Earthline Journal of Mathematical Sciences
E-ISSN: 2581-8147
Volume 11, Number 2, 2023, Pages 199-210
https://doi.org/10.34198/ejms.11223.199210

Maclaurin Coefficient Estimates for a New Subclasses of m-Fold Symmetric Bi-Univalent Functions

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Abstract

In this paper, we obtain upper bounds for the first two Taylor-Maclaurin $|a_{m+1}|$ and $|a_{2m+1}|$ for two new families $\Upsilon_{\Sigma_m}(\eta,\gamma;\alpha)$ and $\Upsilon^*_{\Sigma_m}(\eta,\gamma;\beta)$ of holomorphic and m-fold symmetric bi-univalent functions defined in the open unit disk U. Further, we point out several certain special cases for our results.

1. Introduction

Denote by $\mathcal{A}$ the family of functions f that are holomorphicin the open unit disk $U=\{z\in\mathbb{C} : |z|<1\}$ and normalized by the conditions $f(0)=f'(0)-1=0$ and having the form:

$$f(z)=z+\sum_{k=2}^{\infty}a_k z^k\,. \tag{1.1}$$

We also denote by S the subfamily of $\mathcal{A}$ consisting of functions satisfying (1.1) which are also univalent in U. According to the Koebe one-quarter theorem (see [8]), every function $f\in S$ has an inverse f^{-1} which satisfies

$$f^{-1}\big(f(z)\big)=z,(z\in U)$$

and

$$f\big(f^{-1}(w)\big)=w,\qquad \left(|w|<r_0(f),r_0(f)\geq\frac{1}{4}\right),$$

Received: September 8, 2022; Revised & Accepted: October 12, 2022; Published: October 18, 2022

2020 Mathematics Subject Classification: 30C45, 30C50, 30C80.

Keywords and phrases: analytic functions, univalent functions, bi-univalent functions, m-fold symmetric bi-univalent functions, coefficient estimates.

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where

$$g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots. \quad (1.2)$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U. We denote by Σ the family of bi-univalent functions in U given by (1.1). For a brief history and interesting examples in the family Σ see the pioneering work on this subject by Srivastava et al. [22], which actually revived the study of bi-univalent functions in recent years. In a considerably large number of sequels to the aforementioned work of Srivastava et al. [22], several different subfamilies of the bi-univalent function family Σ were introduced and studied analogously by the many authors (see, for example, [1,2,5,10,11,12,16,18,19,20,25,26,27]).

For each function $f \in S$, the function $h(z) = \sqrt[m]{f(z^m)}$, $(z \in U, m \in \mathbb{N})$ is univalent and maps the unit disk U into a region with m-fold symmetry. A function is said to be m-fold symmetric (see [13]) if it has the following normalized form:

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1}, \quad (z \in U, m \in \mathbb{N}). \quad (1.3)$$

We denote by S_m the family of m-fold symmetric univalent functions in U, which are normalized by the series expansion (1.3). In fact, the functions in the family S are one-fold symmetric.

In [23] Srivastava et al. defined m-fold symmetric bi-univalent functions analogues to the concept of m-fold symmetric univalent functions. They gave some important results, such as each function $f \in \Sigma$ generates an m-fold symmetric bi-univalent function for each $m \in \mathbb{N}$. Furthermore, for the normalized form of f given by (1.3), they obtained the series expansion for f^{-1} as follows:

$$g(w) = w - a_{m+1}w^{m+1} + [(m+1)a_{m+1}^2 - a_{2m+1}]w^{2m+1}$$
$$- \left[\frac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1}\right] w^{3m+1} + \cdots, \quad (1.4)$$

where $f^{-1} = g$. We denote by Σ_m the family of m-fold symmetric bi-univalent functions in U. It is easily seen that for $m = 1$, the formula (1.4) coincides with the formula (1.2) of the family Σ. Some examples of m-fold symmetric bi-univalent functions are given as follows:

$$\left(\frac{z^m}{1-z^m}\right)^{\frac{1}{m}}, \quad \left[\frac{1}{2}\log\left(\frac{1+z^m}{1-z^m}\right)\right]^{\frac{1}{m}} \text{ and } [-\log(1-z^m)]^{\frac{1}{m}}$$

with the corresponding inverse functions

$$\left(\frac{w^m}{1+w^m}\right)^{\frac{1}{m}}, \quad \left(\frac{e^{2w^m}-1}{e^{2w^m}+1}\right)^{\frac{1}{m}} \text{ and } \left(\frac{e^{w^m}-1}{e^{w^m}}\right)^{\frac{1}{m}},$$

respectively.

