Earthline Journal of Mathematical Sciences
E-ISSN: 2581-8147
Volume 11, Number 2, 2023, Pages 249-263
https://doi.org/10.34198/ejms.11223.249263

The Triple Ordinates for 3-variable v-convex Functions in Hadamard's Inequality

Hussein Sami Shihab

Department of Medical Physics, Madenat Alelem University College, Iraq
e-mail: hussein.sami93@mauc.edu.iq

Abstract

In this paper, along with a number of applications, the extension of Hadamard's kind inequality for v-convex functions and v-convex functions on triple ordinates given in 3-variables is shown.

1. Introduction

Let $h: J \subseteq \mathbb{R}^+ \to \mathbb{R}^+$ where $\mathbb{R}^+ = [0, \infty)$ be a convex mapping familiar with real numbers $h: J \subseteq \mathbb{R}^+ \to \mathbb{R}^+$ be a convex mapping on the real numbers' period J and $\mu_1, \mu_2 \in J$ of period J, with $\mu_1 < \mu_2$. There are the following two inequalities:

$$h\left(\frac{\mu_1 + \mu_2}{2}\right) \leq \frac{1}{\mu_2 - \mu_1} \int_{\mu_1}^{\mu_2} h(p)dp \leq \frac{h(\mu_1) + h(\mu_2)}{2} \tag{1}$$

is acknowledged for convex maps, Hadamard's inequality is well-known in the literature.

In [1] among others, Hudzik and Maligrada were seen as belonging to the v-convex functions class. This class is referred to as the following: v-convex is a definition of a function $h: \mathbb{R}^+ \to \mathbb{R}^+$.

$$h[\sigma p + (1-\sigma)q + (2-\sigma)s] \leq \sigma^v h(p) + (1-\sigma)^v h(q) + (2-\sigma)h(s) \tag{2}$$

holds for all $p, q,$ and $s \in \mathbb{R}^+$, $\sigma \in [0,1]$ and for some fixed $v \in (0,1]$. Each 1-convex function is convex, as may be easily proved.

In [3] Dragomir and Fitzpatrick show how v-convex functions satisfy a variable of Hadamard's inequality.

Received: July 14, 2022; Revised & Accepted: August 26, 2022; Published: November 8, 2022

2020 Mathematics Subject Classification: 26D15.

Keywords and phrases: Jensen's inequality, v-inequality, Hadamard's v-convex function.

Theorem A. *Assume that* $h: \mathbb{R}^+ \to \mathbb{R}^+$ *is a* v*-convex function in* (2), *where* $v \in (0,1)$ *and let* $\mu_1, \mu_2 \in [0, \infty), \mu_1 < \mu_2$. *If* $h \in L^1[0,1]$, *then the inequalities are as follows:*

$$2^{v-1} h\left(\frac{\mu_1 + \mu_2}{2}\right) \leq \frac{1}{\mu_2 - \mu_1} \int_{\mu_1}^{\mu_2} h(p)dp \leq \frac{h(\mu_1) + h(\mu_2)}{v+1} \tag{3}$$

the constant $N = \frac{1}{v+1}$ *is the better likely in the* (3). *Sharp inequalities appear on the top.*

For a triple ordinated convex mapping on a rectangle in the plane $\mathbb{R}^+$, Dragomir found the following Hadamard type inequality in [3].

Accurately, the following mapping on $[0,1]^3$ is frequently used if $h: [\mu_1, \mu_2] \times [\eta_1, \eta_2] \times [\zeta_1, \zeta_2] \to \mathbb{R}^+$ is a convex function:

$$H(\chi, \tau, \delta) = \frac{1}{(\mu_2 - \mu_1)(\eta_2 - \eta_1)(\zeta_2 - \zeta_1)} \int_{\mu_1}^{\mu_2} \int_{\eta_1}^{\eta_2} \int_{\zeta_1}^{\zeta_2} h\left[\chi p + \chi^* \left(\frac{\mu_1 + \mu_2}{2}\right),\right.$$
$$\left. \tau q + \tau^* \left(\frac{\eta_1 + \eta_2}{2}\right), \delta s + \delta^* \left(\frac{\zeta_1 + \zeta_2}{2}\right)\right] dsdqdp.$$

In which $\chi^* = \{(1-\chi) + (2-\chi)\}$, $\tau^* = \{(1-\tau) + (2-\tau)\}$, and $\delta^* = \{(1-\delta) + (2-\delta)\}$.

Theorem B. *Suppose that* $h: \mathbb{R}^3 \to \mathbb{R}^+$ *is triple ordinated convex on* $\mathbb{R}^3$. *Thereafter, there are inequalities.*

$$h\left(\frac{\mu_1 + \mu_2}{2}, \frac{\eta_1 + \eta_2}{2}, \frac{\zeta_1 + \zeta_2}{2}\right)$$
$$\leq \frac{1}{(\mu_2 - \mu_1)(\eta_2 - \eta_1)(\zeta_2 - \zeta_1)} \int_{\mu_1}^{\mu_2} \int_{\eta_1}^{\eta_2} \int_{\zeta_1}^{\zeta_2} h(p, q, s) dsdqdp$$
$$\leq \frac{\begin{array}{c} h(\mu_1, \eta_1, \zeta_1) + h(\mu_1, \eta_1, \zeta_2) + h(\mu_1, \eta_2, \zeta_1) + h(\mu_1, \eta_2, \zeta_2) \\ + h(\mu_2, \eta_1, \zeta_1) + h(\mu_2, \eta_1, \zeta_2) + h(\mu_2, \eta_2, \zeta_1) + (\mu_2, \eta_2, \zeta_2) \end{array}}{8} \tag{4}$$

The resulting inequalities are clear.

