Earthline Journal of Mathematical Sciences
E-ISSN: 2581-8147
Volume 11, Number 2, 2023, Pages 183-198
https://doi.org/10.34198/ejms.11223.183198

Bi-univalent Function Subclasses Subordinate to Horadam Polynomials

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Abstract

The object of this article is to explore two subclasses of regular and bi-univalent functions subordinate to Horadam polynomials in the disk $\{\varsigma \in \mathbb{C} : |\varsigma| < 1\}$. We originate upper bounds for the initial Taylor-Maclaurin coefficient estimates of functions in these subclasses. Fekete-Szegö functional problem is also established. Furthermore, we present some new observations and investigate relevant connections to existing results.

1 Preliminaries

Let the set of all complex numbers be denoted by $\mathbb{C}$ and let the unit disk $\{\varsigma \in \mathbb{C} : |\varsigma| < 1\}$ be denoted by $\mathfrak{D}$. Let $\mathbb{N} := \{1, 2, 3, ...\} = \mathbb{N}_0 \backslash \{0\}$ and $\mathbb{R}$ be the sets of real numbers. We signify by $\mathcal{A}$, the set of functions g that are regular in $\mathfrak{D}$ and have the form:

$$g(\varsigma) = \varsigma + \sum_{j=2}^{\infty} d_j \varsigma^j, \qquad (\varsigma \in \mathfrak{D}), \tag{1.1}$$

with $g(0) = 0 = g'(0) - 1$ and we symbolize by $\mathcal{S}$, a sub-set of $\mathcal{A}$ comprising univalent functions in $\mathfrak{D}$. According to Koebe theorem (see[6]), every function g in $\mathcal{S}$ has an inverse given by

$$g^{-1}(\omega) = f(\omega) = \omega - d_2\omega^2 + (2d_2^2 - d_3)\omega^3 - (5d_2^3 - 5d_2 d_3 + d_4)\omega^4 + \dots. \tag{1.2}$$

Received: September 17, 2022; Accepted: October 6, 2022; Published: October 12, 2022
2020 Mathematics Subject Classification: Primary 30C45, 33C45; Secondary 11B39.
Keywords and phrases: regular function, bi-univalent function, Horodam polynomials, Fekete-Szegö functional.

satisfying $\varsigma = g^{-1}(g(\varsigma))$ and $\omega = g(g^{-1}(\omega))$, $|\omega| < r_0(g)$, $r_0(g) \geq 1/4$, $\varsigma, \omega \in \mathfrak{D}$.

We say that a function g of $\mathcal{A}$ is bi-univalent in $\mathfrak{D}$ if $\mathfrak{D} \subset g(\mathfrak{D})$ and if both g and g^{-1} are univalent in $\mathfrak{D}$. Let σ stands for the set of bi-univalent functions in $\mathfrak{D}$ given by (1.1). Some functions in the family σ are given by $-log(1-\varsigma)$, $\frac{1}{2}log\left(\frac{1+\varsigma}{1-\varsigma}\right)$ and $\frac{\varsigma}{1-\varsigma}$. However, the Koebe function $\notin \sigma$. Other functions $\in \mathcal{S}$ such as $\varsigma - \frac{\varsigma^2}{2}$ and $\frac{\varsigma}{1-\varsigma^2}$ are not members of σ.

Coefficient related investigations for elements of the family σ begun around the year 1970. Lewin [16] examined the family σ and claimed that $|d_2| < 1.51$ for elements of σ. Brannan and Clunie [3] proved that $|d_2| < \sqrt{2}$ for members of σ. Subsequently, Tan [31] found coefficient related investigations for functions $\in \sigma$. Brannan and Taha [4] investigated bi-starlike and bi-convex subfamilies of σ in $\mathfrak{D}$. Coefficient bounds for bi-starlike analytic functions of order α, $0 \leq \alpha < 1$, were found by Mishra and Soren [18]. The research trend in the last decade was to investigate coefficient related bounds for elements of certain subfamilies of σ as it can be seen in papers [5, 9, 10, 24, 32].

The trend in recent years is the study of functions belonging to the family σ which are associated with modified sigmoid functions, Horadam polynomials, Fibonacci polynomials, Legendrae polynomials, Gegenbauer polynomials and Lucas polynomials. Interesting results about coefficient estimates and Fekete-Szegö functional $|d_3 - \delta d_2^2|$ for members of some subfamilies of σ linked with any of the aforementioned polynomials have been found by many. We mention here some of them. Srivastava et al. [23] have proposed certain subfamilies of σ subordinate to Horadam polynomials, Frasin et al. [12] examined a comprehensive subfamily of σ associated with k-Fibonacci numbers, a comprehensive subfamily of σ subordinate to Horadam polynomials was investigated by Shammaky et al. in [21], Swamy and Wanas [29] have introduced a subfamily of σ subordinate to (m,n)-Lucas polynomials, Swamy and Yalçin [30], initiated two subclasses of σ linked with Gegenbauer polynomials, Wanas et al. [33] explored a comprehensive subfamily of σ making use of Gegenbauer polynomials, Horadam polynomials were used by Wanas and Lupas [34] to define Bazilevic bi-univalent function class, Frasin et al. [11] investigated coefficient bounds for a subfamily of σ defined by

Horadam polynomials and so on. Coefficient related investigations for elements of certain subclasses of σ linked with any of the aforementioned polynomials and a modified sigmoid function appeared like the ones published in [2, 26, 27].

