence $[\mathcal{E}(ƕ)]^{\grave{}}_c \Longrightarrow [\mathcal{E}(ƕ)]^{\grave{}}_c \subseteq \mathcal{E}(ƕ)$ is c-closed.

The converse of corollary (3.9) is not hold in general from the following example.

Example 3.10. Let $\Omega = (\mho(\Omega), \mathcal{E}(\Omega))$ be an und. g. and $\mho(\Omega) = \{ʋ_1, ʋ_2, ʋ_3, ʋ_4\}$, $\mathcal{E}(\Omega) = \{\varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_5, \varrho_6\}$.

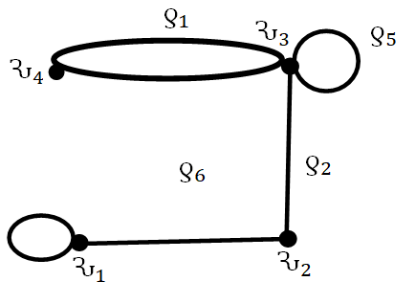

Figure 3.2. und. g. Ω given in Example (3.10).

So, $Þ_c$ is given by:

$$Þ_c(\varrho_1) = \{\{\varrho_2, \varrho_5, \varrho_6\}, \{\varrho_3, \varrho_4\}\}, Þ_c(\varrho_2) = \{\{\varrho_1, \varrho_3, \varrho_5, \varrho_6\}, \{\varrho_4\}\},$$

$$Þ_c(\varrho_3) = \{\{\varrho_2, \varrho_4\}, \{\varrho_1, \varrho_5, \varrho_6\}\}, Þ_c(\varrho_4) = \{\{\varrho_3, \varrho_4\}, \{\varrho_1, \varrho_2, \varrho_5, \varrho_6\}\},$$

$$Þ_c(\varrho_5) = \{\{\varrho_1, \varrho_2, \varrho_5, \varrho_6\}, \{\varrho_3, \varrho_4\}\}, Þ_c(\varrho_6) = \{\{\varrho_1, \varrho_2, \varrho_5\}, \{\varrho_3, \varrho_4\}\}.$$

Accordingly, the family $\mathfrak{T}_{Þ_c}$ of all c-closed of this C-space is given by:

$$\mathfrak{T}_{Þ_c} = \{\mathcal{E}(\Omega), \phi, \{\varrho_1\}, \{\varrho_2\}, \{\varrho_3\}, \{\varrho_4\}, \{\varrho_5\}, \{\varrho_6\}, \{\varrho_1, \varrho_5\}, \{\varrho_1, \varrho_6\}, \{\varrho_5, \varrho_6\}, \{\varrho_1, \varrho_5, \varrho_6\}\}.$$

Let $ƕ \subseteq \Omega$, $ƕ = (Ʊ(ƕ), \mathcal{E}(ƕ))$ be a sub und. g. and $Ʊ(ƕ) = \{ꝩ_2, ꝩ_3, ꝩ_4\}$, $\mathcal{E}(ƕ) = \{\varrho_1, \varrho_2, \varrho_6\}$, then $[\mathcal{E}(ƕ)]^{`}_c = \{\varrho_3\}$ is c-closed, but $\{\varrho_1, \varrho_2, \varrho_6\}$ is not c-closed.

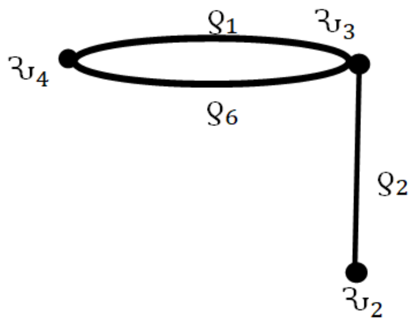

Figure 3.3. sub und. g. ƕ of Ω given in Example (3.10).