In addition, one can have the following characteristics for H (see [4-6]):

(a) On $[0,1]$, H is triple ordinated convex.

(b) There are boundaries.

$$\sup_{(\chi,\tau,\delta)\in[0,1]^3} H(\chi,\tau,\delta) = \frac{1}{(\mu_2-\mu_1)(\eta_2-\eta_1)(\zeta_2-\zeta_1)} \int_{\mu_1}^{\mu_2}\int_{\eta_1}^{\eta_2}\int_{\zeta_1}^{\zeta_2} h(p,q,s)dsdqdp$$

$$= H(1,1,1)$$

and

$$\inf_{(\chi,\tau,\delta)\in[0,1]^3} H(\chi,\tau,\delta) = h\left(\frac{\mu_1+\mu_2}{2},\frac{\eta_1+\eta_2}{2},\frac{\zeta_1+\zeta_2}{2}\right) = H(0,0,0).$$

2. Hadamard's Inequality

Let us define the following concepts first to get things started:

Definition 2.1. Look the 3-dimensional period $\mathbb{R}^3 = [\mu_1,\mu_2]\times[\eta_1,\eta_2]\times[\zeta_1,\zeta_2]$ in $[0,\infty)$ with $\mu_1<\mu_2$, $\eta_1<\eta_2$, and $\zeta_1<\zeta_2$. The mapping $h:\mathbb{R}^3\to\mathbb{R}^+$ is v-convex if

$$h[\sigma p+(1-\sigma)p_1+(2-\sigma)p_2, \sigma q+(1-\sigma)q_1+(2-\sigma)q_2, \sigma s + (1-\sigma)s_1+(2-\sigma)s_2]$$

$$\leq \sigma^v h(p,q,s)+(1-\sigma)^v h(p_1,q_1,s_1)+(2-\sigma)^v h(p_2,q_2,s_2)$$

holds for all (p,q,s), (p_1,q_1,s_1), and $(p_2,q_2,s_2)\in\mathbb{R}^3$ with $\sigma\in[0,1]$ and for some fixed $v\in(0,1]$.

If the partial mappings $h_s:[\mu_1,\mu_2]\times[\eta_1,\eta_2]\to\mathbb{R}^+$, $h_s(j_1,j_2)=h(j_1,j_2,s)$, $h_q:[\mu_1,\mu_2]\times[\zeta_1,\zeta_2]\to\mathbb{R}^+$, $h_q(j_1,j_3)=h(j_1,q,j_3)$, and $h_p:[\eta_1,\eta_2]\times[\zeta_1,\zeta_2]\to\mathbb{R}^+$, $h_p(j_2,j_3)=h(p,j_2,j_3)$, are v-convex on $\mathbb{R}^3$, a function $h:\mathbb{R}^3\to\mathbb{R}^+$ is called triple ordinated v-convex on $\mathbb{R}^3$. For any $s\in[\zeta_1,\zeta_2]$, $q\in[\eta_1,\eta_2]$, and $p\in[\mu_1,\mu_2]$, are v-convex with some fixed $v\in(0,1]$.

Lemma 2.1. *A v-convex mapping is one that is composed of v-convex $h:[\mu_1,\mu_2]\times[\eta_1,\eta_2]\times[\zeta_1,\zeta_2]\subseteq\mathbb{R}^3\to\mathbb{R}^3$ is v-convex on the triple ordinates, but in general, the converse is not true always.*

Proof. Assume that $h:[\mu_1,\mu_2]\times[\eta_1,\eta_2]\times[\zeta_1,\zeta_2]\subseteq\mathbb{R}^3\to\mathbb{R}^+$ is v-convex. Consider the function $h_p:[\eta_1,\eta_2]\times[\zeta_1,\zeta_2]\to[0,\infty)$, $h_p(j_2,j_3)=h(p,j_2,j_3)$. Then for $\sigma\in[0,1]$ and $j_{21},j_{22},j_{23}\in[\eta_1,\eta_2]$, and $j_{31},j_{32},j_{33}\in[\zeta_1,\zeta_2]$, there is:

$$h_p[\sigma j_{21}+(1-\sigma)j_{22}+(2-\sigma)j_{23}, \sigma j_{31}+(1-\sigma)j_{32}+(2-\sigma)j_{33}]$$

$$= h[p, \sigma j_{21} + (1-\sigma) j_{22} + (2-\sigma) j_{23}, \sigma j_{31} + (1-\sigma) j_{32} + (2-\sigma) j_{33}]$$

$$= h[\sigma p + (1-\sigma) p + (2-\sigma) p, \sigma j_{21} + (1-\sigma) j_{22} + (2-\sigma) j_{23}, \sigma j_{31} + (1-\sigma) j_{32} + (2-\sigma) j_{33}]$$

$$\leq \sigma^v h(p, j_{21}, j_{31}) + (1-\sigma)^v h(p, j_{22}, j_{32}) + (2-\sigma)^v h(p, j_{23}, j_{33})$$

$$= \sigma^v h_p(j_{21}, j_{31}) + (1-\sigma)^v h_p(j_{22}, j_{32}) + (2-\sigma)^v h_p(j_{23}, j_{33}).$$

Therefore, $h_p(j_2, j_3) = h(p, j_2, j_3)$ is v-convex on $[\eta_1, \eta_2] \times [\zeta_1, \zeta_2]$.