Recently, the Horadam polynomials $\mathcal{H}_j(\varkappa)$ (or $\mathcal{H}_j(\varkappa, a, b\,; p, q)$), were investigated by Hörçum and Koçer [13] (See also [14]) and are quantified by the recurrence relation

$$\mathcal{H}_j(\varkappa) = p\varkappa\mathcal{H}_{j-1}(\varkappa) + q\mathcal{H}_{j-2}(\varkappa), \tag{1.3}$$

with initial conditions $\mathcal{H}_1(\varkappa) = a,\ \mathcal{H}_2(\varkappa) = b\varkappa$, where $\varkappa,\ p,\ q,\ a,\ b \in \mathbb{R}$, and $j \in \mathbb{N}\backslash\{1, 2\}$. It is seen from (1.3) that $\mathcal{H}_3(\varkappa) = pb\varkappa^2 + qa$. The generating function of the sequence $\mathcal{H}_j(\varkappa)$, $j \in \mathbb{N}$, is as below (see [13]):

$$\mathcal{H}(\varkappa, \varsigma) := \sum_{j=1}^{\infty} \mathcal{H}_j(\varkappa)\varsigma^{j-1} = \frac{(b - ap)\varkappa\varsigma + a}{1 - p\varkappa\varsigma - q\varsigma^2}, \tag{1.4}$$

where $\varsigma \in \mathbb{C}$ is such that $\Re(\varsigma) \neq \varkappa$, $\varkappa \in \mathbb{R}$.

The Horadam polynomial $\mathcal{H}_j(\varkappa, a, b\,; p, q)$ leads to known polynomials for particular choices of a, b, p and q, among them, we illustrate some for example: i). $U_j(\varkappa) := \mathcal{H}_j(\varkappa, 1, 2\,; 2, -1)$, the second type Chebyshev polynomials, ii). $F_j(\varkappa) := \mathcal{H}_j(\varkappa, 1, 1\,; 1, 1)$, the Fibonacci polynomials, iii). $T_j(\varkappa) := \mathcal{H}_j(\varkappa, 1, 1\,; 2, -1)$, the first type Chebyshev polynomials, iv). $L_j(\varkappa) := \mathcal{H}_j(\varkappa, 2, 1\,; 1, 1)$, the Lucas polynomials, v). $Q_j(\varkappa) := \mathcal{H}_j(\varkappa, 2, 2\,; 2, 1)$, the Pell-Lucas polynomials, and vi). $P_j(\varkappa) := \mathcal{H}_j(\varkappa, 1, 2\,; 2, 1)$, the Pell polynomials.

For $\mathfrak{g}_1$, $\mathfrak{g}_2 \in \mathcal{A}$ holomorphic in $\mathfrak{D}$, we say that $\mathfrak{g}_1$ is subordinate to $\mathfrak{g}_2$, if there is a Schwarz function $\psi(\varsigma)$ that is holomorphic in $\mathfrak{D}$ with $\psi(0) = 0$ and $|\psi(\varsigma)| < 1$ $(\varsigma \in \mathfrak{D})$, such that $\mathfrak{g}_1(\varsigma) = \mathfrak{g}_2(\psi(\varsigma))$, $\varsigma \in \mathfrak{D}$. This subordination is indicated as $\mathfrak{g}_1 \prec \mathfrak{g}_2$ or $\mathfrak{g}_1(\varsigma) \prec \mathfrak{g}_2(\varsigma)$ $(\varsigma \in \mathfrak{D})$. In particular, if $\mathfrak{g}_2 \in \mathcal{S}$, then

$$\mathfrak{g}_1(\varsigma) \prec \mathfrak{g}_2(\varsigma) \ \Leftrightarrow\ \mathfrak{g}_1(0) = \mathfrak{g}_2(0) \quad \text{and} \quad \mathfrak{g}_1(\mathfrak{D}) \subset \mathfrak{g}_2(\mathfrak{D}).$$

Motivated by the aforementioned trends on coefficient related problems and the Fekete-Szegö functional [8] on certain subclasses of σ, we present two new families $S\mathfrak{U}_\sigma^\tau(\nu,\varkappa)$ and $S\mathfrak{T}_\sigma^\tau(\gamma,\mu,\varkappa)$ of σ subordinate to Horadam polynomials $\mathcal{H}_j(\varkappa)$ as in (1.3) with the generating function (1.4). $\mathcal{H}(\varkappa,\varsigma)$ is as in (1.4) and $g^{-1}(\omega)=f(\omega)$ an inverse function as in (1.2), are used throughout this paper.

