Earthline Journal of Mathematical Sciences
E-ISSN: 2581-8147
Volume 11, Number 2, 2023, Pages 229-247
https://doi.org/10.34198/ejms.11223.229247

Certain Identities of a General Class of Hurwitz-Lerch Zeta Function of Two Variables

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Abstract

In this paper, we introduce a generalized double Hurwitz-Lerch Zeta function and then systematically investigate its properties and various integral representations. Further, we show that these results provide certain known as well as new extensions of earlier stated results of generalized Hurwitz-Lerch Zeta functions.

1 Introduction

A generalization of Hurwitz Zeta function $\zeta(s,a)$ defined by [1, p. 249] and [6, p. 89]

$$\zeta(s,a) = \sum_{n=0}^{\infty} \frac{1}{(a+n)^s}, \tag{1}$$

Received: August 31, 2022; Revised & Accepted: October 16, 2022; Published: November 2, 2022
2020 Mathematics Subject Classification: 11M35, 33C65.
Keywords and phrases: Appell functions, Hurwitz-Lerch Zeta function, integral representations, identities.
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$\big(a \in \mathbb{C}\backslash\mathbb{Z}_0^-, s \in \mathbb{C}, \mathbb{Z}_0^- = \{0, -1, -2, -3, \ldots\}, \quad \Re(s) > 1$, is set of complex numbers, is given by Hurwitz-Lerch Zeta function in the form [8, p. 27]

$$\phi(z, s, a) = \sum_{m=0}^{\infty} \frac{z^m}{(a+m)^s}, \tag{2}$$

$(a \in \mathbb{C}\backslash\mathbb{Z}_0^-, s \in \mathbb{C}$, when $|z| < 1$, and when $z = 1,\ \Re(s) > 1)$.

In 1997, Goyal and Laddha [12] extended the Hurwitz-Lerch Zeta function (2) in the form

$$\phi_\mu^*(z, s, a) = \sum_{m=0}^{\infty} \frac{(\mu)_m z^m}{(a+m)^s m!'} \tag{3}$$

$(a \in \mathbb{C}\backslash\mathbb{Z}_0^-, s, \mu \in \mathbb{C}$, when $|z| < 1$, and when $z = 1, \Re(s-\mu) > 0, \Re(a) > 0)$.

It is remarked that when N is large, then by the Gaussian formula of gamma function [27, p. 20], there exists a relation $\lim_{n\to\infty} n^s\Gamma(n) = \lim_{n\to\infty}\Gamma(n+s)$, (see also in [15]), thus the formula (3) is written by

$$\Rightarrow \phi_\mu^*(z, s, a) = \sum_{m=0}^{N} \frac{(\mu)_m z^m}{(a+m)^s m!} + \frac{1}{(a)_s} \sum_{m=N+1}^{\infty} \frac{(a)_m(\mu)_m z^m}{(a+s)_m m!}, \tag{4}$$

The last series in Eqn. (4) converges for the conditions $(a+s) \in \mathbb{C}\backslash\mathbb{Z}_0^-, s, \mu \in \mathbb{C}$, when $|z| < 1$, and when $z = 1, \Re(s-\mu) > 0$, $(a+s) \in \mathbb{C}\backslash\mathbb{Z}_0^-$.

Later on Pathan and Daman [20] defined a generalization of (1) to (3) for two variables as

$$\begin{aligned} \phi_{\mu,\lambda}^*(z, t, s, a) &= \sum_{m,k=0}^{\infty} \frac{(\mu)_m(\lambda)_k z^m t^k}{(a+m+k)^s m! k!} \\ &= \sum_{k=0}^{\infty} \frac{(\lambda)_k t^k}{k!} {\phi_\mu}^*(z, s, a+k). \end{aligned} \tag{5}$$

$\big((a+s) \in \mathbb{C}\backslash\mathbb{Z}_0^-, \mu, \lambda, s \in \mathbb{C}$, when $|z| < 1, |t| < 1$ and for $z = t = 1, \Re(s-\mu-\lambda) > 0\big)$.

The *alternating* Hurwitz Zeta function (or, equivalently, Hurwitz-Euler eta function) $\eta(s,a)$ is defined by

$$\eta(s,a)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(a+n)^s}, \Re(s)>0, a>0. \tag{6}$$

The special case of *alternating* Hurwitz Zeta function, when $a=1$, denoted by

$$\eta(s,a)=\eta(s,1)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}, \Re(s)>0, \tag{7}$$

is called the Dirichlet eta function (or the alternating Riemann Zeta function).

It is known that $\eta(s,a)$ and so also $\eta(s)$ can be continued analytically to the whole complex s - plane. Therefore, each of the functions $\eta(s,a)$ and $\eta(s)$ is an entire function of $s, a \in \mathbb{C}$.

Motivated by the researches done in the extensions of Hurwitz-Lerch Zeta function in (for example [2], [3], [4], [5], [6], [7], [12], [15], [17], [18] [19], [20], [22], [23] [26], [28], and others), we further investigate and introduce a general double Hurwitz Lerch Zeta function to study some of its identities with known and new double zeta functions.