Consider the function $h_q: [\mu_1, \mu_2] \times [\zeta_1, \zeta_2] \to \mathbb{R}^+$, $h_q(j_1, j_3) = h(j_1, q, j_3)$. Then for $\sigma \in [0,1]$ and $j_{11}, j_{12}, j_{13} \in [\mu_1, \mu_2]$ and $j_{31}, j_{32}, j_{33} \in [\zeta_1, \zeta_2]$, there is:

$$h_q[\sigma j_{11} + (1-\sigma) j_{12} + (2-\sigma) j_{13}, \sigma j_{31} + (1-\sigma) j_{32} + (2-\sigma) j_{33}]$$

$$= h[\sigma j_{11} + (1-\sigma) j_{12} + (2-\sigma) j_{13}, q, \sigma j_{31} + (1-\sigma) j_{32} + (2-\sigma) j_{33}]$$

$$= h[\sigma j_{11} + (1-\sigma) j_{12} + (2-\sigma) j_{13}, \sigma q + (1-\sigma) q + (2-\sigma) q, \sigma j_{31} + (1-\sigma) j_{32} + (2-\sigma) j_{33}]$$

$$\leq \sigma^v h(j_{11}, q, j_{31}) + (1-\sigma)^v h(j_{12}, q, j_{32}) + (2-\sigma)^v h(j_{13}, q, j_{33})$$

$$= \sigma^v h_q(j_{11}, j_{31}) + (1-\sigma)^v h_q(j_{12}, j_{32}) + (2-\sigma)^v h_q(j_{13}, j_{33}).$$

Therefore, $h_q(j_1, j_3) = h(j_1, q, j_3)$ is v-convex on $[\mu_1, \mu_2] \times [\zeta_1, \zeta_2]$.

We will not go into details about the fact that $h_s: [\mu_1, \mu_2] \times [\eta_1, \eta_2] \times [\zeta_1, \zeta_2] \to \mathbb{R}^+$, $h_s(j_1, j_2) = h(j_1, j_2, s)$ is similarly v-convex and true on $[\mu_1, \mu_2] \times [\eta_1, \eta_2]$ for all $s \in [\zeta_1, \zeta_2]$.

In [4] Dragomir gave a mapping $h_0: [0,1]^2 \to \mathbb{R}^+$, that is convex on the axes but not convex on the coordinates and is defined by the function $h_0(p, q) = pq$. To show that v-convexity on the axis does not necessarily imply v-convexity on the axes, we employ the same function with $v = 1$.

The following inequality is the mapping related to inequality (3).

Theorem 2.1. *Assume that $h: [\mu_1, \mu_2] \times [\eta_1, \eta_2] \times [\zeta_1, \zeta_2] \subseteq \mathbb{R}^3 \to \mathbb{R}^+$, on the triple ordinates of $\mathbb{R}^3$, is a v-convex function. The issue of inequalities is one such:*

$$h\left(\frac{\mu_1 + \mu_2}{2}, \frac{\eta_1 + \eta_2}{2}, \frac{\zeta_1 + \zeta_2}{2}\right)$$

$$\leq 2^{v-2}\left[\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\left(p,\frac{\eta_1+\eta_2}{2},\frac{\zeta_1+\zeta_2}{2}\right)dp+\frac{1}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2}\left(\frac{\mu_1+\mu_2}{2},q,\frac{\zeta_1+\zeta_2}{2}\right)dq\right.$$
$$\left.+\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\left(\frac{\mu_1+\mu_2}{2},\frac{\eta_1+\eta_2}{2},s\right)ds\right]$$

$$\leq \frac{1}{(\mu_2-\mu_1)(\eta_2-\eta_1)(\zeta_2-\zeta_1)}\int_{\mu_1}^{\mu_2}\int_{\eta_1}^{\eta_2}\int_{\zeta_1}^{\zeta_2} h(p,q,s)dsdqdp$$

$$\leq \frac{1}{2(v+1)}\left[\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2} h(p,\eta_1,\zeta_1)dp\right.$$
$$+\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2} h(p,\eta_2,\zeta_1)dp$$
$$+\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2} h(p,\eta_1,\zeta_2)dp+\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2} h(p,\eta_2,\zeta_2)dp$$
$$+\frac{1}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2} h(\mu_1,q,\zeta_1)dq$$
$$+\frac{1}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2} h(\mu_2,q,\zeta_1)dq$$
$$+\frac{1}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2} h(\mu_1,q,\zeta_2)dq$$
$$+\frac{1}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2} h(\mu_2,q,\zeta\_2)dq$$
$$+\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2} h(\mu_1,\eta_1,s)ds$$
$$+\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2} h(\mu_2,\eta_1,s)ds$$
$$\left.+\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2} h(\mu_1,\eta_2,s)ds+\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2} h(\mu_2,\eta_2,s)ds\right]$$

$$\leq \frac{h(\mu_1,\eta_1,\zeta_1)+h(\mu_1,\eta_2,\zeta_1)+h(\mu_1,\eta_1,\zeta_2)+h(\mu_1,\eta_2,\zeta_2)}{(v+1)^2} + \frac{h(\mu_2,\eta_1,\zeta_1)+h(\mu_2,\eta_2,\zeta_1)+h(\mu_2,\eta_1,\zeta_2)+h(\mu_2,\eta_2,\zeta_2)}{(v+1)^2}. \tag{5}$$