Definition 1.1. A function $g\in\sigma$ is said to be in the class $S\mathfrak{U}_\sigma^\tau(\nu,\varkappa)$, $\nu\geq 0,\ \tau\geq 1$ and $\varkappa\in\mathbb{R}$, if

$$(1-\nu)(g'(\varsigma))^\tau+\nu\frac{[(\varsigma g'(\varsigma))']^\tau}{g'(\varsigma)}\prec 1-a+\mathcal{H}(\varkappa,\varsigma),\ \varsigma\in\mathfrak{D},$$

and

$$(1-\nu)(f'(\omega))^\tau+\nu\frac{[(\omega f'(\omega))']^\tau}{f'(\omega)}\prec 1-a+\mathcal{H}(\varkappa,\omega),\ \omega\in\mathfrak{D}.$$

Remark 1.1. We note that $S\mathfrak{U}_\sigma^\tau(0,\varkappa)\equiv\mathfrak{P}_\sigma^\tau(\varkappa)$, $\tau\geq 1$ and $\varkappa\in\mathbb{R}$ is the class of functions $g\in\sigma$ satisfying

$$(g'(\varsigma))^\tau\prec 1-a+\mathcal{H}(\varkappa,\varsigma),\ \varsigma\in\mathfrak{D}\ \text{and}\ (f'(\omega))^\tau\prec 1-a+\mathcal{H}(\varkappa,\omega),\ \omega\in\mathfrak{D}.$$

Definition 1.2. A function g in σ having the power series (1.1) is said to be in the set $S\mathfrak{T}_\sigma^\tau(\gamma,\mu,\varkappa)$, $0\leq\gamma\leq 1$, $\mu\geq\gamma,\tau\geq 1$ and $\varkappa\in\mathbb{R}$, if

$$\frac{\varsigma(g'(\varsigma))^\tau+\mu\varsigma^2g''(\varsigma)}{\gamma\varsigma g'(\varsigma)+(1-\gamma)\varsigma}\prec 1-a+\mathcal{H}(\varkappa,\varsigma),\ \varsigma\in\mathfrak{D}$$

and

$$\frac{\omega(f'(\omega))^\tau+\mu\omega^2f''(\omega)}{\gamma\omega f'(\omega)+(1-\gamma)\omega}\prec 1-a+\mathcal{H}(\varkappa,\omega),\ \omega\in\mathfrak{D}.$$

The family $\mathrm{S}\mathfrak{T}_\sigma^\tau(\gamma,\mu,\varkappa)$ contains new as well as many existing subfamilies of σ for particular choices of γ and μ, as illustrated below:

1. $\mathfrak{J}_\sigma^\tau(\mu,\varkappa)\equiv S\mathfrak{T}_\sigma^\tau(0,\mu,\varkappa),\mu\geq 0$ and $\varkappa\in\mathbb{R}$, is the set of functions $g\in\sigma$ satisfying

$$(g'(\varsigma))^\tau+\mu\varsigma g''(\varsigma)\prec 1-a+\mathcal{H}(\varkappa,\varsigma),\ \varsigma\in\mathfrak{D}$$

and

$$(f'(\omega))^\tau+\mu\omega f''(\omega)\prec 1-a+\mathcal{H}(\varkappa,\omega),\omega\in\mathfrak{D}.$$

2. $\mathfrak{K}_{\sigma}^{\tau}(\mu, x) \equiv S\mathfrak{T}_{\Sigma}^{\tau}(1, \mu, \varkappa), \mu \geq 1$ and $\varkappa \in \mathbb{R}$, is the family of functions $g \in \sigma$ satisfying

$$(g'(\varsigma))^{\tau-1}\left(1+\mu\frac{\varsigma g''(\varsigma)}{(g'(\varsigma))^{\tau}}\right) \prec 1-a+\mathcal{H}(\varkappa,\varsigma),\ \varsigma \in \mathfrak{D}$$

and

$$(f'(\omega))^{\tau-1}\left(1+\mu\frac{\omega f''(\omega)}{(f(\omega))^{\tau}}\right) \prec 1-a+\mathcal{H}(\varkappa,\omega),\ \omega \in \mathfrak{D}.$$

3. $\mathfrak{L}_{\sigma}^{\tau}(\gamma, \varkappa) \equiv S\mathfrak{T}_{\sigma}^{\tau}(\gamma, 1, \varkappa), 0 \leq \gamma \leq 1$ and $\varkappa \in \mathbb{R}$, is the collection of functions $g \in \sigma$ satisfying

$$\frac{\varsigma(g'(\varsigma))^{\tau}+\varsigma^2 g''(\varsigma)}{\gamma\varsigma g'(\varsigma)+(1-\gamma)\varsigma} \prec 1-a+\mathcal{H}(\varkappa,\varsigma),\ \varsigma \in \mathfrak{D}$$

and

$$\frac{\omega(f'(\omega))^{\tau}+\omega^2 f''(\omega)}{\gamma\omega f'(\omega)+(1-\gamma)\omega} \prec 1-a+\mathcal{H}(\varkappa,\omega),\ \omega \in \mathfrak{D}.$$

Remark 1.2. We observe that i) $\mathfrak{J}_{\sigma}^{\tau}(1, \varkappa) \equiv \mathfrak{L}_{\Sigma}^{\tau}(0, \varkappa), \tau \geq 1$ and $\varkappa \in \mathbb{R}$. ii) $\mathfrak{K}_{\sigma}^{\tau}(1, \varkappa) \equiv \mathfrak{L}_{\sigma}^{\tau}(1, \varkappa,), \tau \geq 1$ and $\varkappa \in \mathbb{R}$. iii) $\mathfrak{P}_{\sigma}^{\tau}(\varkappa) \equiv \mathfrak{J}_{\sigma}^{\tau}(0, \varkappa), \varkappa \in \mathbb{R}$.