Proposition 3.11. *If* $\Omega = (Ʊ(\Omega), \mathcal{E}(\Omega))$ *be an antisymmetric und. g. and* $(\Omega, Þ_c)$ *be a C-space and* $ƕ \subseteq \Omega$, *then* ƕ *is c-closed if and only if* $[\mathcal{E}(ƕ)]^{`}_c$ *is c-closed.*

Proof. Let $\Omega = (Ʊ(\Omega), \mathcal{E}(\Omega))$ be an antisymmetric und. g. and $(\Omega, Þ_c)$ be a C-space and $ƕ \subseteq \Omega$.

Suppose that ƕ is c-closed. By Corollary (3.9) we get $[\mathcal{E}(ƕ)]^{`}_c$ is c-closed.

Suppos that $[\mathcal{E}(ƕ)]^{`}_c$ is c-closed. Since Ω is antisymmetric und. g. by Theorem (3.6(b)) we get $[ƕ]^{`}_c$ is c-closed.

Proposition 3.12. *Let* $(\Omega, Þ_c)$ *be a C-space and* $ƕ \subseteq \Omega$ *is c-closed. If* $ƙ \subseteq ƕ$, *then* $[\mathcal{E}(ƙ)]^{`}_c \subseteq ƕ$.

Proof. Let $(\Omega, Þ_c)$ be a C-space and $ƕ \subseteq \Omega$ and $ƙ \subseteq ƕ$. By Proposition (2.12(b)) we get $[\mathcal{E}(ƙ)]^{`}_c \subseteq [\mathcal{E}(ƕ)]^{`}_c$ since ƕ is c-closed, then $[\mathcal{E}(ƕ)]^{`}_c \subseteq ƕ$ and hence $[\mathcal{E}(ƙ)]^{`}_c \subseteq ƕ$.

The opposite is not true, and the follwing example illustrates this.

Example 3.13. In Example (3.5) $\mathcal{E}(ƕ) = \{\varrho_1, \varrho_2, \varrho_3, \varrho_5\}$ is c-closed and let $ƙ = (Ʊ(ƙ), \mathcal{E}(ƙ))$ such that $Ʊ(ƙ) = \{ꝩ_1, ꝩ_2, ꝩ_3\}$, $\mathcal{E}(ƙ) = \{\varrho_1, \varrho_4\}$ so $[\mathcal{E}(ƙ)]^{`}_c = \{\varrho_3\} \subseteq \mathcal{E}(ƕ)$, but $ƙ \nsubseteq ƕ$.

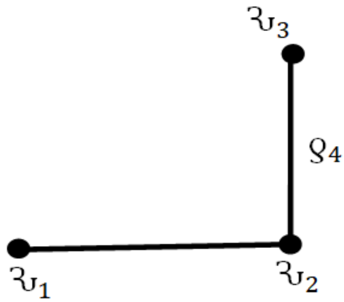

Figure 3.4. sub und. g. ƙ of Ω given in Example (3.13).

Definition 3.14. Let ƕ be a sub und. g. of a C-space $(\Omega, Þ_c)$. The intersection of all c-closed containing ƕ is called c-closure of ƕ and is denoted by $\mathrm{Cl_c}(\mathcal{E}(\text{ƕ}))$, i.e.:

$$\mathrm{Cl_c}(\mathcal{E}(\text{ƕ})) = \cap\{\mathcal{E}(\text{ƙ}) \in \mathfrak{T}_{Þ_c}; \mathcal{E}(\text{ƕ}) \subseteq \mathcal{E}(\text{ƙ})\}.$$

Theorem 3.15. *Let* $(\Omega, Þ_c)$ *be a C-space and let* ƕ $\subseteq \Omega$. *Then*

(a) $\mathcal{E}(\text{ƕ}) \subseteq \mathrm{Cl_c}(\mathcal{E}(\text{ƕ}))$.

(b) $\mathrm{Cl_c}(\mathcal{E}(\text{ƕ})) = \mathcal{E}(\text{ƕ}) \iff \mathcal{E}(\text{ƕ})$ *is c-closed.*

Proof. (a) By Definition (3.14) we get $\mathcal{E}(\text{ƕ}) \subseteq \mathrm{Cl_c}(\mathcal{E}(\text{ƕ}))$.