Recently, many authors investigated bounds for various subfamilies of m-fold bi-univalent functions (see [3,4,7,9,14,17,21,23,24,28,29]).

In order to prove our main results, we require the following lemma.

Lemma 1.1 [3]. *If* $h \in \mathcal{P}$, *then* $|c_k| \leq 2$ *for each* $k \in \mathbb{N}$, *where* $\mathcal{P}$ *is the family of all* $Re\big(h(z)\big) > 0, (z \in U)$.

Here

$$h(z) = 1 + c_1 z + c_2 z^2 + \cdots, \quad (z \in U).$$

2. Coefficient Bounds for the Function Family $\Upsilon_{\Sigma_m}(\eta, \gamma; \alpha)$

Definition 2.1. A function $f \in \Sigma_m$ given by (1.3) is said to be in the family $\Upsilon_{\Sigma_m}(\eta, \gamma; \alpha)(0 \leq \alpha \leq 1, \eta \geq 0, 0 \leq \gamma \leq 1)$ if it satisfies the following conditions:

$$\left|\arg\left((1-\eta)\left[(1-\gamma)\frac{zf'(z)}{f(z)} + \gamma\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] + \eta\frac{\gamma z^2 f''(z) + zf'(z)}{\gamma zf'(z) + (1-\gamma)f(z)}\right)\right| < \frac{\alpha\pi}{2}, \quad (z \in U) \tag{2.1}$$

and

$$\left|\arg\left((1-\eta)\left[(1-\gamma)\frac{wg'(w)}{g'(w)} + \gamma\left(1 + \frac{wg''(w)}{g'(w)}\right)\right] + \eta\frac{\gamma w^2 g''(w) + wg'(w)}{\gamma wg'(w) + (1-\gamma)g(w)}\right)\right| < \frac{\alpha\pi}{2}, \quad (w \in U), \tag{2.2}$$

where the function $g = f^{-1}$ is given by (1.4).

In particular, for one-fold symmetric bi-univalent functions, we denote the family $\Upsilon_{\Sigma_1}(\eta,\gamma;\alpha) = \Upsilon_{\Sigma}(\eta,\gamma;\alpha)$.

Remark 2.1. It should be remarked that the families $\Upsilon_{\Sigma_m}(\eta,\gamma;\alpha)$ and $\Upsilon_{\Sigma}(\eta,\gamma;\alpha)$ are a generalization of well-known families consider earlier. These families are:

1. For $\eta = \gamma = 0$, the family $\Upsilon_{\Sigma_m}(\eta,\gamma;\alpha)$ reduces to the family $S^{\alpha}_{\Sigma_m}$ which was considered by Altinkaya and Yalcin [3].
2. For $\eta = 0$, the family $\Upsilon_{\Sigma}(\eta,\gamma;\alpha)$ reduces to the family $M_{\Sigma}(\alpha,\gamma)$ which was introduced by Liu and Wang [15].
3. For $\eta = \gamma = 0$, the family $\Upsilon_{\Sigma}(\eta,\gamma;\alpha)$ reduces to the family $S^{*}_{\Sigma}(\alpha)$ which was given by Brannan and Taha [6].

Theorem 2.1. *Let* $f \in \Upsilon_{\Sigma_m}(\eta,\gamma;\alpha)(0 < \alpha \leq 1,\ \eta \geq 0,\ 0 \leq \gamma \leq 1,\ m \in \mathbb{N})$ *be given by* (1.3). *Then*

$$|a_{m+1}| \leq \frac{2\alpha}{\sqrt{\alpha[2m(2\gamma m+1) + 2m[\eta\gamma m^2(\gamma-1) + \gamma m(m+2) + 1] + (m^2\alpha - m^2)(1+\gamma m)^2}} \tag{2.3}$$

and

$$|a_{2m+1}| \leq \frac{4\alpha^2}{m^2(1+\gamma m)^2} + \frac{\alpha}{m(2\gamma m+1)}. \tag{2.4}$$