Proof. Since $h:\mathbb{R}^3 \to \mathbb{R}^+$, it implies that $h_p:[\eta_1,\eta_2]\times[\zeta_1,\zeta_2] \to \mathbb{R}^+$, $h_p(q,s) = h(p,q,s)$ is v-convex on $[\eta_1,\eta_2]\times[\zeta_1,\zeta_2]$ for all $p \in [\mu_1,\mu_2]$, is triple ordinated v-convex on $\mathbb{R}^3$. Then there is v-inequality Hadamard's (3), which gives us:

$$2^{v-1} h_p\left(\frac{\eta_1+\eta_2}{2},\frac{\zeta_1+\zeta_2}{2}\right)$$

$$\leq \frac{1}{(\eta_2-\eta_1)(\zeta_2-\zeta_1)} \int_{\eta_1}^{\eta_2}\int_{\zeta_1}^{\zeta_2} h_p(q,s)\,ds\,dq$$

$$\leq \frac{h_p(\eta_1,\zeta_1)+h_p(\eta_2,\zeta_1)+h_p(\eta_1,\zeta_2)+h_p(\eta_2,\zeta_2)}{v+1}, \forall p \in [\mu_1,\mu_2].$$

That is,

$$2^{v-1} h\left(p,\frac{\eta_1+\eta_2}{2},\frac{\zeta_1+\zeta_2}{2}\right)$$

$$\leq \frac{1}{(\eta_2-\eta_1)(\zeta_2-\zeta_1)} \int_{\eta_1}^{\eta_2}\int_{\zeta_1}^{\zeta_1} h(p,q,s)\,ds\,dq$$

$$\leq \frac{h(p,\eta_1,\zeta_1)+h(p,\eta_2,\zeta_1)+h(p,\eta_1,\zeta_2)+h(p,\eta_2,\zeta_2)}{v+1}, \forall p \in [\mu_1,\mu_2].$$

On $[\mu_1,\mu_2]$, we get by integrating this inequality

$$\frac{2^{v-1}}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2} h\left(p,\frac{\eta_1+\eta_2}{2},\frac{\zeta_1+\zeta_2}{2}\right)dp$$

$$\leq \frac{1}{(\mu_2-\mu_1)(\eta_2-\eta_1)(\zeta_2-\zeta_1)} \int_{\mu_1}^{\mu_2}\int_{\eta_1}^{\eta_2}\int_{\zeta_1}^{\zeta_2} h(p,q,s)\,ds\,dq\,dp$$

$$\leq \frac{1}{v+1}\left[\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2} h(p,\eta_1,\zeta_1)dp + \frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2} h(p,\eta_2,\zeta_1)dp + \frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2} h(p,\eta_1,\zeta_2)dp + \frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2} h(p,\eta_2,\zeta_2)dp\right]$$

$$= \frac{1}{(v+1)(\mu_2-\mu_1)}\int_{\mu_1}^{\mu_2} [h(p,\eta_1,\zeta_1)+h(p,\eta_2,\zeta_1)+h(p,\eta_1,\zeta_2)+h(p,\eta_2,\zeta_2)]dp. \quad (6)$$

Applying the same logic to $h_q: [\mu_1,\mu_2]\times[\zeta_1,\zeta_2]\to\mathbb{R}^+$, $h_q(p,s)=h(p,q,s)$, we obtain:

$$\frac{2^{v-1}}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2} h\left(\frac{\mu_1+\mu_2}{2},q,\frac{\zeta_1+\zeta_2}{2}\right)dq$$

$$\leq \frac{1}{(\eta_2-\eta_1)(\mu_2-\mu_1)(\zeta_2-\zeta_1)}\int_{\eta_1}^{\eta_2}\int_{\mu_1}^{\mu_2}\int_{\zeta_1}^{\zeta_2} h(p,q,s)dsdpdq$$

$$\leq \frac{1}{v+1}\left[\frac{1}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2} h(\mu_1,q,\zeta_1)dq + \frac{1}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2} h(\mu_2,q,\zeta_1)dq + \frac{1}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2} h(\mu_1,q,\zeta_2)dq + \frac{1}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2} h(\mu_2,q,\zeta_2)dq\right]$$

$$= \frac{1}{(v+1)(\eta_2-\eta_1)}\int_{\eta_1}^{\eta_2} [h(\mu_1,q,\zeta_1)+h(\mu_2,q,\zeta_1)+h(\mu_1,q,\zeta_2)+h(\mu_2,q,\zeta_2)]dq. \quad (7)$$

And applying the same logic to $h_s: [\mu_1,\mu_2]\times[\eta_1,\eta_2]\to\mathbb{R}^+$, $h_s(p,q)=h(p,q,s)$, we obtain:

$$\frac{2^{v-1}}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2} h\left(\frac{\mu_1+\mu_2}{2},\frac{\eta_1+\eta_2}{2},s\right)ds$$

$$\leq \frac{1}{(\zeta_2-\zeta_1)(\mu_2-\mu_1)(\eta_2-\eta_1)}\int_{\zeta_1}^{\zeta_2}\int_{\mu_1}^{\mu_2}\int_{\eta_1}^{\eta_2} h(p,q,s)dqdpds$$

$$\leq \frac{1}{v+1}\left[\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2} h(\mu_1,\eta_1,s)ds + \frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2} h(\mu_2,\eta_1,s)ds\right.$$

$$\left. + \frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2} h(\mu_1,\eta_2,s)ds + \frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2} h(\mu_2,\eta_2,s)ds\right]$$

$$= \frac{1}{(v+1)(\zeta_2-\zeta_1)}\int_{\zeta_1}^{\zeta_2}[h(\mu_1,\eta_1,s)+h(\mu_2,\eta_1,s)+h(\mu_1,\eta_2,s)+h(\mu_2,\eta_2,s)]ds. \quad (8)$$

By integrating the inequalities in (6), (7), and (8), we can obtain the second and third inequalities in (5).