(b) Let $(\Omega, Þ_c)$ be a C-space and let ƕ $\subseteq \Omega$.

Suppose that ƕ is c-closed. To prove $\mathrm{Cl_c}(\mathcal{E}(\text{ƕ})) = \mathcal{E}(\text{ƕ})$. By Theorem (3.15(a)) we get $\mathcal{E}(\text{ƕ}) \subseteq \mathrm{Cl_c}(\mathcal{E}(\text{ƕ})) - - - (1)$.

Now to prove $\mathrm{Cl_c}(\mathcal{E}(\text{ƕ})) \subseteq \mathcal{E}(\text{ƕ})$.

Let $\varrho \in \mathrm{Cl_c}(\mathcal{E}(\text{ƕ})) \Rightarrow \varrho \in \cap\{\mathcal{E}(\text{ƙ}) \in \mathfrak{T}_{Þ_c}; \mathcal{E}(\text{ƕ}) \subseteq \mathcal{E}(\text{ƙ})\}$ since ƕ is c-closed and $\mathcal{E}(\text{ƕ}) \subseteq \mathcal{E}(\text{ƕ}) \Rightarrow \varrho \in \mathcal{E}(\text{ƕ}) \Rightarrow \mathrm{Cl_c}(\mathcal{E}(\text{ƕ})) \subseteq \mathcal{E}(\text{ƕ}) - - - (2)$.

From (1) and (2) we get $\mathrm{Cl_c}(\mathcal{E}(\text{ƕ})) = \mathcal{E}(\text{ƕ})$.

Suppose that $\mathrm{Cl_c}(\mathcal{E}(\text{ƕ})) = \mathcal{E}(\text{ƕ})$. To prove $\mathcal{E}(\text{ƕ})$ is c-closed.

Since $\mathrm{Cl_c}(\mathcal{E}(\text{ƕ})) = \cap\{\mathcal{E}(\text{ƙ}) \in \mathfrak{T}_{Þ_c}; \mathcal{E}(\text{ƕ}) \subseteq \mathcal{E}(\text{ƙ})\}$, by Theorem (3.3) we get $\mathrm{Cl_c}(\mathcal{E}(\text{ƕ}))$ is c-closed and hence $\mathcal{E}(\text{ƕ})$ is c-closed.

Example 3.16. In Example (3.5), let $\text{ƕ} \subseteq \Omega$, $\text{ƕ} = \big(\mho(\text{ƕ}), \mathcal{E}(\text{ƕ})\big)$ such that $\mho(\text{ƕ}) = \{\mathfrak{V}_1, \mathfrak{V}_3, \mathfrak{V}_4\}$, $\mathcal{E}(\text{ƕ}) = \{\varrho_2, \varrho_3\}$ of this C-space is given by:

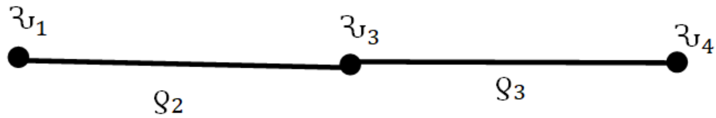

Figure 3.5. sub und. g. ƕ of Ω given in Example (3.16).

So, $\mathrm{Cl_c}\big(\mathcal{E}(\text{ƕ})\big) = \{\varrho_1, \varrho_2, \varrho_3, \varrho_5\}$.

Proposition 3.17. *Let* $(\Omega, \text{Þ}_c)$ *be a C-space and* $\text{ƕ} \subseteq \Omega$. *Then* $\mathcal{E}(\text{ƕ}) \cup [\mathcal{E}(\text{ƕ})]^{\grave{}}_c \subseteq Cl_c\big(\mathcal{E}(\text{ƕ})\big)$.