2 The General Hurwitz-Lerch Zeta Function of Two Variables and Related Double Zeta Functions

In this section, we define a general Hurwitz-Lerch Zeta function of two variables consisting of a double sequence of arbitrary complex numbers and then making an appeal to the convergence conditions of double series in ([13], [14], [22], [23], [24], and [25]), we obtain convergence conditions for general Hurwitz-Lerch Zeta function of two variables and its related double zeta functions.

Theorem 2.1. *Let $a \in \mathbb{C}\backslash\mathbb{Z}_0^-, s \in \mathbb{C}, \{A_{m,n}\}_{m,n=0}^{\infty}$ be a double sequence of arbitrary complex numbers, suitably exists and follows Horn theorem. Then the* *general Hurwitz-Lerch Zeta function of two variables defined by*

$$\phi(z,t,s,a)=\sum_{m,n=0}^{\infty} A_{m,n}\frac{z^m t^n}{(a+m+n)^s}, \tag{8}$$

converges for $a\in\mathbb{C}\backslash\mathbb{Z}_0^-, s\in\mathbb{C}$, *when* $|z|<1, |t|<1, \Re(s)>0$, *and for* $z=t=1$, $\Re(s)>1$.

Proof. On applying techniques given in Eqn. (4), for large M, N, the formula (8) is written in the form

$$\begin{aligned}\phi(z,t,s,a)=&\sum_{m=0,n=0}^{m=M,n=N} A_{m,n}\frac{z^m t^n}{(a+m+n)^s}\\ &+\sum_{m=M+1,n=N+1}^{\infty} A_{m,n}\frac{\Gamma(a+m+n)}{\Gamma(a+s+m+n)}z^m t^n. \end{aligned}\tag{9}$$

Now in last series of (9), use the Horn theorem due to [25] together with the formulae of ([9], [10], [13], [14], [23], [25]) to get rational functions of m and n, given by

$$\mathcal{F}(m,n)=\frac{\mathcal{M}(m+1,n)}{\mathcal{M}(m,n)}z \text{ and } \mathcal{G}(m,n)=\frac{\mathcal{M}(m,n+1)}{\mathcal{M}(m,n)}t,$$

where $\mathcal{M}(m,n)=A_{m,n}\frac{\Gamma(a+m+n)}{\Gamma(a+s+m+n)}z^m t^n$

$$\begin{aligned}\Rightarrow \lim_{u\to\infty}|\mathcal{F}(mu,nu)|=&\lim_{u\to\infty}\left|\frac{A_{mu+1,nu}}{A_{mu,nu}}\right|\lim_{u\to\infty}\left|\frac{\Gamma(a+1+u(m+n))}{\Gamma(a+s+1+u(m+n))}\right|\\ &\times\lim_{u\to\infty}\left|\frac{\Gamma(a+s+u(m+n))}{\Gamma(a+u(m+n))}\right||z|. \end{aligned}\tag{10}$$

Then in formula (10), in the region $|\arg(u)|\le\pi-\epsilon, |\arg(u+a)|<\pi-\epsilon, 0<\epsilon<\pi$, use the formula [27, p. 24]

$$\lim_{v\to\infty}\frac{\Gamma(\mathrm{v}+\mathrm{a})}{\Gamma(\mathrm{v}+\mathrm{b})}=\lim_{v\to\infty}v^{a-b}\left[1+\frac{(a-b)(a+b-1)}{2v}+O\left(v^{-2}\right)\right], \forall m>0, n\ge 0,$$

to find that

$$\lim_{u\to\infty} |\mathcal{F}(mu, nu)| = \lim_{u\to\infty} \left|\frac{A_{mu+1,nu}}{A_{mu,nu}}\right| |z|. \tag{11}$$

Since $\forall m \geq 0, n \geq 0, A_{m,n}$ follows Horn Theorem [23], hence $\lim_{u\to\infty}\left|\frac{A_{mu+1,nu}}{A_{mu,nu}}\right| < 1$, and thus due to Eqn. (11), we get

$$\lim_{u\to\infty} |\mathcal{F}(mu, nu)| < |z|, \forall m > 0, n \geq 0. \tag{12}$$

In (12) if $\Re(s) > 0, |z| < 1$, then for any large m or n, we have

$$\lim_{u\to\infty} |\mathcal{F}(mu, nu)| < 1. \tag{13}$$

Again, in a similar way, as we have done in Eqns. (10) to (13), that if $\Re(s) > 0, |t| < 1$, then for any large m or n,

$$\lim_{u\to\infty} |\mathcal{G}(mu, nu)| < 1. \tag{14}$$

Hence, making an appeal to the Eqns. (9) to (14), the series (8) converges for $a \in \mathbb{C}\backslash\mathbb{Z}_0^-, s \in \mathbb{C}$, when $|z| < 1, |t| < 1$, and $\Re(s) > 0$. Further for $z = 1$, by Eqns. (10) we have

$$\lim_{u\to\infty}\left|\sum_{m,n=1}^{\infty} \mathcal{M}(mu+1, nu)\right| < \sum_{m,n=1}^{\infty} \lim_{u\to\infty} |\mathcal{F}(mu, nu)| \lim_{u\to\infty} |\mathcal{M}(mu, nu)|$$
$$< \sum_{m,n=1}^{\infty} \lim_{u\to\infty} \{u\}^{\Re(s)} \lim_{u\to\infty} |A_{mu,nu}| \lim_{u\to\infty} \left|\frac{\Gamma(a+mu+nu)}{\Gamma(a+s+mu+nu)}\right|$$

