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Results of Semigroup of Linear Equation Generating a Wave Equation

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Abstract

In this paper, we present results of ω-order preserving partial contraction mapping generating a wave equation. We use the theory of semigroup to generate a wave equation by showing that the operator $\begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}$, which is A, is the infinitesimal generator of a C_0-semigroup of operators in some appropriately chosen Banach of functions. Furthermore we show that the operator A is closed, unique and that operator A is the infinitesimal generator of a wave equation.

1 Introduction

Consider the initial value problem for the wave equation in $\mathbb{R}^n$, that is the initial value problem

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = \Delta u & for \quad x \in \mathbb{R}^n, \ t > 0 \\ u(0,x) = u_1(x), \ \frac{\partial u}{\partial t}(0,x) = u_2(x), & for \quad x \in \mathbb{R}^n. \end{cases} \tag{1.1}$$

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The problem is equivalent to the first order system:

$$\left\{ \frac{\partial}{\partial t} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} for\ x \in \mathbb{R}^n,\ t > 0 \right. \tag{1.2}$$

and

$$\left\{ \begin{pmatrix} u_1(0,x) \\ u_2(0,x) \end{pmatrix} = \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix} \quad for\ x \in \mathbb{R}^n. \right.$$

Suppose X is a Banach space, $X_n \subseteq X$ is a finite set, $\omega - OCP_n$ is the ω-order preserving partial contraction mapping, M_m is a matrix, $L(X)$ is a bounded linear operator on X, P_n is a partial transformation semigroup, $\rho(A)$ is a resolvent set, $\sigma(A)$ is a spectrum of A and A is a generator of C_0-semigroup. This paper consists of results of ω-order preserving partial contraction mapping generating a wave equation. Akinyele *et al.* [1], generated a continuous time Markov semigroup of linear operators and also in [2], Akinyele *et al.*, obtained results of ω-order reversing partial contraction mapping generating a differential operator. Balakrishnan [3], introduced an operator calculus for infinitesimal generators of semigroup. Banach [4], established and introduced the concept of Banach spaces. Brezis and Gallouet [5], obtained nonlinear Schrodinger evolution equation. Chill and Tomilov [6], presented some resolvent approach to stability operator semigroup. Davies [7], obtained linear operators and their spectra. Engel and Nagel [8], deduced one-parameter semigroup for linear evolution equations. Omosowon *et al.* [9], established some analytic results of semigroup of linear operator with dynamic boundary conditions, and also in [10], Omosowon *et al.*, obtained dual Properties of ω-order Reversing Partial Contraction Mapping in Semigroup of Linear Operator. Omosowon *et al.* [11], established a regular weak*-continuous semigroup of linear operators, and also in [12], Omosowon *et al.*, generated a quasilinear equations of evolution on semigroup of linear operator. Pazy [13], presented asymptotic behavior of the solution of an abstract evolution and some applications and also in [14], obtained a class of semi-linear equations of evolution. Rauf and Akinyele [15], established ω-order preserving partial contraction mapping and obtained its properties, also in [16], Rauf *et al.*, presented some results of stability and spectra properties on semigroup of linear operator.

Vrabie [17], proved some results of C_0-semigroup and its applications. Yosida [18], established some results on differentiability and representation of one-parameter semigroup of linear operators.

2 Preliminaries

Definition 2.1 (C_0-semigroup) [17]
A C_0-*semigroup* is a strongly continuous one parameter semigroup of bounded linear operator on Banach space.

Definition 2.2 (ω-OCP_n) [15]
A transformation $\alpha \in P_n$ is called ω-*order preserving partial contraction mapping* if $\forall x, y \in$ Dom$\alpha : x \leq y \implies \alpha x \leq \alpha y$ and at least one of its transformation must satisfy $\alpha y = y$ such that $T(t+s) = T(t)T(s)$ whenever $t, s > 0$ and otherwise for $T(0) = I$.