Proof. It follows from conditions (2.1) and (2.2) that

$$\left((1-\eta)\left[(1-\gamma)\frac{zf'(z)}{f(z)} + \gamma\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] + \eta\frac{\gamma z^2 f''(z) + zf'(z)}{\gamma zf'(z) + (1-\gamma)f(z)}\right) = [p(z)]^{\alpha} \tag{2.5}$$

and

$$\left((1-\eta)\left[(1-\gamma)\frac{wg'(w)}{g(w)} + \gamma\left(1 + \frac{wg''(w)}{g'(w)}\right)\right] + \eta\frac{\gamma w^2 g''(w) + wg'(w)}{\gamma wg'(w) + (1-\gamma)g(w)}\right) = [q(w)]^{\alpha}, \tag{2.6}$$

where $g = f^{-1}$ and p, q in $\mathcal{P}$ have the following series representations:

$$p(z) = 1 + p_m z^m + p_{2m} z^{2m} + p_{3m} z^{3m} + \cdots \tag{2.7}$$

and

$$q(w) = 1 + q_m w^m + q_{2m} w^{2m} + q_{3m} w^{3m} + \cdots. \tag{2.8}$$

Comparing the corresponding coefficients of (2.5) and (2.6) yields

$$m(\gamma m+1)a_{m+1}=\alpha p_m \tag{2.9}$$

$$m[2(2\gamma m+1)a_{2m+1}-(\eta\gamma m^2(\gamma-1)+\gamma m(m+2)+1)a_{m+1}^2]$$
$$=\alpha p_{2m}+\frac{\alpha(\alpha-1)}{2}p_m^2, \tag{2.10}$$

$$-m(\gamma m+1)a_{m+1}=\alpha q_m \tag{2.11}$$

and

$$2m\big[(2\gamma m+1)(2a^2m+1-a_{2m+1})-m(\eta\gamma m^2(\gamma-1)+\gamma m(m+1))\big]a_{m+1}^2$$
$$=\alpha q_{2m}+\frac{\alpha(\alpha-1)}{2}q_m^2. \tag{2.12}$$

Making use of (2.9) and (2.11), we obtain

$$p_m=-q_m \tag{2.13}$$

and

$$2m^2(\gamma m+1)^2a_{m+1}^2=\alpha^2(p_m^2+q_m^2). \tag{2.14}$$

Also, from (2.10), (2.12) and (2.14), we find that

$$[2m(2\gamma m+1)+m[\eta\gamma m^2(\gamma-1)+\gamma m(m+1)]a_{m+1}^2$$
$$=\alpha(p_{2m}+q_{2m})+\frac{\alpha(\alpha-1)}{2}(p_m^2+q_m^2)$$
$$=\alpha(p_{2m}+q_{2m})+\frac{m^2(\alpha-1)(1+\gamma m)^2}{\alpha}a_{m+1}^2.$$

Therefore, we have

$$a_{m+1}^2=\frac{\alpha^2(p_{2m}+q_{2m})}{\gamma[2m(2\gamma m+1)+m[\eta\gamma m^2(\gamma-1)+\gamma m(m+1)]-m^2(\gamma-1)(1+\gamma m)^2}. \tag{2.15}$$

Now, taking the absolute value of (2.15) and applying Lemma 1.1 for the coefficients p_{2m} and q_{2m}, we obtain

$$|a_{m+1}|\leq\frac{2\alpha}{\sqrt{\alpha[2m(2\gamma m+1)+2m[\eta\gamma m^2(\gamma-1)+\gamma m(m+2)+1]-m^2(\gamma-1)(1+\gamma m)^2}}.$$

This gives the desired estimate for $|a_{m+1}|$ as asserted in (2.3).

In order to find the bound on $|a_{2m+1}|$, by subtracting (2.12) from (2.10), we get

$$2m(2\gamma m+1)[2a_{m+1}^2-2a_{2m+1}]=\alpha(p_{2m}-q_{2m})+\frac{\alpha(\alpha-1)}{2}(p_m^2-q_m^2). \quad (2.16)$$

It follows from (2.13), (2.14) and (2.16) that

$$a_{2m+1}=\frac{2\alpha^2(p_m^2+q_m^2)}{4m^2(1+\gamma m)^2}+\frac{\alpha(p_{2m}-q_{2m})}{4m(2\gamma m+1)}. \quad (2.17)$$

Taking the absolute value of (2.17) and applying Lemma 1.1 once again for the coefficients p_m, p_{2m}, q_m and q_{2m} , we obtain

$$|a_{2m+1}|\leq\frac{2\alpha^2(m+1)}{m^2(1+\gamma m)^2}+\frac{\alpha}{m(2\gamma m+1)},$$

which completes the proof of Theorem 2.1.