(Since by (12), when $z = 1, \lim_{u\to\infty} |\mathcal{F}(mu, nu)| < 1, \forall m > 0, n \geq 0$) and again by (10), we have

$$\lim_{u\to\infty}\left|\sum_{m,n=1}^{\infty} \mathcal{M}(mu+1, nu)\right| < \sum_{m,n=1}^{\infty} \lim_{u\to\infty} \{u\}^{\Re(s)} \lim_{u\to\infty} |A_{mu,nu}|$$
$$< \sum_{m,n=1}^{\infty} \lim_{u\to\infty} |A_{mu,nu}| (m+n)^{-\Re(s)} < M_1 \sum_{m,n=1}^{\infty} (m+n)^{-\Re(s)}$$
$$\left(\text{Let } \forall m > 0, n \geq 0, \text{ there exists } \lim_{u\to\infty} |A_{mu,nu}| < M_1, M_1 > 0\right).$$

Therefore, $\forall m > 0, n \geq 0$,

$$\lim_{u\to\infty}\left|\sum_{m,n=1}^{\infty}\mathcal{M}(mu+1,nu)\right| < M_1\sum_{N=1}^{\infty}N^{-\Re(s)}.$$

Certainly, the series in right hand side converges when

$$\Re(s) > 1. \tag{15}$$

In a similar manner using Eqn. (15) for $t = 1$, we have

$$\lim_{u\to\infty}\left|\sum_{m,n=1}^{\infty}\mathcal{M}(mu,nu+1)\right| < \mathcal{M}_1, \text{ if } \Re(s) > 1. \tag{16}$$

Therefore when $t = z = 1$, then by Eqns. (15) and (16), the series (8) converges for $\Re(s) > 1$.

Finally, we are led to the assertion of Theorem 2.1. □

Example 2.2

Let $m > 0, n \geq 0$ and $A_{m,n} = \frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_m(\gamma')_n m!n!}, \forall \alpha, \beta, \beta', \gamma, \gamma' \in \mathbb{C}$ and if $\Re(\alpha) = \alpha_1$, $\Re(\beta) = \beta_1, \Re(\gamma) = \gamma_1, \Re(\gamma') = \gamma_1'$, $\left|\frac{\Gamma(\gamma)\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\beta)}\right| < N_1, m!n! < (m+n)!$, then we get

$$\lim_{u\to\infty}|A_{mu,nu}| < N_1\lim_{u\to\infty}\left|\frac{\Gamma(\alpha+u(m+n))\Gamma(\beta+mu)\Gamma(\beta'+nu)}{\Gamma(1+u(m+n))\Gamma(\gamma+mu)\Gamma(\gamma'+nu)}\right|\left|\frac{\Gamma(u(m+n)+1)}{\Gamma(um+1)\Gamma(un+1)}\right|,$$

$$\Rightarrow \lim_{u\to\infty}|A_{mu,nu}| < N_1C_1\lim_{u\to\infty}(u(m+n))^{\alpha_1-1}(mu)^{\beta_1-\gamma_1}(nu)^{\beta_1'-\gamma_1'},$$

(since by Stirling formula $\lim_{u\to\infty}\left|\frac{\Gamma(u(m+n)+1)}{\Gamma(um+1)\Gamma(un+1)}\right| \leq C_1,\ 0 < C_1 < \infty$),

$$\Rightarrow \lim_{u\to\infty}|A_{mu,nu}| < N_1C_1(m+n)^{\alpha_1-1}(m)^{\beta_1-\gamma_1}(n)^{\beta_1'-\gamma_1'}\lim_{u\to\infty}u^{\alpha_1-1+\beta_1-\gamma_1+\beta_1'-\gamma_1'}. \tag{17}$$

The limit in (17) exists if

$$\alpha_1+\beta_1+\beta'{}_1-\gamma_1-\gamma_1'-1=0. \tag{18}$$

$$\Rightarrow \lim_{u\to\infty}|A_{mu,nu}|<N_1C_1(m+n)^{\alpha_1-1}(m)^{\beta_1-\gamma_1}(n)^{\beta_1'-\gamma_1'}$$

$$\lim_{u\to\infty}|A_{mu,nu}|<N_1C_1(m+n)^{\alpha_1-1}(m)^{\beta_1-\gamma_1}(n)^{\beta'{}_1-\gamma'{}_1}<\frac{N_1}{4^{\varepsilon}}C_1(m+n)^{\alpha_1+2\varepsilon-1}, \tag{19}$$

(on letting $\varepsilon>\max\left(\beta_1-\gamma_1,\beta_1'-\gamma_1'\right),\varepsilon>0$ and using the concept of Geometric Mean < Harmonic Mean).