Proof. Let $(\Omega, \text{Þ}_c)$ be a C-space and $\text{ƕ} \subseteq \Omega$. Since $\mathcal{E}(\text{ƕ}) \subseteq \mathrm{Cl_c}\big(\mathcal{E}(\text{ƕ})\big)$, then $[\mathcal{E}(\text{ƕ})]^{\grave{}}_c \subseteq [\mathrm{Cl_c}\big(\mathcal{E}(\text{ƕ})\big)]^{\grave{}}_c$, but $[\mathrm{Cl_c}\big(\mathcal{E}(\text{ƕ})\big)]^{\grave{}}_c \subseteq \mathrm{Cl_c}\big(\mathcal{E}(\text{ƕ})\big)$ because $\mathrm{Cl_c}\big(\mathcal{E}(\text{ƕ})\big)$ is c-closed and so $[\mathcal{E}(\text{ƕ})]^{\grave{}}_c \subseteq \mathrm{Cl_c}\big(\mathcal{E}(\text{ƕ})\big)$. Accordingly, $\mathcal{E}(\text{ƕ}) \cup [\mathcal{E}(\text{ƕ})]^{\grave{}}_c \subseteq \mathrm{Cl_c}\big(\mathcal{E}(\text{ƕ})\big)$.

Remark 3.18. If $(\Omega, \text{Þ}_c)$ is a C-space and $\text{ƕ} \subseteq \Omega$, then the relation $\mathcal{E}(\text{ƕ}) \cup [\mathcal{E}(\text{ƕ})]^{\grave{}}_c = \mathrm{Cl_c}\big(\mathcal{E}(\text{ƕ})\big)$ is not necessarily true in general.

The next example is employed as a counter example to show the above remark.

Example 3.19. In Example (3.10), let $\text{ƕ} \subseteq \Omega$, $\text{ƕ} = \big(\mho(\text{ƕ}), \mathcal{E}(\text{ƕ})\big)$ such that $\mho(\text{ƕ}) = \{\mathfrak{V}_2, \mathfrak{V}_3, \mathfrak{V}_4\}$, $\mathcal{E}(\text{ƕ}) = \{\varrho_1, \varrho_2, \varrho_5\}$, then $[\mathcal{E}(\text{ƕ})]^{\grave{}}_c = \{\varrho_3\}$ and $\mathrm{Cl_c}\big(\mathcal{E}(\text{ƕ})\big) = \mathcal{E}(\Omega)$.

Obviously, $\mathcal{E}(\text{ƕ}) \cup [\mathcal{E}(\text{ƕ})]^{\grave{}}_c \neq \mathrm{Cl_c}\big(\mathcal{E}(\text{ƕ})\big)$.

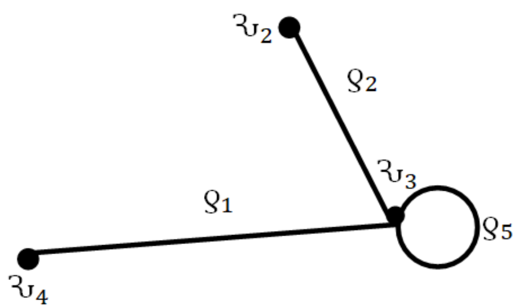

Figure 3.6. sub und. g. ƕ of Ω given in Example(3.19).

Corollary 3.20. *If* $\Omega = (\mho(\Omega), \mathcal{E}(\Omega))$ *is antisymmetric und. g. and* $(\Omega, \text{Þ}_c)$ *is a C-space and* $\text{ƕ} \subseteq \Omega$, *then* $\mathcal{E}(\text{ƕ}) \cup [\mathcal{E}(\text{ƕ})]_c^{`} = Cl_c(\mathcal{E}(\text{ƕ}))$.

Proof. Let $\Omega = (\mho(\Omega), \mathcal{E}(\Omega))$ be an antisymmetric und. g. and $(\Omega, \text{Þ}_c)$ be a C-space and $\text{ƕ} \subseteq \Omega$.

By Proposition (3.17), we get $\mathcal{E}(\text{ƕ}) \cup [\mathcal{E}(\text{ƕ})]_c^{`} \subseteq \mathrm{Cl_c}(\mathcal{E}(\text{ƕ})) - - - (1)$.