Remark 1.3. i) When $\tau = 1$, the family $S\mathfrak{U}_{\sigma}^{1}(\nu, \varkappa)$ was examined by Orhan et al.[19].
ii) When $\tau = 1$, the family $\mathrm{S}\mathfrak{T}_{\sigma}^{1}(\gamma, \mu, \varkappa)$ was investigated by Swamy and Sailaja [28].

In Section 2, we find estimations for $|d_2|$, $|d_3|$ and $|d_3 - \delta d_2^2|, \delta \in \mathbb{R}$ for functions $\in S\mathfrak{U}_{\sigma}^{\tau}(\nu, \varkappa)$. In Section 3, we derive the upper bounds for $|d_2|$, $|d_3|$ and $|d_3 - \delta d_2^2|, \delta \in \mathbb{R}$ for functions $\in S\mathfrak{T}_{\sigma}^{\tau}(\gamma, \mu, \varkappa)$. Interesting consequences and relevant connections to the known results are presented.

2 Bi-univalent Function Class $S\mathfrak{U}_{\sigma}^{\tau}(\nu, \varkappa)$

First, We find the coefficient related estimates for $g \in S\mathfrak{U}_{\sigma}^{\tau}(\nu, \varkappa)$, the class given in Definition 1.1.

Theorem 2.1. *Let* $0 \le \nu \le 1,\ \tau \ge 1$ *and* $\varkappa \in \mathbb{R}$. *If* $g \in S\mathfrak{U}_\sigma^\tau(\nu,\varkappa)$, *then*

$$|d_2| \le \frac{|b\varkappa|\sqrt{|b\varkappa|}}{\sqrt{|(2\tau(3\nu+1)(\tau-1)+3\tau-\nu(2\tau-1))(b\varkappa)^2-4(\tau(\nu+1)-\nu)^2(pb\varkappa^2+qa)|}}, \tag{2.1}$$

$$|d_3| \le \frac{(b\varkappa)^2}{4(\tau(\nu+1)-\nu)^2} + \frac{|b\varkappa|}{3(\tau(2\nu+1)-\nu)} \tag{2.2}$$

and for $\delta \in \mathbb{R}$

$$|d_3-\delta d_2^2| \le \begin{cases} \frac{|b\varkappa|}{3(\tau(2\nu+1)-\nu)} & ;\ |1-\delta| \le\ J \\ \frac{|b\varkappa|^3\,|1-\delta|}{|(2\tau(3\nu+1)(\tau-1)+3\tau-\nu(2\tau-1))(b\varkappa)^2-4(\tau(\nu+1)-\nu)^2(pb\varkappa^2+qa)|} & ;\ |1-\delta| \ge\ J, \end{cases} \tag{2.3}$$

where

$$J = \left|\frac{(2\tau(3\nu+1)(\tau-1)+3\tau-\nu(2\tau-1))b^2\varkappa^2-4(\tau(\nu+1)-\nu)^2(pb\varkappa^2+qa)}{3(\tau(2\nu-1)+\nu)b^2\varkappa^2}\right|. \tag{2.4}$$

Proof. Let $g \in S\mathfrak{U}_\sigma^\tau(\nu,\varkappa)$. Then, on account of Definition 1.1, we get

$$(1-\nu)(g'(\varsigma))^\tau + \nu\frac{[(\varsigma g'(\varsigma))']^\tau}{g'(\varsigma)} = 1-a+\mathcal{H}(\varkappa,\mathfrak{m}(\varsigma)) \tag{2.5}$$

and

$$(1-\nu)(f'(\omega))^\tau + \nu\frac{[(\omega f'(\omega))']^\tau}{f'(\omega)} = 1-a+\mathcal{H}(\varkappa,\mathfrak{n}(\omega)), \tag{2.6}$$

where

$$\mathfrak{m}(\varsigma) = m_1\varsigma + m_2\varsigma^2 + m_3\varsigma^3 + ... \tag{2.7}$$

and

$$\mathfrak{n}(\omega) = n_1\omega + n_2\omega^2 + n_3\omega^3 + ..., \tag{2.8}$$

are some analytic functions with $\mathfrak{m}(0) = 0 = \mathfrak{n}(0), |\mathfrak{m}(\varsigma)| < 1$ and $|\mathfrak{n}(\omega)| < 1$, $\varsigma, \omega \in \mathfrak{D}$. It is known that

$$|\mathfrak{m}_i| \le 1, (i \in \mathbb{N}) \text{ and } |\mathfrak{n}_i| \le 1, (i \in \mathbb{N}). \tag{2.9}$$