Definition 2.3 (Evolution Equation) [13]
An *evolution equation* is an equation that can be interpreted as the differential law of the development (evolution) in time of a system. The class of evolution equations includes, first of all, ordinary differential equations and systems of the form

$$u' = f(t, u), u'' = f(t, u, u'),$$

etc., in the case where $u(t)$ can be regarded naturally as the solution of the Cauchy problem; these equations describe the evolution of systems with finitely many degrees of freedom.

Definition 2.4 (Mild Solution) [14]
A continuous solution u of the integral equation.

$$u(t) = T(t - t_0)u_0 + \int_{t_0}^{t} T(t - s)f(s, u(s))ds \tag{2.1}$$

will be called a *mild solution* of the initial value problem

$$\begin{cases} \frac{du(t)}{dt} + Au(t) = f(t, u(t)),\ t > t_0 \\ u(t_0) = u_0 \end{cases} \tag{2.2}$$

if the solution is a Lipschitz continuous function.

Definition 2.5 [13]
Let

$$D(A) = H^2(\mathbb{R}^n) \times H'(\mathbb{R}^n) \tag{2.3}$$

and for $U = [u_1, u_2] \in D(A)$ we have that

$$AU = A[u_1, u_2] = [u_2, \Delta u_1]. \tag{2.4}$$

Definition 2.6 (Wave Equation) [19]
The (two-way) *wave equation* is a second-order partial differential equation describing waves, including traveling and standing waves; the latter can be considered as linear superpositions of waves traveling in opposite directions.

Example 1
For every 2×2 matrix in $[M_m(\mathbb{R}^n)]$.
Suppose

$$A = \begin{pmatrix} 2 & 0 \\ \Delta & 2 \end{pmatrix}$$

and let $T(t) = e^{tA}$, then

$$e^{tA} = \begin{pmatrix} e^{2t} & I \\ e^{\Delta t} & e^{2t} \end{pmatrix}.$$

Example 2
For every 3×3 matrix in $[M_m(\mathbb{C})]$.
For each $\lambda > 0$ such that $\lambda \in \rho(A)$ where $\rho(A)$ is a resolvent set on X.
Suppose

$$A = \begin{pmatrix} 2 & 2 & I \\ 2 & 2 & 2 \\ \Delta & 2 & 2 \end{pmatrix}$$

and let $T(t) = e^{tA_\lambda}$, then

$$e^{tA_\lambda} = \begin{pmatrix} e^{2t\lambda} & e^{2t\lambda} & I \\ e^{2t\lambda} & e^{2t\lambda} & e^{2t\lambda} \\ e^{\Delta t\lambda} & e^{2t\lambda} & e^{2t\lambda} \end{pmatrix}.$$

Example 3

Let $X = C_{ub}(\mathbb{N} \cup \{0\})$ be the space of all bounded and uniformly continuous function from $\mathbb{N} \cup \{0\}$ to $\mathbb{R}$, endowed with the sup-norm $\|\cdot\|_\infty$ and let $\{T(t); t \in \mathbb{R}_+\} \subseteq L(X)$ be defined by

$$[T(t)f](s) = f(t+s)$$

For each $f \in X$ and each $t, s \in \mathbb{R}_+$, one may easily verify that $\{T(t); t \in \mathbb{R}_+\}$ satisfies Examples 1 and 2.

3 Main Results

This section present results of semigroup of linear operator by using ω-OCP_n to generates a wave equation:

Theorem 3.1

Suppose $A \in \omega - OCP_n$ is the infinitesimal generator of a C_0-semigroup $\{T(t);\ t \geq 0\}$. If $v > 0$ and $f \in H^k(\mathbb{R}^n)$, $k \geq 0$, then there is a unique function $u \in H^{k+2}(\mathbb{R}^n)$ satisfying

$$u - v\Delta u = f. \tag{3.1}$$

Proof:

Let $\hat{f}(\xi)$ be the Fourier transform of f and let

$$\overline{u}(\xi) = (1 + v|\xi|^2)^{-1}\hat{f}(\xi).$$

Since $f \in H^k(\mathbb{R}^n)$, $(1+|\xi|^2)^{k/2}\hat{f}(\xi) \in L^2(\mathbb{R}^n)$ and therefore $(1+|\xi|^2)^{(k+2)/2}\overline{u}(\xi) \in L^2(\mathbb{R}^n)$. If u is defined by

$$u(x) = (2\pi)^{-\pi/2} \int_{\mathbb{R}^n} e^{ix\xi}\overline{u}(\xi)d\xi$$

then $u \in H^{k+2}(\mathbb{R}^n)$ and u is a solution of (3.1). The uniqueness of the solution u of (3.1) follows from the fact that if $w \in H^{k+2}(\mathbb{R}^n)$ satisfies $w - v\Delta w = 0$, then $\hat{w} = 0$ and therefore $w = 0$. Hence the proof is complete.

Theorem 3.2

Assume A is the infinitesimal generator of a C_0-semigroup $\{T(t);\ t \geq 0\}$. Then for every $F = [f_1, f_2] \in C_0^\infty(\mathbb{R}^n) \times C_0^\infty(\mathbb{R}^n)$ and real $\lambda \neq 0$, the equation

$$u - \lambda AU = F \tag{3.2}$$

has a unique solution $U = [u_1, u_2] \in H^k(\mathbb{R}^n) \times H^{k-2}(\mathbb{R}^n)$ for every $k \geq 2$ such that $A \in \omega - OCP_n$. Moreover,

$$\|U\| \leq (1 - 2|\lambda|)^{-1}\|F\| \quad for\ 0 < |\lambda| < \frac{1}{2}. \tag{3.3}$$

Proof:

Let $\lambda \neq 0$ be real and let w_1, w_2 be solutions of

$$w_i - \lambda^2 \Delta w_i = f_i, \quad i = 1, 2. \tag{3.4}$$

From Theorem 3.1 it is clear that such solutions exist and that $w_1 \in H^k(\mathbb{R}^n)$ for every $k \geq 0$. Set

$$u_1 = w_1 + \lambda w_2, \quad u_2 = w_2 + \lambda \Delta w_1.$$

It is easy to check that $U = [u_1, u_2]$ is a solution of (3.2) and therefore $u_1 - \lambda u_2 = f_1$ and $u_2 - \lambda \Delta u_1 = f_2$. Moreover, $U \in H^k(\mathbb{R}^n) \times H^{k-2}(\mathbb{R}^n)$ for every $k \geq 2$. Denoting $\langle \cdot \rangle_o$ the scalar product in $L^2(\mathbb{R}^n)$ we have

$$\begin{aligned}
\|f\|^2 =& \langle f_1 - \Delta f_1, f_1 \rangle_o + \langle f_2, f_2 \rangle_o \\
=& \langle u_1 - \lambda u_2 - \Delta u_1 + \lambda \Delta u_2, u_1 - \lambda u_2 \rangle_o + \langle u_2 - \lambda \Delta u_1, u_2 - \lambda \Delta u_1 \rangle_o \\
\geq& \langle u_1 - \Delta u_1, u_1 \rangle_o + \|u_2\|_{0,2}^2 - 2|\lambda| Re \langle u_1, u_2 \rangle_o \\
\geq& \langle 1 - |\lambda| \rangle \|u\|^2.
\end{aligned}$$

Therefore if $0 < |\lambda| < \frac{1}{2}$, we have

$$\|F\|^2 \geq (1 - 2|\lambda|)^2 \|u\|^2, \tag{3.5}$$

and it shows that the range of operator $I - \lambda A$ contains $C_0^\infty(\mathbb{R}^n) \times C_0^\infty(\mathbb{R}^n)$ for all real λ satisfying $0 < |\lambda| < \frac{1}{2}$. Since the operator A defined by Definition 2.5 is closed, the range of $I - \lambda A$ belongs to $H = H'(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ for all $A \in \omega - OCP_n$. Then for every $F \in H'(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ and real λ satisfying $0 < |\lambda| < \frac{1}{2}$, the equation

$$U - \lambda AU = F \tag{3.6}$$

has a unique solution $u \in H^2(\mathbb{R}^n) \times H'(\mathbb{R}^n)$ and

$$\|U\| \leq (1 - 2|\lambda|)^{-1}\|F\|. \tag{3.7}$$

Hence the proof is complete.