For one-fold symmetric bi-univalent functions, Theorem 2.1 reduces to the following corollary:

Corollary 2.1. *Let* $f\in\Upsilon_\Sigma(\eta,\gamma;\alpha)(0<\alpha\leq 1,\eta\geq 0,0\leq\gamma\leq 1)$ *be given by* (1.1). *Then*

$$|a_2|\leq\frac{2\alpha}{\sqrt{\alpha[2(2\gamma+1)+2[\eta\gamma(\gamma-1)+3\gamma+1]+(\alpha-1)(1+\gamma)^2}}$$

and

$$|a_3|\leq\frac{4\alpha^2}{(1+\gamma)^2}+\frac{\alpha}{2\gamma+1}.$$

3. Coefficient Bounds for the Function Family $\Upsilon_{\Sigma_m}^*(\eta,\gamma;\beta)$

Definition 3.1. A function $f\in\Sigma_m$ given by (1.3) is said to be in the family $\Upsilon_{\Sigma_m}^*(\eta,\gamma;\beta)$ $(0\leq\beta<1,\eta\geq 0,0\leq\gamma\leq 1)$ if it satisfies the following conditions:

$$Re\left\{(1-\eta)\left[(1-\gamma)\frac{zf'(z)}{f(z)}+\gamma\left(1+\frac{zf''(z)}{f'(z)}\right)\right]+\eta\frac{\gamma z^2f''(z)+zf'(z)}{\gamma zf'(z)+(1-\gamma)f(z)}\right\}>\beta, \quad (3.1)$$

and

$$Re\left\{(1-\eta)\left[(1-\gamma)\frac{wg'(w)}{g(w)}+\gamma\left(1+\frac{wg''(w)}{g'(w)}\right)\right]+\eta\frac{\gamma w^2g''(w)+wg'(w)}{\gamma wg'(w)+(1-\gamma)g(w)}\right\}>\beta, \quad (3.2)$$

$$(0\leq\beta<1,\eta\geq\ 0,0\leq\gamma\leq 1,m\in\mathbb{N},z,w\in U),$$

where the function $g=f^{-1}$ is given by (1.4).

In particular, for one-fold symmetric bi-univalent functions, we denote the family $\Upsilon^*_{\Sigma_1}(\eta,\gamma;\beta) = \Upsilon^*_{\Sigma}(\eta,\gamma;\beta)$.

Remark 3.1. It should be remarked that the families $\Upsilon^*_{\Sigma_m}(\eta,\gamma;\beta)$ and $\Upsilon^*_{\Sigma}(\eta,\gamma;\beta)$ are a generalization of well-known families consider earlier. These families are:

1. For $\eta = \gamma = 0$, the family $\Upsilon^*_{\Sigma_m}(\eta,\gamma;\beta)$ reduce to the family $S^{\beta}_{\Sigma_m}$ which was considered by Altinkaya and Yalcin [3].
2. For $\eta = 0$, the family $\Upsilon^*_{\Sigma}(\eta,\gamma;\beta)$ reduce to the family $B_{\Sigma}(\beta,\gamma)$ which was introduced by Liu and Wang [15].
3. For $\eta = \gamma = 0$ and $\gamma = 1$, the family $\Upsilon^*_{\Sigma}(\eta,\gamma;\beta)$ reduce to the family $S^*_{\Sigma}(\beta)$ which was given by Brannan and Taha [6].

Theorem 3.1. *Let* $f \in \Upsilon^*_{\Sigma_m}(\eta,\gamma;\beta)(0 \le \beta < 1, \eta \ge 0, 0 \le \gamma \le 1, m \in \mathbb{N})$, *be given by* (1.3). *Then*

$$|a_{m+1}| \le \frac{2}{m}\sqrt{\frac{1-\beta}{(1+\delta)\big(\delta+2\gamma(1+\lambda m)\big)+\gamma(\gamma-1)(1+\lambda m)^2}} \tag{3.3}$$

and

$$|a_{2m+1}| \le \frac{2(m+1)(1-\beta)^2}{m^2\big(\delta+\gamma(1+\lambda m)\big)^2} + \frac{1-\beta}{m\big(\delta+\gamma(1+2\lambda m)\big)}\,. \tag{3.4}$$