It follows from (2.5)-(2.8) with (1.4) that

$$(1-\nu)(g'(\varsigma))^\tau + \nu\frac{[(\varsigma g'(\varsigma))']^\tau}{g'(\varsigma)} = 1-a+\mathcal{H}_1(\varkappa)+\mathcal{H}_2(\varkappa)\mathfrak{m}(\varsigma)+\mathcal{H}_3(\varkappa)\mathfrak{m}^2(\varsigma)+... \tag{2.10}$$

and

$$(1-\nu)(f'(\omega))^{\tau}+\nu\frac{[(\omega f'(\omega))']^{\tau}}{f'(\omega)}=1-a+\mathcal{H}_1(\varkappa)+\mathcal{H}_2(\varkappa)\mathfrak{n}(\omega)+\mathcal{H}_3(\varkappa)\mathfrak{n}^2(\omega)+\ldots. \tag{2.11}$$

In view of (1.3), we find from (2.10) and (2.11) that

$$(1-\nu)(g'(\varsigma))^{\tau}+\nu\frac{[(\varsigma g'(\varsigma))']^{\tau}}{g'(\varsigma)}=1+\mathcal{H}_2(\varkappa)\mathfrak{m}_1\varsigma+[\mathcal{H}_2(\varkappa)\mathfrak{m}_2+\mathcal{H}_3(\varkappa)\mathfrak{m}_1^2]\varsigma^2+\ldots \tag{2.12}$$

and

$$(1-\nu)(f'(\omega))^{\tau}+\nu\frac{[(\omega f'(\omega))']^{\tau}}{f'(\omega)}=1+\mathcal{H}_2(\varkappa)\mathfrak{n}_1\omega+[\mathcal{H}_2(\varkappa)\mathfrak{n}_2+\mathcal{H}_3(\varkappa)\mathfrak{n}_1^2]\omega^2+\ldots. \tag{2.13}$$

Comparing (2.12) and (2.13), we have

$$2(\tau(\nu+1)-\nu)d_2=\mathcal{H}_2(\varkappa)\mathfrak{m}_1, \tag{2.14}$$

$$3(\tau(2\nu+1)-\nu)d_3-2(2\nu(2\tau-1)-\tau(3\nu+1)(\tau-1))d_2^2=\mathcal{H}_2(\varkappa)\mathfrak{m}_2+\mathcal{H}_3(\varkappa)\mathfrak{m}_1^2, \tag{2.15}$$

$$-2(\tau(\nu+1)-\nu)\,d_2=\mathcal{H}_2(\varkappa)\mathfrak{n}_1 \tag{2.16}$$

and

$$3(\tau(2\nu+1)-\nu)(2d_2^2-d_3)-2(2\nu(2\tau-1)-\tau(3\nu+1)(\tau-1))d_2^2=\mathcal{H}_2(\varkappa)\mathfrak{n}_2+\mathcal{H}_3(\varkappa)\mathfrak{n}_1^2. \tag{2.17}$$

From (2.14) and (2.16), we easily obtain

$$\mathfrak{m}_1=-\mathfrak{n}_1 \tag{2.18}$$

and also

$$8(\tau(\nu+1)-\nu)^2d_2^2=(\mathfrak{m}_1^2+\mathfrak{n}_1^2)(\mathcal{H}_2(\varkappa))^2. \tag{2.19}$$

To obtain the bound on $|d_2|$, we add (2.15) and (2.17):

$$[(2\tau(3\nu+1)(\tau-1)+3\tau-\nu(2\tau-1))]2d_2^2=\mathcal{H}_2(\varkappa)(\mathfrak{m}_2+\mathfrak{n}_2)+\mathcal{H}_3(\varkappa)(\mathfrak{m}_1^2+\mathfrak{n}_1^2). \tag{2.20}$$

Substituting the value of $\mathfrak{m}_1^2+\mathfrak{n}_1^2$ from (2.19) in (2.20), we get

$$d_2^2=\frac{\mathcal{H}_2^3(\varkappa)(\mathfrak{m}_2+\mathfrak{n}_2)}{2\left[(2\tau(3\nu+1)(\tau-1)+3\tau-\nu(2\tau-1))\mathcal{H}_2^2(\varkappa)-4(\tau(\nu+1)-\nu)^2\mathcal{H}_3(\varkappa)\right]}. \tag{2.21}$$

Applying (2.9) for the coefficients $\mathfrak{m}_2$ and $\mathfrak{n}_2$, we obtain (2.1).

To obtain the bound on $|d_3|$, we subtract (2.17) from (2.15):

$$d_3 = d_2^2 + \frac{\mathcal{H}_2(\varkappa)(\mathfrak{m}_2 - \mathfrak{n}_2)}{6(\tau(2\nu+1)-\nu)}. \tag{2.22}$$

Then in view of (2.18) and (2.19), (2.22) becomes

$$d_3 = \frac{\mathcal{H}_2^2(\varkappa)(\mathfrak{m}_1^2 + \mathfrak{n}_1^2)}{8(\tau(\nu+1)-\nu)^2} + \frac{\mathcal{H}_2(\varkappa)(\mathfrak{m}_2 - \mathfrak{n}_2)}{6(\tau(2\nu+1)-\nu)},$$

and applying (2.9) for the coefficients $\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{n}_1$ and $\mathfrak{n}_2$ we get (2.2).