Hence by (15) and (19), we find that

$$\lim_{u\to\infty}\left|\sum_{m,n=1}^{\infty}\mathcal{M}(mu+1,nu)\right|<\sum_{m,n=1}^{\infty}\lim_{u\to\infty}|A_{mu,nu}|\,(m+n)^{-\Re(s)}$$

$$<\frac{N_1}{4^{\varepsilon}}C_1\sum_{m,n=1}^{\infty}(m+n)^{\alpha_1+2\varepsilon-1-\Re(s)}<\frac{N_1}{4^{\varepsilon}}C_1\sum_{N=1}^{\infty}(N)^{\alpha_1+2\varepsilon-1-\Re(s)},$$

here, the series $\sum_{N=1}^{\infty}(N)^{\alpha_1+2\varepsilon-1-\Re(s)}$ converges when $\Re(s)>\alpha_1+2\varepsilon$.

Similarly, we get

$$\lim_{u\to\infty}\left|\sum_{m,n=1}^{\infty}\mathcal{M}(mu,nu+1)\right|<\frac{N_1}{4\varepsilon}C_1\sum_{N=1}^{\infty}(N)^{\alpha_1+2\varepsilon-1-\Re(s)},\ \text{when } \Re(s)>\alpha_1+2\varepsilon. \tag{20}$$

Since for special values, a double sequence of arbitrary complex numbers $\{A_{m,n}\}$ reduces to all the extended generalized Hurwitz-Lerch zeta functions defined in Section 1 and other known zeta functions studied in the work due to ([6] and [3]), we shall call it the unified Hurwitz-Lerch zeta function of two variables.

Again consider $A_{m,n}=\frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_m(\gamma')_n m!n!}$ in the double series (8). It yields a Hurwitz-Lerch Zeta function due to ([6] and [3]), given by

$$\phi_{\alpha,\beta,\beta';\gamma,\gamma'}(z,t,s,a)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_m\,(\beta')_n}{(\gamma)_m\,(\gamma')_n\,m!n!}\frac{z^mt^n}{(a+m+n)^s}, \tag{21}$$

where, $\alpha, \beta, \beta' \in \mathbb{C}, \gamma, \gamma', a \in \mathbb{C}\backslash\mathbb{Z}_0^-, s, z, t \in \mathbb{C}$ and $\Re(s) > 0$, when $|z| < 1, |t| < 1$ and for $z = t = 1$, $\Re(\gamma + \gamma' + s - \alpha - \beta - \beta') > 0$.

Remark 2.1. It is remarked that the conditions found by Theorem 2.1, are equivalent to the constraints given in (21) in the researches of ([6] and [3]).

Remark 2.2. Amongst various other specializations, we consider in our present investigation, the following special cases and limiting cases of extended Hurwitz-Lerch Zeta Function $\phi_{\alpha,\beta,\beta';\gamma,\gamma'}(z, t, s, a)$ given by (21), as

Case 2.1. *Consider the following limiting case of* (21) *to get the generalized Hurwitz-Lerch Zeta function of Pathan and Daman* [20]

$$\lim_{\alpha,\gamma,\gamma'\to\infty} \left\{\phi_{\alpha,\beta,\beta';\gamma,\gamma'}\left(\frac{\gamma z}{\alpha}, \frac{\gamma' t}{\alpha}, s, a\right)\right.$$
$$= \lim_{\alpha,\gamma,\gamma'\to\infty} \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_m(\gamma')_n m!n!} \frac{\gamma^m z^m \gamma'^n t^n}{\alpha^{m+n}(a+m+n)^s}$$

$$= \sum_{m,n=0}^{\infty} \frac{(\beta)_m(\beta')_n}{m!n!} \frac{z^m t^n}{(a+m+n)^s} = \phi^*_{\beta,\beta'}(z, t, s, a), \tag{22}$$

where, $((a+s) \in \mathbb{C}\backslash\mathbb{Z}_0^-, \beta, \beta', s \in \mathbb{C}$, *when* $|z| < 1, |t| < 1$ *and for* $z = t = 1$, $\Re(s - \beta - \beta') > 0)$.

Case 2.2. *The following is the limiting case of extended generalized Hurwitz Zeta function of two variables*

$$\lim_{\alpha\to\infty} \left\{\phi_{\alpha,\beta,\beta';\gamma,\gamma'}\left(\frac{z}{\alpha}, \frac{t}{\alpha}, s, a\right)\right.$$
$$= \lim_{\alpha\to\infty} \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_m(\gamma')_n m!n!} \frac{z^m t^n}{\alpha^m \alpha^n(a+m+n)^s}$$

$$= \sum_{m,n=0}^{\infty} \frac{(\beta)_m(\beta')_n}{(\gamma)_m(\gamma')_n m!n!} \frac{z^m t^n}{(a+m+n)^s} = \phi^*_{\beta,\beta';\gamma,\gamma'}(z, t, s, a) \text{ (let)}. \tag{23}$$

Case 2.3. *The following limiting case is found as*

$$\lim_{\beta'\to\infty}\left\{\phi_{\alpha,\beta,\beta';\gamma,\gamma'}\left(z,\frac{t}{\beta'},s,a\right)\right.$$
$$=\lim_{\beta'\to\infty}\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_m\,(\beta')_n}{(\gamma)_m\,(\gamma')_n\,m!n!}\frac{z^m t^n}{\beta'^n(a+m+n)^s}$$
$$=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_m}{(\gamma)_m\,(\gamma')_n\,m!n!}\frac{z^m t^n}{(a+m+n)^s}=\phi^{*}_{\alpha,\beta;\gamma,\gamma'}(z,t,s,a), \tag{24}$$

where, $\alpha,\beta\in\mathbb{C},\gamma,\gamma',a\in\mathbb{C}\backslash\mathbb{Z}_0^-,s,z,t\in\mathbb{C}$ *and* $\Re(s)>0,$ *when* $|z|<1,|t|<1$ *and for* $z=t=1,\ \Re(\gamma+\gamma'+s-\alpha-\beta)>0.$