As a result of the v-Hadamard inequality (3), we also have:

$$h\left(\frac{\mu_1+\mu_2}{2},\frac{\eta_1+\eta_2}{2},\frac{\zeta_1+\zeta_2}{2}\right) \leq \frac{2^{v-1}}{(\eta_2-\eta_1)(\zeta_2-\zeta_1)}\int_{\eta_1}^{\eta_2}\int_{\zeta_1}^{\zeta_2} h\left(\frac{\mu_1+\mu_2}{2},q,s\right)dsdq \quad (9)$$

and

$$h\left(\frac{\mu_1+\mu_2}{2},\frac{\eta_1+\eta_2}{2},\frac{\zeta_1+\zeta_2}{2}\right) \leq \frac{2^{v-1}}{(\mu_2-\mu_1)(\zeta_2-\zeta_1)}\int_{\mu_1}^{\mu_2}\int_{\zeta_1}^{\zeta_2} h\left(p,\frac{\eta_1+\eta_2}{2},s\right)dsdp \quad (10)$$

and

$$h\left(\frac{\mu_1+\mu_2}{2},\frac{\eta_1+\eta_2}{2},\frac{\zeta_1+\zeta_2}{2}\right) \leq \frac{2^{v-1}}{(\mu_2-\mu_1)(\eta_2-\eta_1)}\int_{\mu_1}^{\mu_2}\int_{\eta_1}^{\eta_2} h\left(p,q,\frac{\zeta_1+\zeta_2}{2}\right)dqdp \quad (11)$$

which provides the first inequality in (5). The same inequality can also be used to state the following:

$$\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2} h(p,\eta_1,\zeta_1)dp \leq \frac{h(\mu_1,\eta_1,\zeta_1)+h(\mu_2,\eta_1,\zeta_1)}{v+1}$$

$$\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2} h(p,\eta_2,\zeta_1)dp \leq \frac{h(\mu_1,\eta_2,\zeta_1)+h(\mu_2,\eta_2,\zeta_1)}{v+1}$$

$$\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2} h(p,\eta_1,\zeta_2)dp \le \frac{h(\mu_1,\eta_1,\zeta_2)+h(\mu_2,\eta_1,\zeta_2)}{v+1}$$

$$\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2} h(p,\eta_2,\zeta_2)dp \le \frac{h(\mu_1,\eta_2,\zeta_2)+h(\mu_2,\eta_2,\zeta_2)}{v+1}$$

$$\frac{1}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2} h(\mu_1,q,\zeta_1)dq \le \frac{h(\mu_1,\eta_1,\zeta_1)+h(\mu_1,\eta_2,\zeta_1)}{v+1}$$

$$\frac{1}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2} h(\mu_2,q,\zeta_1)dq \le \frac{h(\mu_2,\eta_1,\zeta_1)+h(\mu_2,\eta_2,\zeta_1)}{v+1}$$

$$\frac{1}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2} h(\mu_1,q,\zeta_2)dq \le \frac{h(\mu_1,\eta_1,\zeta_2)+h(\mu_1,\eta_2,\zeta_2)}{v+1}$$

$$\frac{1}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2} h(\mu_2,q,\zeta_2)dq \le \frac{h(\mu_2,\eta_1,\zeta_2)+h(\mu_2,\eta_2,\zeta_2)}{v+1}$$

$$\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2} h(\mu_1,\eta_1,s)ds \le \frac{h(\mu_1,\eta_1,\zeta_1)+h(\mu_1,\eta_1,\zeta_2)}{v+1}$$

$$\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2} h(\mu_2,\eta_1,\zeta_1)ds \le \frac{h(\mu_2,\eta_1,\zeta_1)+h(\mu_2,\eta_1,\zeta_2)}{v+1}$$

$$\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2} h(\mu_1,\eta_2,s)ds \le \frac{h(\mu_1,\eta_2,\zeta_1)+h(\mu_1,\eta_2,\zeta_2)}{v+1}$$

and

$$\frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2} h(\mu_2,\eta_2,s)ds \le \frac{h(\mu_2,\eta_2,\zeta_1)+h(\mu_2,\eta_2,\zeta_2)}{v+1}$$

it, in addition, provides the final inequality in (5).

Note. The inequality is reduced to inequality (4) when the condition $v = 1$, in (5).

For the mapping H, we currently have the following result(s):

Theorem 2.2. *Assume that* $h: \mathbb{R}^3 \to \mathbb{R}^3$ *is triple ordinated* v*-convex on* $\mathbb{R}^3$. *The inequalities for* $H(\chi, \tau, \delta)$ *are as follows*:

(a) H *is* v*-convex on the triple ordinates* $[0,1]^3$.

(b) *On the triple ordinates, H is monotonic and non-decreasing.*

(c) *There are boundaries.*

$$\inf_{(\chi,\tau,\delta)\in\mathbb{R}^3} H(\chi,\tau,\delta) = H(0,0,0)$$

and

$$\sup_{(\chi,\tau,\delta)\in\mathbb{R}^3} H(\chi,\tau,\delta) = H(1,1,1).$$

Proof.