To prove $\mathrm{Cl_c}(\mathcal{E}(\text{ƕ})) \subseteq \mathcal{E}(\text{ƕ}) \cup [\mathcal{E}(\text{ƕ})]_c^{`}$. Let $\varrho \in \mathrm{Cl_c}(\mathcal{E}(\text{ƕ}))$ since Ω is antisymmetric und. g. by Theorem (3.6(b)) we get $\mathcal{E}(\text{ƕ})$ is c-closed and by Theorem (3.15(b)) we get $\mathcal{E}(\text{ƕ}) = \mathrm{Cl_c}(\mathcal{E}(\text{ƕ}))$ then

$\varrho \in \mathcal{E}(\text{ƕ}) \Longrightarrow \varrho \in \mathcal{E}(\text{ƕ}) \cup [\mathcal{E}(\text{ƕ})]_c^{`} \Longrightarrow \mathrm{Cl_c}(\mathcal{E}(\text{ƕ})) \subseteq \mathcal{E}(\text{ƕ}) \cup [\mathcal{E}(\text{ƕ})]_c^{`} - - - - (2)$.

From (1) and (2) we get $\mathcal{E}(\text{ƕ}) \cup [\mathcal{E}(\text{ƕ})]_c^{`} = \mathrm{Cl_c}(\mathcal{E}(\text{ƕ}))$.

Proposition 3.21. *If* $(\Omega, \text{Þ}_c)$ *is a C-space, then*

(a) $\mathrm{Cl_c}(\phi) = \phi$.

(b) $\mathrm{Cl_c}(\Omega) = \Omega$.

Proof. Let $(\Omega, \text{Þ}_c)$ be a C-space.

(a) By Theorem (3.2) we get ϕ is c-closed and by Theorem (3.15(b)) we get $\mathrm{Cl_c}(\phi) = \phi$.

(b) By Theorem (3.2) we get Ω is c-closed and by Theorem (3.15(b)) we get $\mathrm{Cl_c}(\Omega) = \Omega$.

Proposition 3.22. *If* $(\Omega, \text{Þ}_c)$ *is a C-space and let* $\text{ƕ} \subseteq \Omega$, *then*

(a) $\mathrm{Cl_c}(\mathrm{Cl_c}(\text{ƕ})) = \mathrm{Cl_c}(\text{ƕ})$.

(b) $If\ \text{ƙ} \subseteq \text{ƕ}, then\ \mathrm{Cl_c}(\text{ƙ}) \subseteq \mathrm{Cl_c}(\text{ƕ})$.

(c) $\mathrm{Cl_c}(\text{ƕ} \cap \text{ƙ}) \subseteq \mathrm{Cl_c}(\text{ƕ}) \cap \mathrm{Cl_c}(\text{ƙ})$.

(d) $\mathrm{Cl_c}(\text{ƕ}) \cup \mathrm{Cl_c}(\text{ƙ}) \subseteq \mathrm{Cl_c}(\text{ƕ} \cup \text{ƙ})$.

Proof. (a) Let $(\Omega, \text{Þ}_c)$ be a C-space and let $\text{ƕ} \subseteq \Omega$. To prove that $\mathrm{Cl_c}(\mathrm{Cl_c}(\text{ƕ})) = \mathrm{Cl_c}(\text{ƕ})$ by Definition (3.14) and Theorem (3.3) and Theorem (3.15(b)) we get that $\mathrm{Cl_c}(\mathrm{Cl_c}(\text{ƕ})) = \mathrm{Cl_c}(\text{ƕ})$.