Case 2.4. *The following limiting case of extended generalized Hurwitz Zeta Function of two variables is obtained as*

$$\lim_{\beta,\beta'\to\infty}\left\{\phi_{\alpha,\beta,\beta';\gamma,\gamma'}\left(\frac{z}{\beta},\frac{t}{\beta'},s,a\right)\right.$$
$$=\lim_{\beta,\beta'\to\infty}\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_m\,(\beta')_n}{(\gamma)_m\,(\gamma')_n\,m!n!}\frac{z^m t^n}{\beta^m\beta'^n(a+m+n)^s}$$
$$=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n}}{(\gamma)_m\,(\gamma')_n\,m!n!}\frac{z^m t^n}{(a+m+n)^s}=\phi^{*}{}_{\alpha;\gamma,\gamma'}(z,t,s,a). \tag{25}$$

Here, $\alpha\in\mathbb{C},\gamma,\gamma',a\in\mathbb{C}\backslash\mathbb{Z}_0^-,s,z,t\in\mathbb{C}$ *and* $\Re(s)>0,$ *when* $|z|<1,|t|<1$ *and for* $z=t=1,\ \Re(\gamma+\gamma'+s-\alpha)>0.$

Theorem 2.2. *Let* $\{B_{m,n}\}$ *be a sequence of arbitrary complex numbers, suitably convergent and follows Horn Theorem. Then the alternating Hurwitz zeta function (or, equivalently, Hurwitz-Euler eta function)* $\eta(z,t,s,a)$ *function of two variables exits as*

$$\eta(z,t,s,a)=\sum_{m,n=0}^{\infty}B_{m,n}\frac{(-1)^{m+n}z^m t^n}{(a+m+n)^s},\Re(s)>0,a>0. \tag{26}$$

Proof. Following the methods used to prove Theorem 2.1, we can evaluate existence conditions of (26) and prove the Theorem 2.2. □

3 Integral Representations of $\phi_{\alpha,\beta,\beta';\gamma,\gamma'}(z,t,s,a)$

In this section, under the conditions $\alpha,\beta,\beta',x,y\in\mathbb{C},\gamma,\gamma',a\in\mathbb{C}\backslash\mathbb{Z}_0^-$, we recall Appell's hypergeometric function of two variables $F_2(.)$ ([27, p. 53], [9, p. 23]) defined by

$$F_2\left[\alpha,\beta,\beta';\gamma,\gamma';x,y\right]=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_m\,(\beta')_n}{(\gamma)_m\,(\gamma')_n}\frac{x^m}{m!}\frac{y^n}{n!},\quad (|x|+|y|)<1. \tag{27}$$

Also, the following are the confluent forms of Appell hypergeometric function F_2 (see [9], [21], [25] and [27])

$$\psi_1\left[\alpha,\beta;\gamma,\gamma';x,y\right]=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_m}{(\gamma)_m\,(\gamma')_n}\frac{x^m}{m!}\frac{y^n}{n!} \tag{28}$$

$$\psi_2\left[\alpha;\gamma,\gamma';x,y\right]=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n}}{(\gamma)_m\,(\gamma')_n}\frac{x^m}{m!}\frac{y^n}{n!}. \tag{29}$$

Theorem 3.1. *If $(|z|+|t|)<e^x\ \forall x\in(0,\infty)$, and $\alpha,\beta,\beta',z,t,s\in\mathbb{C}$, $\gamma,\gamma'\in\mathbb{C}\backslash\mathbb{Z}_0^-$, $a>0$, $\Re(s)>0$, then, following integral representation of $\phi_{\alpha,\beta,\beta';\gamma,\gamma'}(z,t,s,a)$ in (21) holds true*

$$\phi_{\alpha,\beta,\beta';\gamma,\gamma'}(z,t,s,a)=\frac{1}{\Gamma(s)}\int_0^\infty x^{s-1}e^{-ax}F_2\left(\alpha,\beta,\beta';\gamma,\gamma';ze^{-x},te^{-x}\right)dx. \tag{30}$$

Proof. In the integrand of (30), under the conditions $(|z|+|t|)<e^x\ \forall\ x\in(0,\infty)$, $\alpha,\beta,\beta',z,t,s\in\mathbb{C},\gamma,\gamma'\in\mathbb{C}\backslash\mathbb{Z}_0^-,a>0,\Re(s)>0$, expand the Appell function $F_2(.)$ in the series form, then change the order of integration and summation and then applying a formula

$$\frac{1}{(a+m+n)^s}=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-(a+m+n)t}dt,\ \text{(see in [10] and [20])}$$

and finally making an appeal to the formula (21), we obtain left hand side of (30). □

Remark 3.1. Setting $\gamma = \beta$ and $\gamma' = \beta'$ in (30) and using [27, p. 54(9)], particularly, we get the integral formula of Hurwitz-Lerch zeta function as in [23, Eqn. (1.4)]

$$\phi_{\alpha,\beta,\beta';\beta,\beta'}(z,t,s,a) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}e^{-ax}}{(1-(z+t)e^{-x})^\alpha} dx = \phi(z+t,s,a). \tag{31}$$

Remark 3.2. On applying suitably known linear transformations for the Appell series $F_2(.)$, available in ([9], [10], [16], [21], [25] and [27]) it is not difficult to deduce several further results equivalent to those which are presented in this paper.