(a) Confirm $\chi \in [0,1]$. Then for all $w, y, z \geq 0$ with $w + y + z = 1$ and $\tau_i, \delta_i \in [0,1] \times [0,1]$; $i = 1,2,3$ we have:

$$H(\chi, w\tau_1 + y\tau_2 + z\tau_3, w\delta_1 + y\delta_2 + z\delta_3)$$

$$= \frac{1}{(\mu_2-\mu_1)(\eta_2-\eta_1)(\zeta_2-\zeta_1)}$$

$$\times \int_{\mu_1}^{\mu_2}\int_{\eta_1}^{\eta_2}\int_{\zeta_1}^{\zeta_2} h\left[\chi p + \chi^* \frac{\mu_1+\mu_2}{2}, (w\tau_1 + y\tau_2 + z\tau_3)q\right.$$

$$\left. + \tau^{**}\frac{\eta_1 + _2}{2}, (w\delta_1 + y\delta_2 + z\delta_3)s + \delta^{**}\frac{\zeta_1+\zeta_2}{2}\right] dsdqdp.$$

In which $\tau^{**} = [\{1 - (w\tau_1 + y\tau_2 + z\tau_3)\} + \{2 - (w\tau_1 + y\tau_2 + z\tau_3)\}]$ and $\delta^{**} = [\{1 - (w\delta_1 + y\delta_2 + z\delta_3)\} + \{2 - (w\delta_1 + y\delta_2 + z\delta_3)\}]$

$$= \frac{1}{(\mu_2-\mu_1)(\eta_2-\eta_1)(\zeta_2-\zeta_1)}\int_{\mu_1}^{\mu_2}\int_{\eta_1}^{\eta_2}\int_{\zeta_1}^{\zeta_2} h\left[\chi p + \chi^*\frac{\mu_1+\mu+2}{2}, w\left(\tau_1 q + \tau_1^*\frac{\eta_1+\eta_2}{2}\right)\right.$$

$$+ y\left(\tau_2 q + \tau_2^*\frac{\eta_1+\eta_2}{2}\right) + z\left(\tau_3 q + \tau_3^*\frac{\eta_1+\eta_2}{2}\right), w\left(\delta_1 s + \delta_1^*\frac{\zeta_1+\zeta_2}{2}\right)$$

$$\left. + y\left(\delta_2 s + \delta_2^*\frac{\zeta_1+\zeta_2}{2}\right) z\left(\delta_3 s + \delta_3^*\frac{\zeta_1+\zeta_2}{2}\right)\right] dsdqdp$$

$$= w^v . \frac{1}{(\mu_2-\mu_1)(\eta_2-\eta_1)(\zeta_2-\zeta_1)} \int_{\mu_1}^{\mu_2}\int_{\eta_1}^{\eta_2}\int_{\zeta_1}^{\zeta_2} h\Big(\chi p + \chi^* \frac{\mu_1+\mu_2}{2}, \tau_1 q + \tau_1^* \frac{\eta_1+\eta_2}{2}, \delta_1 s + \delta_1^* \frac{\zeta_1+\zeta_2}{2}\Big) dsdqdp$$

$$+ y^v . \frac{1}{(\mu_2-\mu_1)(\eta_2-\eta_1)(\zeta_2-\zeta_1)} \int_{\mu_1}^{\mu_2}\int_{\eta_1}^{\eta_2}\int_{\zeta_1}^{\zeta_2} h\Big(\chi p + \chi^* \frac{\mu_1+\mu_2}{2}, \tau_2 q + \tau_2^* \frac{\eta_1+\eta_2}{2}, \delta_2 s + \delta_2^* \frac{\zeta_1+\zeta_2}{2}\Big) dsdqdp$$

$$+ z^v . \frac{1}{(\mu_2-\mu_1)(\eta_2-\eta_1)(\zeta_2-\zeta_1)} \int_{\mu_1}^{\mu_2}\int_{\eta_1}^{\eta_2}\int_{\zeta_1}^{\zeta_2} h\Big(\chi p + \chi^* \frac{\mu_1+\mu_2}{2}, \tau_3 q + \tau_3^* \frac{\eta_1+\eta_2}{2}, \delta_3 s + \delta_3^* \frac{\zeta_1+\zeta_2}{2}\Big) dsdqdp$$

$$= w^* H(\chi, \tau_1, \delta_1) + y^* H(\chi, \tau_2, \delta_2) + z^*(\chi, \tau_3, \delta_3).$$

In which $\tau_1^* = \{(1-\tau_1) + (2-\tau_1)\}$, $\tau_2^* = \{(1-\tau_2) + (2-\tau_2)\}$, $\tau_3^* = \{(1-\tau_3) + (2-\tau_3)\}$ and $\delta_1^* = \{(1-\delta_1) + (2-\delta_1)\}$, $\delta_2^* = \{(1-\delta_2) + (2-\delta_2)\}$, $\delta_3^* = \{(1-\delta_3) + (2-\delta_3)\}$.