From (2.21) and (2.22), for $\delta \in \mathbb{R}$, we get in view of (1.3) that

$$|d_3 - \delta d_2^2| = \frac{|\mathcal{H}_2(\varkappa)|}{2} \left| \left(\mathfrak{B}(\delta,\varkappa) + \frac{1}{3(\tau(2\nu+1)-\nu)} \right) \mathfrak{m}_2 + \left(\mathfrak{B}(\delta,\varkappa) - \frac{1}{3(\tau(2\nu+1)-\nu)} \right) \mathfrak{n}_2 \right|,$$

where

$$\mathfrak{B}(\delta,\varkappa) = \frac{(1-\delta)\mathcal{H}_2^2(\varkappa)}{\left[(2\tau(3\nu+1)(\tau-1) + 3\tau - \nu(2\tau-1))\mathcal{H}_2^2(\varkappa) - 4(\tau(\nu+1)-\nu)^2\mathcal{H}_3(\varkappa)\right]}.$$

Clearly

$$|d_3 - \delta d_2^2| \le \begin{cases} \frac{|\mathcal{H}_2(\varkappa)|}{3(\tau(2\nu+1)-\nu)} & ;\ 0 \le |\mathfrak{B}(\delta,\varkappa)| \le \frac{1}{3(\tau(2\nu+1)-\nu)} \\ |\mathcal{H}_2(\varkappa)||\mathfrak{B}(\delta,\varkappa)| & ;\ |\mathfrak{B}(\delta,\varkappa)| \ge \frac{1}{3(\tau(2\nu+1)-\nu)}, \end{cases}$$

from which we conclude (2.3) with J as in (2.4). □

Remark 2.1. i) Taking $\tau = 1$ in Theorem 2.1, we get a result of Orhan et al. [19, Theorem 1]. Further by letting $\nu = 1$, we obtain Corollary 2.3 of Magesh et al. [17], which is also stated as Corollary 1 in Orhan et al. [19]. Also, we get Corollary 2 of Orhan et al. [19], if we let $\nu = 0$ instead of $\nu = 1$.
ii) Taking $\nu = 1$ in Theorem 2.1, we get Corollary 2.4 of Swamy and Sailaja [28].

Corollary 2.1. *Let $\tau \ge 1$ and $\varkappa \in \mathbb{R}$. If $g \in \mathfrak{P}_\sigma^\tau(\varkappa)$, then*

$$|d_2| \le \frac{|b\varkappa|\sqrt{|b\varkappa|}}{\sqrt{|(\tau(2\tau+1))(b\varkappa)^2 - 4\tau^2(pb\varkappa^2 + qa)|}},$$

$$|d_3| \leq \frac{(b\varkappa)^2}{4\tau^2} + \frac{|b\varkappa|}{3\tau}$$

and for $\delta \in \mathbb{R}$

$$|d_3 - \delta d_2^2| \leq \begin{cases} \frac{|b\varkappa|}{3\tau} & ;\ |1-\delta| \leq J_1 \\ \frac{|b\varkappa|^3\,|1-\delta|}{|(\tau(2\tau+1))(b\varkappa)^2 - 4\tau^2(pb\varkappa^2+qa)|} & ;\ |1-\delta| \geq J_1, \end{cases}$$

where

$$J_1 = \left| \frac{(\tau(2\tau+1))b^2\varkappa^2 - 4\tau^2(pb\varkappa^2+qa)}{3\tau b^2\varkappa^2} \right|.$$

Remark 2.2. Letting $\tau = 1$ in Corollary 2.1, we get Theorem 2.2 of Alamoush [1].

3 Bi-univalent Function Class $S\mathfrak{T}_\sigma^\tau(\nu, \varkappa)$

Next, we state and prove the coefficient related estimates for $g \in S\mathfrak{T}_\sigma^\tau(\nu, \varkappa)$, the class defined in Definition 1.2.

Theorem 3.1. *Let* $0 \leq \gamma \leq 1$, $\mu \geq \gamma$, $\tau \geq 1$ *and* $\varkappa \in \mathbb{R}$. *If* $g \in S\mathfrak{T}_\sigma^\tau(\gamma, \mu, \varkappa)$, *then*

$$|d_2| \leq \frac{|b\varkappa|\sqrt{|b\varkappa|}}{\sqrt{|((3-4\gamma)(\mu+\tau-\gamma)+2\tau(\tau-1)+3\mu))(b\varkappa)^2 - 4(\mu+\tau-\gamma)^2(pb\varkappa^2+qa)|}}, \tag{3.1}$$

$$|d_3| \leq \frac{(b\varkappa)^2}{2(\mu+\tau-\gamma)^2} + \frac{|b\varkappa|}{3(2\mu+\tau-\gamma)} \tag{3.2}$$

and for $\delta \in \mathbb{R}$

$$|d_3 - \delta d_2^2| \leq \begin{cases} \frac{|b\varkappa|}{3(2\mu+\tau-\gamma)} & ;\ |1-\delta| \leq Q \\ \frac{|b\varkappa|^3\,|1-\delta|}{|((3-4\gamma)(\mu+\tau-\gamma)+2\tau(\tau-1)+3\mu))(b\varkappa)^2 - 4(\mu+\tau-\gamma)^2(pb\varkappa^2+qa)|} & ;\ |1-\delta| \geq Q, \end{cases} \tag{3.3}$$

where

$$Q = \left| \frac{((3-4\gamma)(\mu+\tau-\gamma)+2\tau(\tau-1)+3\mu))b^2\varkappa^2 - 4(\mu+\tau-\gamma)^2\left(pb\varkappa^2+qa\right)}{3(2\mu+\tau-\gamma)b^2\varkappa^2} \right|. \tag{3.4}$$