Similarly, employing Theorem 3.1, and Remark 2.2 and then applying the formulae (28) and (29), we get the integral representations asserted by the corollaries stated below:

Corollary 3.1. *If $x \in (0,\infty)$, $\Re(s) > 0$, $\Re(a) > 0$, then following integral representations for $\phi^*_{\alpha,\beta;\gamma,\gamma'}(z,t,s,a)$ and ${\phi^*}_{\alpha;\gamma,\gamma'}(z,t,s,a)$ hold true*

$$\phi^*_{\alpha,\beta;\gamma,\gamma'}(z,t,s,a) = \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1}e^{-ax}\psi_1\left(\alpha,\beta;\gamma,\gamma';ze^{-x},te^{-x}\right) dx, ze^{-x} < 1, te^{-x} < \infty. \tag{32}$$

$$\phi^*_{\alpha;\gamma,\gamma'}(z,t,s,a) = \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1}e^{-a}\psi_2\left(\alpha;\gamma,\gamma';ze^{-x},te^{-x}\right) dx, ze^{-x} < \infty, te^{-x} < \infty. \tag{33}$$

Corollary 3.2. *In the formula* (30), *if $\gamma = \alpha$, by the transformation formula [25, p. 305]*

$$F_2\left(\sigma,\alpha_1,\alpha_2;\sigma,\beta_2;x,y\right) = (1-x)^{-\alpha_1} F_1\left(\alpha_2,\alpha_1,\sigma-\alpha_1;\beta_2;\tfrac{y}{1-x},y\right),$$

we get

$$\phi_{\alpha,\beta,\beta';\alpha,\gamma'}(z,t,s,a) = \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1}e^{-ax}(1-ze^{-x})^{-\beta} \times F_1\left(\beta',\beta,\alpha-\beta;\gamma';\frac{t}{e^x-z},te^{-x}\right) dx. \tag{34}$$

4 Relationship between $\phi_{\alpha,\beta,\beta';\gamma,\gamma'}(z,t,s,a)$ and Generalized Hypergeometric Functions

In this section, we establish certain relationship between extension of generalized Hurwitz-Lerch Zeta function $\phi_{\alpha,\beta,\beta';\gamma,\gamma'}(z,t,s,a)$ and the generalized hypergeometric functions ${}_pF_q$ for $p=2$ and $q=1$.

Theorem 4.1. *If* $\Re(\alpha)>0, \Re(\beta)>0, \Re(\beta')>0, \Re(\gamma-\beta)>0, \Re(\beta-\alpha)>0,$ $\Re(\beta'-\gamma')>0,$ $\gamma,\gamma',a\in\mathbb{C}\backslash\mathbb{Z}_0^-$, *then for* $|z|+|t|<|e^x|$, *following Barnes type integral representation of* $\phi_{\alpha,\beta,\beta';\gamma,\gamma'}(z,t,s,a)$ *holds true*

$$\phi_{\alpha,\beta,\beta';\gamma,\gamma'}(z,t,s,a)=\frac{1}{2\pi i}\frac{\Gamma(\gamma')}{\Gamma(\beta')}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(\alpha+u)\Gamma(-u)}{\Gamma(\alpha)}\phi_{k+u,\beta;\gamma}(z,a+u,s)(-t)^u du, \tag{35}$$

where,

$$\phi_{\alpha,\beta;\gamma}(z,a,s)=\sum_{r=0}^{\infty}\frac{(\alpha)_r(\beta)_r}{(\gamma)_r r!}\frac{(z)^r}{(a+r)^s},$$

$$\forall\alpha,\beta,s\in\mathbb{C},\Re(\alpha)>0,\Re(s)>0,\gamma,a\in\mathbb{C}\backslash\mathbb{Z}_0^-.$$

Proof. For $|x|+|y|<1$, we recall an Euler type integral of $F_2(.)$ due to Slater [21], given by

$$F_2(\alpha,\beta,\beta';\gamma,\gamma';x,y)=\frac{\Gamma(\gamma)\Gamma(\gamma')}{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma-\beta)\Gamma(\gamma'-\beta')}$$
$$\times\int_0^1\int_0^1\mu^{\beta-1}\tau^{\beta'-1}(1-\mu)^{\gamma-\beta-1}(1-\tau)^{\gamma'-\beta'-1}(1-x\mu-y\tau)^{-\alpha}d\mu d\tau, \tag{36}$$

provided that $\Re(\alpha)>0, \Re(\beta)>0, \Re(\beta')>0, \Re(\gamma-\beta)>0, \Re(\beta-\alpha)>0, \Re(\beta'-\gamma')>0, \gamma,\gamma'\in\mathbb{C}\backslash\mathbb{Z}_0^-$, and use it when $|ze^{-x}|+|te^{-x}|<1$, in the formula (30) to get

$$\phi_{\alpha,\beta,\beta';\gamma,\gamma'}(z,t,s,a) = \frac{\Gamma(\gamma)\Gamma(\gamma)}{\Gamma(s)\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma-\beta)} \times \int_0^\infty \int_0^1 \int_0^1 x^{s-1}e^{-ax}\mu^{\beta-1}\tau^{\beta'-1} \times \left(1+\frac{t\tau e^{-x}}{z\mu e^{-x}-1}\right)^{-\alpha} d\mu d\tau dx. \tag{37}$$