Similarly, if $\tau \in [0,1]$ is stable, then for all $\chi_1, \chi_2, \chi_3, \delta_1, \delta_2$, and $\delta_3 \in [0,1] \times [0,1]$, and $w, y, z \geq 0$ with $w + y + z = 1$. And if $\delta \in [0,1]$ is stable, then for all $\chi_i, \tau_i \in [0,1] \times [0,1]$; $i = 1,2,3$, and $w, y, z \geq 0$ with $w + y + z = 1$, we also have:

$$H(w\chi_1 + y\chi_2 + z\chi_3, \tau, w\delta_1 + y\delta_2 + z\delta_3) \leq w^v H(\chi_1, \tau, \delta_1) + y^v H(\chi_2, \tau, \delta_2) + z^v H(\chi_3, \tau, \delta_3)$$

and

$$H(w\chi_1 + y\chi_2 + z\chi_3, w\tau_1 + y\tau_2 + z\tau_3, \delta) \leq w^v H(\chi_1, \tau_1, \delta) + y^v H(\chi_2, \tau_2, \delta) + z^v H(\chi_3, \tau_3, \delta)$$

and the statement is proved.

(b) First, we will show that

$$H(\chi, \tau, \delta) \geq H(0,0,\delta), \forall \chi, \tau, \delta \in [0,1]^3. \tag{12}$$

By means of v-Hadamard's inequality (3), we have:

$$H(\chi,\tau,\delta) \geq \frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2} h\left(\frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\left[\chi p+\chi^*\frac{\mu_1+\mu_2}{2}\right]dp, \frac{1}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2}\left[\tau q + \tau^*\frac{\eta_1+\eta_2}{2}\right]dq, \delta s+\delta^*\frac{\zeta_1+\zeta_2}{2}\right)ds$$

$$= \frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2} h\left(\frac{\mu_1+\mu_2}{2}, \frac{\eta_1+\eta_2}{2}, \delta s+\delta^*\frac{\zeta_1+\zeta_2}{2}\right)ds = H(0,0,\delta),$$

$$\forall \chi,\tau,\delta \in [0,1]^3.$$

Now, let $0 \leq \chi_1 < \chi_2 < \chi_3 \leq 1$ and $0 \leq \tau_1 < \tau_2 < \tau_3 \leq 1$. Due to $H(0,0,\delta)$ v-convexity for any $\delta \in [0,1]$, we have:

$$\frac{H(\chi_3,\tau_3,\delta)-H(\chi_2,\tau_2,\delta)}{(\chi_3,\tau_3)-(\chi_2,\tau_2)} \geq \frac{H(\chi_2,\tau_2,\delta)-H(\chi_1,\tau_1,\delta)}{(\chi_2,\tau_2)-(\chi_1,\tau_1)} \geq \frac{H(\chi_1,\tau_1,\delta)-H(0,0,\delta)}{(\chi_1,\tau_1)} \geq 0$$

We will see that we utilized a different inequality for the last one (12).

(c) We can deduce from Jensen's inequality for integrals that g is s-convex on the triple ordinates:

$$H(\chi,\tau,\delta) = \frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2}\left[\frac{1}{(\eta_2-\eta_1)(\zeta_2-\zeta_1)}\int_{\eta_1}^{\eta_2}\int_{\zeta_1}^{\zeta_2} h\left(\chi p+\chi^*\frac{\mu_1+\mu_2}{2}, \tau q + \tau^*\frac{\eta_1+\eta_2}{2}, \delta s+\delta^*\frac{\zeta_1+\zeta_2}{2}\right)dsdq\right]dp$$

$$\geq \frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2} h\left(\chi p+\chi^*\frac{\mu_1+\mu_2}{2}, \frac{1}{\eta_2-\eta_1}\int_{\eta_1}^{\eta_2}\left[\tau q+\tau^*\frac{\eta_1+\eta_2}{2}\right]dq, \frac{1}{\zeta_2-\zeta_1}\int_{\zeta_1}^{\zeta_2}\left[\delta s+\delta^*\frac{\zeta_1+\zeta_2}{2}\right]ds\right)dp$$

$$= \frac{1}{\mu_2 - \mu_1} \int_{\mu_1}^{\mu_2} h\left(\chi p + \chi^* \frac{\mu_1 + \mu_2}{2}, \frac{\eta_1 + \eta_2}{2}, \frac{\zeta_1 + \zeta_2}{2}\right) dp$$

$$\geq h\left(\frac{1}{\mu_2 - \mu_1} \int_{\mu_1}^{\mu_2} \left[\chi p + \chi^* \frac{\mu_1 + \mu_2}{2}\right] dp, \frac{\eta_1 + \eta_2}{2}, \frac{\zeta_1 + \zeta_2}{2}\right)$$

$$= h\left(\frac{\mu_1 + \mu_2}{2}, \frac{\eta_1 + \eta_2}{2}, \frac{\zeta_1 + \zeta_2}{2}\right) = H(0,0,0).$$

H is v-convexity on the triple ordinates gives us

$$H(\chi, \tau, \delta) = \frac{1}{\mu_2 - \mu_1} \int_{\mu_1}^{\mu_2} \left[\frac{1}{(\eta_2 - \eta_1)(\zeta_2 - \zeta_1)} \int_{\eta_1}^{\eta_2} \int_{\zeta_1}^{\zeta_2} h\left(\chi p + \chi^* \frac{\mu_1 + \mu_2}{2}, q, s\right) dsdq \right.$$
$$+ \tau^*.\delta^* \frac{1}{(\eta_2 - \eta_1)(\zeta_2 - \zeta_1)} \int_{\eta_1}^{\eta_2} \int_{\zeta_1}^{\zeta_2} h\left(\chi p + \chi^* \frac{\mu_1 + \mu_2}{2}, \right.$$
$$\left.\left. \frac{\eta_1 + \eta_2}{2}, \frac{\zeta_1 + \zeta_2}{2}\right) dsdq \right] dp$$