(b) Let $(\Omega, \text{Þ}_c)$ be a C-space and let $\text{ƙ} \subseteq \text{ƕ} \subseteq \Omega$. To prove that $\mathrm{Cl_c}(\text{ƙ}) \subseteq \mathrm{Cl_c}(\text{ƕ})$ let

$\varrho \in \mathrm{Cl}_c(\text{ƙ})$

$$\Rightarrow \varrho \in \cap \{\mathcal{E}(\mathcal{A}) \in \mathfrak{T}_{\text{Þ}_c}; \mathcal{E}(\mathcal{A}) \subseteq \mathcal{E}(\text{ƙ})\}$$

$$\Rightarrow \varrho \in \cap \{\mathcal{E}(\mathcal{A}) \in \mathfrak{T}_{\text{Þ}_c}; \mathcal{E}(\mathcal{A}) \subseteq \mathcal{E}(\text{ƙ}) \subseteq \mathcal{E}(\text{hʋ})\}$$

$$\Rightarrow \varrho \in \cap \{\mathcal{E}(\mathcal{A}) \in \mathfrak{T}_{\text{Þ}_c}; \mathcal{E}(\mathcal{A}) \subseteq \mathcal{E}(\text{hʋ})\}$$

$$\Rightarrow \varrho \in \mathrm{Cl}_c(\text{hʋ}) \text{ thus } \mathrm{Cl}_c(\text{ƙ}) \subseteq \mathrm{Cl}_c(\text{hʋ}).$$

(c) Let $(\Omega, \text{Þ}_c)$ be a C-space and let $\text{ƙ}, \text{hʋ} \subseteq \Omega$. To prove that $\mathrm{Cl}_c(\text{hʋ} \cap \text{ƙ}) \subseteq \mathrm{Cl}_c(\text{hʋ}) \cap \mathrm{Cl}_c(\text{ƙ})$.

Since $\text{hʋ} \cap \text{ƙ} \subseteq \text{hʋ}$ by Proposition (3.22(b)) we get $\mathrm{Cl}_c(\text{hʋ} \cap \text{ƙ}) \subseteq \mathrm{Cl}_c(\text{hʋ}) - - - (1)$.

Since $\text{hʋ} \cap \text{ƙ} \subseteq \text{ƙ}$ by Proposition (3.22(b)) we get $\mathrm{Cl}_c(\text{hʋ} \cap \text{ƙ}) \subseteq \mathrm{Cl}_c(\text{ƙ}) - - - (2)$.

From (1) and (2) we get $\mathrm{Cl}_c(\text{hʋ} \cap \text{ƙ}) \subseteq \mathrm{Cl}_c(\text{hʋ}) \cap \mathrm{Cl}_c(\text{ƙ})$.

(d) Let $(\Omega, \text{Þ}_c)$ be a C-space and let $\text{ƙ}, \text{hʋ} \subseteq \Omega$. To prove that $\mathrm{Cl}_c(\text{hʋ}) \cup \mathrm{Cl}_c(\text{ƙ}) \subseteq \mathrm{Cl}_c(\text{hʋ} \cup \text{ƙ})$.

Since $\text{hʋ} \subseteq \text{hʋ} \cup \text{ƙ}$ by Proposition (3.22(b)) we get $\mathrm{Cl}_c(\text{hʋ}) \subseteq \mathrm{Cl}_c(\text{hʋ} \cup \text{ƙ}) - - - (1)$.

Since $\text{ƙ} \subseteq \text{hʋ} \cup \text{ƙ}$ by Proposition (3.22(b)) we get $\mathrm{Cl}_c(\text{ƙ}) \subseteq \mathrm{Cl}_c(\text{hʋ} \cup \text{ƙ}) - - - (2)$.

From (1) and (2) we get $\mathrm{Cl}_c(\text{hʋ}) \cup \mathrm{Cl}_c(\text{ƙ}) \subseteq \mathrm{Cl}_c(\text{hʋ} \cup \text{ƙ})$.

Remark 3.23. In above Proposition we note that

1. $\mathrm{Cl}_c(\text{hʋ}) \cap \mathrm{Cl}_c(\text{ƙ}) \nsubseteq \mathrm{Cl}_c(\text{hʋ} \cap \text{ƙ})$.

2. $\mathrm{Cl}_c(\text{hʋ} \cup \text{ƙ}) \nsubseteq \mathrm{Cl}_c(\text{hʋ}) \cup \mathrm{Cl}_c(\text{ƙ})$.