Now let $\forall \alpha, z, t \in \mathbb{C}, \left(\frac{t\tau}{z\mu - e^x}\right) \in \mathbb{C}, \left|\arg\left(\frac{t\tau}{z\mu-e^x}\right)\right| < \pi$, and in the complex plane, if path is the vertical line from $c - i\infty$ to $c + i\infty$, where $-\Re(\alpha) < c < 0$, then for the given conditions $\Re(\alpha) > 0, |z| + |t| < e^x$, we have (see in [15])

$$\left(1+\frac{t\tau e^{-x}}{z\mu)^{-x}-1}\right)^{-\alpha} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma(\alpha+u)\Gamma(-u)}{\Gamma(\alpha)}\left(\frac{t\tau}{z\mu-e^x}\right)^u du. \tag{38}$$

Upon using (38) in the formula (37), we get

$$\phi_{\alpha,\beta,\beta';\gamma,\gamma'}(z,t,s,a) = \frac{1}{2\pi i}\frac{\Gamma(\gamma')}{\Gamma(\beta')}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma(\alpha+u)\Gamma(-u)}{\Gamma(\alpha)} \times \left\{\sum_{r=0}^\infty \frac{(\alpha+u)_r(\beta)_r}{(\gamma)_r r!}\frac{(z)^r}{(a+u+r)^s}\right\}(-t)^u du,$$

now let there exists a generalized zeta function defined by [11]

$$\phi_{\alpha,\beta;\gamma}(z,a,s) = \sum_{r=0}^\infty \frac{(\alpha)_r(\beta)_r}{(\gamma)_r r!}\frac{(z)^r}{(a+r)^s}, \forall \alpha,\beta,s \in \mathbb{C}, \Re(\alpha) > 0, \Re(s) > 0, \gamma, a \in \mathbb{C}\backslash\mathbb{Z}_0^-. \tag{39}$$

Then finally, we are led to Barnes type integral

$$\phi_{\alpha,\beta,\beta';\gamma,\gamma'}(z,t,s,a) = \frac{1}{2\pi i}\frac{\Gamma(\gamma')}{\Gamma(\beta')}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma(\alpha+u)\Gamma(-u)}{\Gamma(\alpha)}\phi_{\alpha+u,\beta;\gamma}(z,a+u,s)(-t)^u du. \tag{40}$$

□

Theorem 4.2. *If* $\Re(\alpha) > 0, \Re(\beta) > 0, \Re(\beta') > 0, \Re(\gamma-\beta) > 0, \Re(\beta-\alpha) > 0, \Re(\beta'-\gamma') > 0,\ \gamma, \gamma', a \in \mathbb{C}\backslash\mathbb{Z}_0^-$, *then for* $|z| + |t| < |e^x|$, *following integral* *representation for $\phi_{\alpha,\beta,\beta';\gamma,\gamma'}(z,t,s,a)$ holds true*

$$\phi_{\alpha,\beta,\beta';\gamma,\gamma'}(z,t,s,a) = \frac{1}{\Gamma(s)\Gamma(\alpha)} \int_0^\infty \int_0^\infty x^{s-1} e^{-ax} e^{-t} t^{\alpha-1} {}_1F_1\left(\beta;\gamma;ze^{-x}y\right) \times {}_1F_1\left(\beta';\gamma';te^{-x}y\right) dxdy. \quad (41)$$

Proof. For $\Re(\alpha) > 0$, consider the Eulerian integral (see in [10] and [20])

$$(1 - wu - zv)^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-(1-wu-zv)t} t^{\alpha-1} dt. \quad (42)$$

Under the conditions given in Theorem 4.2, use the above relation (42) in (37) and then on changing the order of integration, we get

$$\begin{aligned}\phi_{\alpha,\beta,\beta';\gamma,\gamma'}(z,t,s,a) &= \frac{\Gamma(\gamma)\Gamma(\gamma)}{\Gamma(s)\Gamma(\beta)\Gamma\left(\beta'\right)\Gamma(\gamma-\beta)\Gamma\left(\gamma'-\beta'\right)\Gamma(\alpha)} \\ &\times \int_0^\infty \int_0^\infty \int_0^1 \int_0^1 x^{s-1} e^{-ax} \mu^{\beta-1} \tau^{\beta'-1} \\ &\times (1-\mu)^{\gamma-\beta-1} (1-\tau)^{\gamma'-\beta'-1} e^{-\left(1-ze^{-x}\mu-te^{-x}\tau\right)y} y^{\alpha-1} d\mu d\tau dxdy. \quad (43)\end{aligned}$$

For $\Re(a) > 0$ and $\Re(c-a) > 0$, an integral representation of confluent hypergeometric function [21] is given as

$${}_1F_1(a;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1} (1-t)^{c-a-1} e^{zt} dt. \quad (44)$$