Proof. It follows from conditions (3.1) and (3.2) that there exist $p, q \in \mathcal{P}$ such that

$$\left((1-\eta)\left[(1-\gamma)\frac{zf'(z)}{f(z)}+\gamma\left(1+\frac{zf''(z)}{f'(z)}\right)\right]+\eta\frac{\gamma z^2 f''(z)+zf'(z)}{\gamma z f'(z)+(1-\gamma)f(z)}\right)$$
$$= \beta + (1-\beta)p(z) \tag{3.5}$$

and

$$\left((1-\eta)\left[(1-\gamma)\frac{wg'(w)}{g(w)}+\gamma\left(1+\frac{wg''(w)}{g'(w)}\right)\right]+\eta\frac{\gamma w^2 g''(w)+wg'(w)}{\gamma w g'(w)+(1-\gamma)g(w)}\right)$$
$$= \beta + (1-\beta)q(w), \tag{3.6}$$

where $p(z)$ and $q(w)$ have the forms (2.7) and (2.8), respectively. Equating coefficients (3.5) and (3.6) yields

$$m(\gamma m+1)a_{m+1}=(1-\beta)p_m \tag{3.7}$$

$$m[2(2\gamma m+1)a_{2m+1}-(\eta\gamma m^2(\gamma-1)+\gamma m(m+2)+1)a_{m+1}^2]=(1-\beta)p_{2m}, \tag{3.8}$$

$$-m(\gamma m+1)a_{m+1}=(1-\beta)q_m \tag{3.9}$$

and

$$2m(2\gamma m+1)(2a_{m+1}^2-a_{2m+1})-m(\eta\gamma m^2(\gamma-1)+\gamma m(m+2)+1)a_{m+1}^2$$
$$=(1-\beta)q_{2m}. \tag{3.10}$$

From (3.7) and (3.9), we get

$$p_m=-q_m \tag{3.11}$$

and

$$2m^2(\gamma m+1)^2a_{m+1}^2=(1-\beta)^2(p_m^2+q_m^2). \tag{3.12}$$

Adding (3.8) and (3.10), we obtain

$$[4m(2\gamma m+1)+2m(\eta\gamma m^2(\gamma-1)+\gamma m(m+2)+1]a_{m+1}^2=(1-\beta)(p_{2m}+q_{2m}). \tag{3.13}$$

Therefore, we have

$$a_{m+1}^2=\frac{(1-\beta)(p_{2m}+q_{2m})}{4m(2\gamma m+1)+2m(\eta\gamma m^2(\gamma-1)+\gamma m(m+2)+1)}.$$

Applying Lemma 1.1 for the coefficients p_{2m} and q_{2m}, we obtain

$$|a_{m+1}|\leq\sqrt{\frac{1-\beta}{m(2\gamma m+1)+2m(\eta\gamma m^2(\gamma-1)+\gamma m(m+2)+1}}.$$

This gives the desired estimate for $|a_{m+1}|$ as asserted in (3.3).

In order to find the bound on $|a_{2m+1}|$, by subtracting (3.10) from (3.8), we get

$$2m(2\gamma m+1)[2a_{m+1}^2-2a_{2m+1}]=(1-\beta)(p_{2m}-q_{2m}),$$

or equivalently

$$a_{2m+1}=\frac{2}{2}a_{m+1}^2+\frac{(1-\beta)(p_{2m}-q_{2m})}{4m(2\gamma m+1)}.$$

Upon substituting the value of a_{m+1}^2 from (3.12), it follows that

$$a_{2m+1} = \frac{2(1-\beta)^2(p_m^2+q_m^2)}{4m^2(\gamma m+1)^2} + \frac{(1-\beta)(p_{2m}-q_{2m})}{4m(2\gamma m+1)}.$$

Applying Lemma 1.1 once again for the coefficients p_m, p_{2m}, q_m and q_{2m} , we obtain

$$|a_{2m+1}| \leq \frac{4(1-\beta)^2}{m^2(\gamma m+1)^2} + \frac{1-\beta}{m(2\gamma m+1)},$$

which completes the proof of Theorem 3.1.

For one-fold symmetric bi-univalent functions, Theorem 3.1 reduces to the following corollary:

Corollary 3.1. *Let* $f \in \Upsilon_{\Sigma}^{*}(\eta,\gamma;\beta)(0 \leq \beta < 1, \eta \geq 0, 0 \leq \gamma \leq 1)$, *be given by* (1.1). *Then*

$$|a_2| \leq 2\sqrt{\frac{1-\beta}{(2\gamma+1)+2(\eta\gamma(\gamma-1)+3\gamma+1}}$$

and

$$|a_3| \leq \frac{4(1-\beta)^2}{(\gamma+1)^2} + \frac{1-\beta}{2\gamma+1}.$$

4. Conclusion

This paper has introduced a new subfamilies $\Upsilon_{\Sigma_m}(\eta,\gamma;\alpha)$ and $\Upsilon_{\Sigma_m}^{*}(\eta,\gamma;\beta)$ of Σ_m and find estimates on the coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ for functions in each of these new subfamilies.

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