The following examples shows the remark above

Example 3.24. In Example (3.5) let $\text{hʋ} = \big(\text{Ʊ}(\text{hʋ}), \mathcal{E}(\text{hʋ})\big)$ such that $\text{Ʊ}(\text{hʋ}) = \{\text{ვ}_1, \text{ვ}_2, \text{ვ}_3\}$, $\mathcal{E}(\text{hʋ}) = \{\varrho_2, \varrho_5\}$ and $\text{ƙ} = \big(\text{Ʊ}(\text{ƙ}), \mathcal{E}(\text{ƙ})\big)$ such that $\text{Ʊ}(\text{ƙ}) = \{\text{ვ}_1, \text{ვ}_3, \text{ვ}_4\}$, $\mathcal{E}(\text{ƙ}) = \{\varrho_2, \varrho_3\}$.

Figure 3.7. sub und. g. $\text{hʋ}, \text{ƙ}$ of Ω given in Example (3.24).

Then we note that $\mathrm{Cl_c}(ƕ) = \{ϱ_1, ϱ_2, ϱ_3, ϱ_5\}$, $\mathrm{Cl_c}(ƙ) = \{ϱ_1, ϱ_2, ϱ_3, ϱ_5\}$, $\mathrm{Cl_c}(ƕ \cap ƙ) = \{ϱ_2\}$, $\mathrm{Cl_c}(ƕ) \cap \mathrm{Cl_c}(ƙ) = \{ϱ_1, ϱ_2, ϱ_3, ϱ_5\}$ and hence $\mathrm{Cl_c}(ƕ) \cap \mathrm{Cl_c}(ƙ) \nsubseteq \mathrm{Cl_c}(ƕ \cap ƙ)$.

Example 3.25. In Example (3.5) let $ƕ = \big(Ʊ(ƕ), \mathcal{E}(ƕ)\big)$ such that $Ʊ(ƕ) = \{Ꝩ_1, Ꝩ_2\}$, $\mathcal{E}(ƕ) = \{ϱ_1\}$ and $ƙ = \big(Ʊ(ƙ), \mathcal{E}(ƙ)\big)$ such that $Ʊ(ƙ) = \{Ꝩ_1, Ꝩ_3\}, \mathcal{E}(ƙ) = \{ϱ_2\}$.

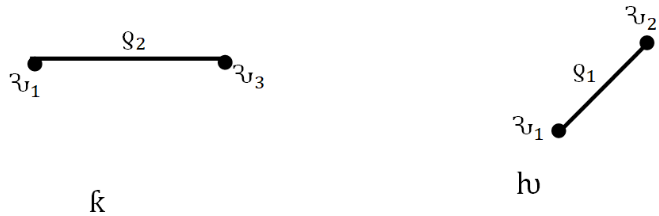

Figure 3.8. sub und. g. $ƕ, ƙ$ of Ω given in Example (3.25).

Then we note that $\mathrm{Cl_c}(ƕ) = \{ϱ_1\}$, $\mathrm{Cl_c}(ƙ) = \{ϱ_2\}$, $\mathrm{Cl_c}(ƕ \cup ƙ) = \{ϱ_1, ϱ_2, ϱ_3, ϱ_5\}$, $\mathrm{Cl_c}(ƕ) \cup \mathrm{Cl_c}(ƙ) = \{ϱ_1, ϱ_2\}$ and hence $\mathrm{Cl_c}(ƕ \cup ƙ) \nsubseteq \mathrm{Cl_c}(ƕ) \cup \mathrm{Cl_c}(ƙ)$.

Corollary 3.26. *If* $\Omega = (Ʊ(\Omega), \mathcal{E}(\Omega))$ *is antisymmetric und. g. and* $(\Omega, Þ_c)$ *is a C-space and* $ƕ, ƙ \subseteq \Omega$, *then*

(a) $\mathrm{Cl_c}(ƕ \cap ƙ) = \mathrm{Cl_c}(ƕ) \cap \mathrm{Cl_c}(ƙ)$.

