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Some Spectrum Estimates of the αq-Cesàro Matrices with $0 < \alpha, q < 1$ on c

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Abstract

q-Calculus Theory is rapidly growing in various directions. The goal of this paper is to collect and underline recent results on αq-analogs of the Cesáro matrix andemphasize various generalizations. One αq-analogs of the Cesáro matrix of order one is the triangular matrix with nonzero entries $c_{nk}^{\alpha}(q) = \frac{(\alpha q)^{n-k}}{1+q+\cdots+q^n}$, $0 \leq k \leq n$, where $\alpha, q \in (0,1)$. The purpose of this article examines various spectral decompositions of $C_q^{\alpha} = (c_{nk}^{\alpha}(q))$ such as the spectrum, the fine spectrum, the approximate point spectrum, the defect spectrum, and the compression spectrum on the sequence space c.

1 Introduction

Spectral theory is a sub-branch of functional analysis and its applications. It deals with the general properties of the inverse operator. The principal operator also allows us to understand the relationship between spectrum and set point spectrums. It consists of three discrete sets, the continuous spectrum and the residual spectrum. Spectral theory also plays an important role in physics. For example, in quantum mechanics The point spectrum of the Hamiltonian transformations is the energy at the boundary state of the system. The continuous and residual spectrum play a significant role in distribution theory.

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The studies on q-mathematics have many applications in branches such as mathematics and engineering in recent years, and the history of studies on these subjects are actually so old that they go back to the time of Euler. With $q \neq 1$, the q-analogue of the integer n is given by the following expression:

$$[n]_q = \frac{1-q^n}{1-q}.$$

The transformation given by

$$s_n(x) = \frac{q^n x_0 + q^{n-1}x_1 + q^{n-2}x_2 + \cdots + x_n}{1+q+\cdots+q^n} \tag{1.1}$$

is called the q-Cesàro mean and denoted by $C(q)$. If we take $q \to 1^-$ in (1.1), the C_1 Cesàro transformation given by

$$t_n(x) = \frac{x_0 + x_1 + x_2 + \cdots + x_n}{n+1}. \tag{1.2}$$

In [9], Bekar studied the q-analogue of this Cesàro transformation given by (1.2). Also [2], Aktuğ and Bekar compared q-Cesàro transform and q-statistical convergence. In [17] , [21] and [36] , the spectrum and spectral decomposition of the q-Cesàro transform on various spaces are given.

For $0 < \alpha$, $q < 1$, the transformation given by

$$z_n(x) = \frac{(\alpha q)^n x_0 + (\alpha q)^{n-1} x_1 + (\alpha q)^{n-2} x_2 + \cdots + x_n}{1+q+\cdots+q^n} \tag{1.3}$$

is called the genaralized αq-Cesàro mean C_q^α or simply the C_q^α mean. The matrix of the C_q^α method is given by

$$c_{nk}^\alpha(q) = \begin{cases} \frac{(\alpha q)^{n-k}}{1+q+\cdots+q^n} & , \ 0 < k \leq n \\ 0 & , \ k > n. \end{cases}$$

In this case, $\alpha \to 1^-$ in this matrix, the q-Cesàro matrix is obtained. The spectra and spectral decompositions of the αq-Cesàro matrix over the c_0 sequence space are investigated in [34].

This study is about the spectrum and spectral decompositions of the αq-Cesàro operator on Banach space c. Here c denotes the space of convergent sequences, which is considered with the supremum norm. This study is expected to broaden the applicability of q-calculus.

2 Preliminaries and Notation

Let us give a brief information on the spectrum of a bounded linear operator $T : X \to X$, defined in an infinite dimensional X Banach space. Let $T_\lambda = T - \lambda I$ for $\lambda \in \mathbb{C}$. If there is an inverse operator T_λ^{-1} for $\lambda \in \mathbb{C}$, then $R(\lambda; T) := R_\lambda(T) := T_\lambda^{-1}$ is called the resolvent operator of T. If 1) $R_\lambda(T)$ exists, 2) $R_\lambda(T)$ is bounded, and 3) $R_\lambda(T)$ is defined on a dense set at X, then this λ is called a regular value of T and $\rho(T, X)$ denotes the set of all regular values of T. The set $\sigma(T, X) = \mathbb{C} \backslash \rho(T, X)$ is also called the spectrum of T. If we are working on a particular Banach space X, we will write $\sigma(T)$ for short instead of $\sigma(T, X)$.