$$\leq \tau\delta \frac{1}{(\eta_2 - \eta_1)(\zeta_2 - \zeta_1)} \int_{\eta_1}^{\eta_2} \int_{\zeta_1}^{\zeta_2} \left[\chi \frac{1}{\mu_2 - \mu_1} \int_{\mu_1}^{\mu_2} h(p, q, s) dpdsdq \right.$$
$$\left. + \chi^* \frac{1}{\mu_2 - \mu_1} \int_{\mu_1}^{\mu_2} h\left(\frac{\mu_1 + \mu_2}{2}, q, s\right) dp \right] dsdq$$

$$+ \tau^* \delta^* \frac{1}{(\eta_2 - \eta_1)(\zeta_2 - \zeta_1)} \int_{\eta_1}^{\eta_2} \int_{\zeta_1}^{\zeta_2} \left[\chi \frac{1}{\mu_2 - \mu_1} \int_{\mu_1}^{\mu_2} h\left(p, \frac{\eta_1 + \eta_2}{2}, \frac{\zeta_1 + \zeta_2}{2}\right) dp \right.$$
$$\left. + \chi^* h\left(\frac{\mu_1 + \mu_2}{2}, \frac{\eta_1 + \eta_2}{2}, \frac{\zeta_1 + \zeta_2}{2}\right)\right] dsdq$$

$$= \tau\delta\chi \frac{1}{(\mu_2-\mu_1)(\eta_2-\eta_1)(\zeta_2-\zeta_1)} \int_{\mu_1}^{\mu_2}\int_{\eta_1}^{\eta_2}\int_{\zeta_1}^{\zeta_2} h(p,q,s)dpdqds$$

$$+ \tau\delta\chi^* \frac{1}{(\eta_2-\eta_1)(\zeta_2-\zeta_1)} \int_{\eta_1}^{\eta_2}\int_{\zeta_1}^{\zeta_2} h\left(\frac{\mu_1+\mu_2}{2},q,s\right)dsdq$$

$$+ \chi\tau^*\delta^* \frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2} h\left(p,\frac{\eta_1+\eta_2}{2},\frac{\zeta_1+\zeta_2}{2}\right)dp$$

$$+ \chi^*\tau^*\delta^* h\left(\frac{\mu_1+\mu_2}{2},\frac{\eta_1+\eta_2}{2},\frac{\zeta_1+\zeta_2}{2}\right).$$

Therefore, by (6), (7), (8) and (9) we conclude that

$$H(\chi,\tau,\delta) \le (\tau\delta\chi + \tau\delta\chi^* + \chi\tau^*\delta^* + \chi^*\tau^*\delta^*)$$

$$\times \frac{1}{(\mu_2-\mu_1)(\eta_2-\eta_1)(\zeta_2-\zeta_1)} \int_{\mu_1}^{\mu_2}\int_{\eta_1}^{\eta_2}\int_{\zeta_1}^{\zeta_2} h(p,q,s)\, dsdqdp$$

$$= \frac{1}{(\mu_2-\mu_1)(\eta_2-\eta_1)(\zeta_2-\zeta_1)} \int_{\mu_1}^{\mu_2}\int_{\eta_1}^{\eta_2}\int_{\zeta_1}^{\zeta_2} h(p,q,s)dsdqdp$$

$$= H(1,1,1).$$

The (c) is therefore established.

Note. If we put $v = 1$ in Theorem 2.2, we get the conclusion obtained in Theorem B.

Corollary 2.1. *Assume that* $h: \mathbb{R}^3 \to \mathbb{R}^+$ *is v-convex and triple ordinated on* $\mathbb{R}^3$. *Make the mapping* $g: [0,1]\times[0,1] \to \mathbb{R}$, $g(\chi,\chi) = G(\chi,\chi,\chi)$. *As a result, g is non-decreasing convex monotonic on* $[0,1]\times[0,1]$, *and one has the following boundaries:*

$$\inf_{\chi\in\mathbb{R}^2} g(\chi,\chi) = g(0,0) = G(0,0,0),$$

and

$$\sup_{\chi\in\mathbb{R}^2} g(\chi,\chi) = g(1,1) = G(1,1,1).$$

Proof. There is an immediate outcome from Theorem 2.2.

References

[1] H. Hudzik and L. Maligranda, Some remarks on *s*-convex functions, *Aequationes Math.* 48 (1994), 100-111. https://doi.org/10.1007/bf01837981

[2] M. Alomari and M. Darus, The Hadamard's inequality for *s*-convex function of 2-variables on the co-ordinates, *Int. J. Math. Anal.* 2(13) (2008), 629-638.

[3] S.S. Dragomir and S. Fitzpatrick, The Hadamard's inequality for *s*-convex functions in the second sense, *Demonstratio Math.* 32(4) (1999), 687-696. https://doi.org/10.1515/dema-1999-0403

[4] S. S. Dragomir, On Hadamard's inequality for convex functions on the co-ordinates in a rectangle from the plane, *Taiwanese Journal of Mathematics* 5 (2001), 775-788. https://doi.org/10.11650/twjm/1500574995

[5] S. S. Dragomir, A mapping in connection to Hadamard's inequality, *An Ostro. Akad. Wiss. Math. -Natur* (*Wien*) 128 (1991), 17-20.

[6] S. S. Dragomir, Two mappings in connection to Hadamard's inequality, *J. Math. Anal. Appl.* 167 (1992), 49-56.