Theorem 3.3

Let A be the infinitesimal generator of a C_0-semigroup $\{T(t);\ t \geq 0\}$ such that $A \in \omega - OCP_n$. For every $f_1 \in H^2(\mathbb{R}^n)$, $f_2 \in H'(\mathbb{R}^n)$ there exists a unique $u(t,x) \in C'([0,\infty) : H^2\mathbb{R}^n)$ satisfying the initial value problem

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = \Delta u \\ u(0,x) = f_1(x) \\ u_1'(0,x) = f_2(x). \end{cases} \tag{3.8}$$

Proof:

Let $T(t)$ be the semigroup generated by A and set

$$[u_1(t,x), u_2(t,x)] = T(t)[f_1(x), f_2(x)]$$

then

$$\frac{\partial}{\partial t}[u_1, u_2] = A[u_1 u_2] = [u_2, \Delta u_1]$$

where $A \in \omega - OCP_n$ and u_1 is the desired solution. Then we have that the operator A, defined in Definition 2.5 is the infinitesimal generator of a C_0-semigroup on $H = H'(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ satisfying

$$\|T(t)\| \leq e^{2|t|}. \tag{3.9}$$

We have that the domain of A, $H^2(\mathbb{R}^n) \times H'(\mathbb{R}^n)$ is clearly dense in H. From (3.6) it follows that $(\mu I - A)^{-1}$ exists for $|\mu| > 2$ and satisfies

$$\|(\mu I - A)^{-1}\| \leq \frac{1}{|\mu|^{-2}} \tag{3.10}$$

for $|\mu| > 2$ and $A \in \omega - OCP_n$. Then it follows that A is the infinitesimal generator of a C_0-smigroup satisfying (3.9) and this achieved the proof.

Theorem 3.4

Assume $A \in \omega - OCP_n$ is the infinitesimal generator of a C_0-semigroup. Then for $0 \leq m < k - \frac{n}{2}$, we have

$$H^k(\mathbb{R}^n) \subset C^m(\mathbb{R}^n). \tag{3.11}$$

Proof:

Let $v \in C_0^\infty(\mathbb{R}^n)$. Then as it is well known, $\xi^\alpha \hat{v}(\xi) \in L^2(\mathbb{R}^n)$ for every α and

$$D^\alpha v(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} i^{|\alpha|} \xi^\alpha e^{ix\xi} \hat{v}(\xi) d\xi.$$

Estimating $D^\alpha v(x)$, by the Cauchy-Schwartz inequality, we find for every $N > \frac{n}{2}$,

$$\begin{aligned} |D^\alpha v(x)|^2 &\leq (2\pi)^{-n} \int_{\mathbb{R}^n} (1 + |\xi|^2)^{-N} d\xi \int_{\mathbb{R}^n} |\xi|^{2|\alpha|} (1 + |\xi|^2)^N |\hat{v}(\xi)|^2 d\xi \\ &\leq C_1 \int_{\mathbb{R}^n} (1 + |\xi|^2)^{N+|\alpha|} |\hat{v}(\xi)|^2 d\xi \\ &\leq C_2 \|v\|^2_{N+|\alpha|,2} \end{aligned} \tag{3.12}$$

where C_1 and C_2 are constants depending on N and $|\alpha|$. Let $u \in H^k(\mathbb{R}^n)$ and let $u_n \in C_0^\infty(\mathbb{R}^n)$ be such that $u_n \to u$ in $H^k(\mathbb{R}^n)$. Then from (3.12) it follows that $D^\alpha u_n \to D^\alpha u$ uniformly in $\mathbb{R}^n$ for all α satisfying $|\alpha| \leq m < k - n/2$ and therefore $u \in C^m(\mathbb{R}^n)$ as desired. Hence the proof is complete.

Conclusion

In this paper, it has been established that ω-order preserving partial contraction mapping generates some results of a wave equation.

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