Using (44) in (43) and after further simplifying we get

$$\phi_{\alpha,\beta,\beta';\gamma,\gamma'}(z,t,s,a) = \frac{1}{\Gamma(s)\Gamma(\alpha)} \int_0^\infty \int_0^\infty x^{s-1} e^{-ax} y^{\alpha-1} e^{-y} \times {}_1F_1\left(\beta;\gamma;ze^{-x}y\right)_1 F_1\left(\beta';\gamma';te^{-x}y\right) dxdy. \quad (45)$$

□

5 Concluding Remarks and Observations

Theorem 4.1 is applicable in computation of analytic continuation theory and analytic number theory [1]. An integral representation of the Appell series $F_2(\alpha, \beta, \beta'; \gamma, \gamma'; x, y)$ is used in ([16], [21]) to obtain several finite-sum expansions in terms of the less cumbersome hypergeometric functions ${}_2F_1$ and ${}_3F_2$. In the case when the parameters $\alpha, \beta, \beta'\gamma, \gamma'$ are all positive integers, some of our results obtained in Section 3, may be seen as a generalization of the finite-sum expansions of ${}_2F_1$ and ${}_3F_2$. Reduction formulas of Appell hypergeometric function $F_2(.)$ involving the Gauss ${}_2F_1$ and the Clausen ${}_3F_2$, hypergeometric functions are available in [16]. The following assertions given in [16, Theorem 1, Eqn. (2.2) and (2.3), p.182], are useful from this point of view.

$$F_2\left[\alpha_1, \alpha_2, 1; \beta_1, 2; x, y\right] = \frac{1}{\alpha_1 y(1-y)^{\alpha_1}} {}_2F_1\left[\alpha_1, \alpha_2; \beta_1; \frac{x}{1-y}\right] - \frac{1}{\alpha_1 y} {}_2F_1\left[\alpha_1, \alpha_2; \beta_1; x\right], \left(\alpha_1 \neq 0, \alpha_2 \in \mathbb{C}, \beta_1 \in \mathbb{C}\backslash\mathbb{Z}_0^-, |x|+|y|<1\right), \tag{46}$$

and

$$F_2\left[1, \alpha_1, 1; \beta_1, 2; x, y\right] = \frac{\alpha_1 x}{\beta_1 y(1-y)} {}_3F_2\left[\alpha_1, 1, 1; \beta_1+1, 2; \frac{x}{1-y}\right] - \frac{\alpha_1 x}{\beta_1 y} {}_3F_2\left[\alpha_1, 1, 1; \beta_1+1, 2; x\right] - \frac{\ln(1-y)}{y}, \left(\alpha_1 \in \mathbb{C}, \beta_1 \in \mathbb{C}\backslash\mathbb{Z}_0^-, |x|+|y|<1\right). \tag{47}$$

Result 5.1. *For all $|z|+|t|<1$, and $a, \alpha_1 \in \mathbb{C}, \beta_1 \in \mathbb{C}\backslash\mathbb{Z}_0^-, \Re(a)>1, \Re(\alpha_1)>0$, there exists an integral representation*

$$\phi_{\alpha_1,\alpha_2,1;\beta_1,2}(z,t,s,a) = \frac{1}{\alpha_1 t\Gamma(s)} \int_0^\infty x^{s-1} e^{-(a-1)x} \times \left\{\left(1-te^{-x}\right)^{-\alpha_1} {}_2F_1\left[\alpha_1, \alpha_2; \beta_1; \frac{ze^{-x}}{1-te^{-x}}\right] - {}_2F_1\left[\alpha_1, \alpha_2; \beta_1; ze^{-x}\right]\right\} dx. \tag{48}$$

Result 5.2. *For all $|z|+|t|<1$, and $a, \alpha_1 \in \mathbb{C}, \beta_1 \in \mathbb{C}\backslash\mathbb{Z}_0^-, \Re(a)>1, \Re(\alpha_1)>0$,* *there exists another integral representation*

$$\phi_{1,\alpha_1,1;\beta_1,2}(z,t,s,a) = \frac{\alpha_1 z}{\beta_1 t\Gamma(s)} \int_0^\infty x^{s-1} e^{-ax}$$
$$\times \left\{ \left(1 - te^{-x}\right)^{-1} {}_3F_2 \left[\alpha_1, 1, 1; \beta_1 + 1, 2; \frac{ze^{-x}}{1 - te^{-x}}\right] - {}_3F_2 \left[\alpha_1, 1, 1; \beta_1 + 1, 2; ze^{-x}\right] \right\} dx$$
$$- \frac{1}{t\Gamma(s)} \int_0^\infty x^{s-1} e^{-(a-1)x} \ln\left(1 - te^{-x}\right) dx. \quad (49)$$

Additionally, it may be shown that an integral representation of Appell's $F_2(.)$ series can be used to generate many integral representations like (48),(49) in terms of the hypergeometric functions ${}_2F_1$ and ${}_3F_2$ which are very useful in applications. A listing of some useful reduction (and transformation) formulas is provided in [16]. As a consequence of the present work, special cases of the integrals containing the Appell hypergeometric function $F_2(.)$ can now be expressed in terms of elementary hypergeometric and algebraic functions.

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