(b) $\mathrm{Cl_c}(ƕ) \cup \mathrm{Cl_c}(ƙ) = \mathrm{Cl_c}(ƕ \cup ƙ)$.

Proof.

(a) Let Ω be an antisymmetric und. g. and $(\Omega, Þ_c)$ be a C-space and $ƕ, ƙ \subseteq \Omega$ by Proposition (3.22(c)) we get $\mathrm{Cl_c}(ƕ \cap ƙ) \subseteq \mathrm{Cl_c}(ƕ) \cap \mathrm{Cl_c}(ƙ) ----(1)$.

Let $ϱ \in [\mathrm{Cl_c}(ƕ) \cap \mathrm{Cl_c}(ƙ)] \Longrightarrow ϱ \in \mathrm{Cl_c}(ƕ)$ and $ϱ \in \mathrm{Cl_c}(ƙ)$.

Since $ƕ, ƙ \subseteq \Omega$ by Theorem (3.6(b)) we get $ƕ, ƙ$ are c-closed and by Theorem (3.15(b)) we get $ƕ = \mathrm{Cl_c}(ƕ)$ and $ƙ = \mathrm{Cl_c}(ƙ) \Longrightarrow ϱ \in ƕ$ and $ϱ \in ƙ \Longrightarrow ϱ \in (ƕ \cap ƙ)$ by Theorem (3.6(b)) and by Theorem (3.15(b)) we get

$$ϱ \in \mathrm{Cl_c}(ƕ \cap ƙ) \Longrightarrow \mathrm{Cl_c}(ƕ) \cap \mathrm{Cl_c}(ƙ) \subseteq \mathrm{Cl_c}(ƕ \cap ƙ) ----(2).$$

From (1) and (2) we get $\mathrm{Cl_c}(ƕ \cap ƙ) = \mathrm{Cl_c}(ƕ) \cap \mathrm{Cl_c}(ƙ)$.

(b) Let Ω be an antisymmetric und. g. and $(\Omega, Þ_c)$ be a C-space and $ƕ, ƙ \subseteq \Omega$ by Proposition (3.22(d)) we get $\mathrm{Cl_c}(ƕ) \cup \mathrm{Cl_c}(ƙ) \subseteq \mathrm{Cl_c}(ƕ \cup ƙ) ----(1)$.

Let $ϱ \in \mathrm{Cl_c}(ƕ \cup ƙ)$. Since $(ƕ \cup ƙ) \subseteq \Omega$ by Theorem (3.6(b)) we get $(ƕ \cup ƙ)$ is c-closed and by Theorem (3.15(b)) we get $\mathrm{Cl}_c(\mathfrak{h} \cup \mathfrak{k}) = (\mathfrak{h} \cup \mathfrak{k}) \Longrightarrow \varrho \in (\mathfrak{h} \cup \mathfrak{k}) \Longrightarrow \varrho \in \mathfrak{h}$ or $\varrho \in \mathfrak{k}$ by Theorem (3.6(b)) and by Theorem (3.15(b)) we get $\varrho \in \mathrm{Cl}_c(\mathfrak{h})$ or $\varrho \in \mathrm{Cl}_c(\mathfrak{k})$

$$\Longrightarrow \varrho \in \mathrm{Cl}_c(\mathfrak{h}) \cup \mathrm{Cl}_c(\mathfrak{k}) \Longrightarrow \mathrm{Cl}_c(\mathfrak{h} \cup \mathfrak{k}) \subseteq \mathrm{Cl}_c(\mathfrak{h}) \cup \mathrm{Cl}_c(\mathfrak{k}) ----(2).$$

From (1) and (2) we get $\mathrm{Cl}_c(\mathfrak{h} \cup \mathfrak{k}) = \mathrm{Cl}_c(\mathfrak{h}) \cup \mathrm{Cl}_c(\mathfrak{k})$.

Conclusion

Based on the findings of this study, various characteristics of the C-space employing combined edges systems are explored. In this paper, new ideas including the C-subspace, c-equivalent, c-derived undirected graphs, c-closed and c-closure.

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