Proof. Let $g \in S\mathfrak{T}_\sigma^\tau(\gamma,\mu,\varkappa)$. Then, on account of Definition 1.2, we get

$$\frac{\varsigma(g'(\varsigma))^\tau + \mu\varsigma^2 g''(\varsigma)}{\gamma\varsigma g'(\varsigma) + (1-\gamma)\varsigma} = 1 - a + \mathcal{H}(\varkappa, \mathfrak{m}(\varsigma)) \tag{3.5}$$

and

$$\frac{\omega(f'(\omega))^\tau + \mu\omega^2 f''(\omega)}{\gamma\omega f'(\omega) + (1-\gamma)\omega} = 1 - a + \mathcal{H}(\varkappa, \mathfrak{n}(\omega)), \tag{3.6}$$

where $\mathfrak{m}(\varsigma)$ and $\mathfrak{n}(\omega)$ are given by (2.7) and (2.8), respectively. It is known that $|\mathfrak{m}_i| \le 1, (i \in \mathbb{N})$ and $|\mathfrak{n}_i| \le 1\,(i \in \mathbb{N})$ if $|\mathfrak{m}(\varsigma)| = |\mathfrak{m}_1\varsigma + \mathfrak{m}_2\varsigma^2 + \mathfrak{m}_3\varsigma^3 + ...| < 1, \varsigma \in \mathfrak{D}$ and $|\mathfrak{n}(\omega)| = |\mathfrak{n}_1\omega + \mathfrak{n}_2\omega^2 + \mathfrak{n}_3\omega^3 + ...| < 1, \omega \in \mathfrak{D}$. Following (2.10)-(2.13) in the proof of Theorem 2.1, one gets in view of (3.5) and (3.6)

$$2(\mu+\tau-\gamma)d_2 = \mathcal{H}_2(\varkappa)\mathfrak{m}_1, \tag{3.7}$$

$$3(2\mu+\tau-\gamma)d_3 - 2(2\gamma(\mu+\tau-\gamma) - \tau(\tau-1))d_2^2 = \mathcal{H}_2(\varkappa)\mathfrak{m}_2 + \mathcal{H}_3(\varkappa)\mathfrak{m}_1^2, \tag{3.8}$$

$$-\,2(\mu+\tau-\gamma)\,d_2 = \mathcal{H}_2(\varkappa)\mathfrak{n}_1 \tag{3.9}$$

and

$$3(2\mu+\tau-\gamma)(2d_2^2 - d_3) - 2(2\gamma(\mu+\tau-\gamma) - \tau(\tau-1))d_2^2 = \mathcal{H}_2(\varkappa)\mathfrak{n}_2 + \mathcal{H}_3(\varkappa)\mathfrak{n}_1^2. \tag{3.10}$$

The results (3.1)-(3.3) with Q as in (3.4) of this theorem will now follow from (3.7)-(3.10) by applying the technique as in Theorem 2.1 with respect to (2.14)-(2.17). □

Remark 3.1. Theorem 2.1 of Swamy and Sailaja [28] is a case of Theorem 3.1, when $\tau = 1$.

When $\gamma = 0$, Theorem 3.1 would yield:

Corollary 3.1. *Let* $\mu \ge 0, \tau \ge 1, \delta \in \mathbb{R}$ *and* $\varkappa \in \mathbb{R}$. *If* $g \in \mathfrak{J}_\sigma^\tau(\mu,\varkappa) \equiv S\mathfrak{T}_\sigma^\tau(0,\mu,\varkappa)$, *then*

$$|d_2| \le \frac{|b\varkappa|\sqrt{|b\varkappa|}}{\sqrt{|(\tau(2\tau+1)+6\mu)(b\varkappa)^2 - 4(\mu+\tau)^2(pb\varkappa^2 + qa)|}},$$

$$|d_3| \le \frac{b^2\varkappa^2}{4(\mu+\tau)^2} + \frac{|b\varkappa|}{3(2\mu+\tau)}$$

and

$$|d_3-\delta d_2^2| \leq \begin{cases} \frac{|b\varkappa|}{3(2\mu+\tau)} & ;\ |1-\delta| \leq Q_1 \\ \frac{|b\varkappa|^3\,|1-\delta|}{|(\tau(2\tau+1)+6\mu)(b\varkappa)^2-4(\mu+\tau)^2(pb\varkappa^2+qa)|} & ;\ |1-\delta| \geq Q_1, \end{cases}$$

where $Q_1 = \frac{1}{3(2\mu+\tau)}\left|\tau(2\tau+1)+6\mu-4(\mu+\tau)^2\left(\frac{pb\varkappa^2+qa}{b^2\varkappa^2}\right)\right|$.

Remark 3.2. Corollary 2.1 of Swamy and Sailaja [28] is a case of Corollary 3.1, when $\tau = 1$. Also, Theorem 2.2 of Alamoush [1] is another case of Corollary 3.1 and is got when $\mu = 0$ and $\tau = 1$.

Allowing $\gamma = 1$ in Theorem 3.1, we obtain

Corollary 3.2. *Let* $\mu \geq 1, \tau \geq 1, \delta \in \mathbb{R}$ *and* $\varkappa \in \mathbb{R}$. *If* $g \in \mathfrak{J}_\sigma^\tau(\mu,\varkappa) \equiv S\mathfrak{T}_\sigma^\tau(1,\mu,\varkappa)$, *then*

$$|d_2| \leq \frac{|b\varkappa|\sqrt{|b\varkappa|}}{\sqrt{|(\tau^2-3\tau+1+2\mu)(b\varkappa)^2-4(\mu+\tau-1)^2(pb\varkappa^2+qa)|}},$$

$$|d_3| \leq \frac{b^2\varkappa^2}{4(\mu+\tau-1)^2}+\frac{|b\varkappa|}{3(2\mu+\tau-1)}$$

and

$$|d_3-\delta d_2^2| \leq \begin{cases} \frac{|b\varkappa|}{3(2\mu+\tau-1)} & ;\ |1-\delta| \leq Q_2 \\ \frac{|b\varkappa|^3\,|1-\delta|}{|(\tau^2-3\tau+1+2\mu)(b\varkappa)^2-4(\mu+\tau-1)^2(pb\varkappa^2+qa)|} & ;\ |1-\delta| \geq Q_2, \end{cases}$$

where $Q_2 = \frac{1}{3(2\mu+\tau-1)}\left|(\tau^2-3\tau+1+2\mu)-4(\mu+\tau-1)^2\left(\frac{pb\varkappa^2+qa}{b^2\varkappa^2}\right)\right|$.

Remark 3.3. Allowing $\tau = 1$ in Corollary 3.2, we obtain a result of Swamy and Sailaja [28, Corollary 2.2].

Setting $\mu = 1$ in Theorem 3.1, we have

Corollary 3.3. *Let* $0 \leq \gamma \leq 1, \tau \geq 1, \delta \in \mathbb{R}$ *and* $\varkappa \in \mathbb{R}$. *If* $g \in \mathfrak{L}_\sigma^\tau(\gamma,\varkappa) \equiv S\mathfrak{T}_\sigma^\tau(\gamma,1,\varkappa)$, *then*

$$|d_2| \leq \frac{|b\varkappa|\sqrt{|b\varkappa|}}{\sqrt{|(6(1-\gamma)+(\tau-\gamma)(1-4\gamma)+2\tau^2)(b\varkappa)^2-4(1+\tau-\gamma)^2(pb\varkappa^2+qa)|}},$$

$$|d_3| \leq \frac{(b\varkappa)^2}{4(1+\tau-\gamma)^2} + \frac{|b\varkappa|}{3(2+\tau-\gamma)}$$

and

$$|d_3 - \delta d_2^2| \leq \begin{cases} \frac{|b\varkappa|}{3(2+\tau-\gamma)} & ; |1-\delta| \leq Q_3 \\ \frac{|b\varkappa|^3 \, |1-\delta|}{|(6(1-\gamma)+(\tau-\gamma)(1-4\gamma)+2\tau^2)(b\varkappa)^2 - 4(1+\tau-\gamma)^2(pb\varkappa^2+qa)|} & ; |1-\delta| \geq Q_3, \end{cases}$$

where

$$Q_3 = \frac{1}{3(2+\tau-\gamma)} \left| 6(1-\gamma) + (\tau-\gamma)(1-4\gamma) + 2\tau^2 - 4(1+\tau-\gamma)^2 \left(\frac{pb\varkappa^2+qa}{b^2\varkappa^2} \right) \right|.$$

Remark 3.4. Allowing $\gamma = \tau = 1$ in Corollary 3.3, we obtain a result of Magesh et al. [17, Corollary 2.3], which is also stated as Corollary 1 in Orhan et al. [19].

4 Conclusion

In the current paper, two subfamilies of σ associated with Horadam polynomials are defined and the upper bounds of $|d_2|$ and $|d_3|$ for functions belonging to these subfamilies are obtained. Furthermore, we have found the Fekete-Szegö functional $|d_3 - \delta d_2^2|, \delta \in \mathbb{R}$ for functions in these subfamilies. We have pointed out several consequences by fixing the paramerters in Theorem 2.1 and Theorem 3.1. Relevant connections to the existing results are also identified. Problem to estimate bound of $|d_j|, (j \in \mathbb{R} - \{1,2,3\})$ for the families that have been defined in this paper remain open. Subfamilies explored in this investigations could inspire many researchers to focus on large number of recent publications based on i) q-derivative operator [7, 25], ii) q-integral operator [15], iii) operators on fractional q-calculus[22], iv) integro-differential operator [20] and so on.

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