The set in which $R_\lambda(T)$ does not exist is called the point spectrum or discrete spectrum, and is denoted by $\sigma_p(T)$. Also, a number $\lambda \in \sigma_p(T)$ is called the eigen-value of T. The set of $\lambda \in \sigma(T)$ elements, where $R_\lambda(T)$ exists and will provide 3) but not 2), that is, $R_\lambda(T)$ is unbounded, is called the continuous spectrum and it is denoted by $\sigma_c(T)$. The set of $\lambda \in \sigma(T)$ elements such that $R_\lambda(T)$ exists (bounded or unbounded) and 3) cannot be realized is called the residue spectrum of T and it is denoted by $\sigma_r(T)$. Thus $\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T)$ and these sets are binary discrete. This creates the natural spectral decomposition of the $T \in B(X)$ operator [31].

Now let's give the Goldberg decomposition of the $\sigma(T)$ spectrum. Let $\lambda \in \sigma(T)$. Then there are three possibilities for $\mathcal{R}(T_\lambda)$, the range of $T_\lambda = T - \lambda I$: I) T_λ is surjective, II) $\overline{\mathcal{R}(T_\lambda)} = X$, but $\mathcal{R}(T_\lambda) \neq X$, III) $\overline{\mathcal{R}(T_\lambda)} \neq X$; and three possibilities for $R_\lambda = T_\lambda^{-1}$: 1) T_λ is injective and R_λ is bounded, 2) T_λ is injective and R_λ is unbounded, and 3) T_λ is not injective.

There are nine different states for a $\lambda \in \mathbb{C}$: I_1, I_2, I_3, II_1, II_2, II_3, III_1, III_2, III_3. If $T_\lambda \in I_1$ or $T_\lambda \in II_1$ then it is $\lambda \in \rho(T)$. From the closed graph theorem, there is no λ so that $T_\lambda \in I_2$, so $I_2 = \emptyset$. Therefore, we get a decomposition of the spectrum as

$$\begin{aligned} \sigma(T) &= I_3\sigma(T) \cup II_2\sigma(T) \cup II_3\sigma(T) \\ &\quad \cup III_1\sigma(T) \cup III_2\sigma(T) \cup III_3\sigma(T). \end{aligned}$$

This is called the Goldberg decomposition of the spectrum.

We can summarize these two decompositions of the spectrum in with the Table below [24].

Table 1. Goldberg's and fine decomposition of the spectrum

		1	2	3
		R_λ exists and is bounded	R_λ exists and is unbounded	R_λ does not exists
I	$\mathcal{R}(T_\lambda) = X$	$\lambda \in \rho(T)$	–	$\lambda \in \sigma_p(T)$
II	$\overline{\mathcal{R}(T_\lambda)} = X$	$\lambda \in \rho(T)$	$\lambda \in \sigma_c(T)$	$\lambda \in \sigma_p(T)$
III	$\overline{\mathcal{R}(T_\lambda)} \neq X$	$\lambda \in \sigma_r(T)$	$\lambda \in \sigma_r(T)$	$\lambda \in \sigma_p(T)$

The fine spectrum of some bounded linear operators over various spaces has been specified by many authors ([1], [12], [18], [19], [20], [22], [23], [25]-[28], [30], [32], [33], [35], [37]-[41]).

Let X be Banach space on a field $\mathbb{K}$ and $T \in B(X)$. If $(x_n) \subset X$ is a sequence such that $\|Tx_n\| \to 0$ while $n \to \infty$ and $\|x_n\| = 1$, then (x_n) is called a Weyl sequence for T.

The set

$$\sigma_{ap}(T, X) := \{\lambda \in \mathbb{K} : \text{ there is a Weyl sequence for } \lambda I - T\} \tag{2.1}$$

is called as the approximate point spectrum of T. Moreover, the subspectrum

$$\sigma_\delta(T, X) := \{\lambda \in \mathbb{K} : \lambda I - T \text{ is not surjective}\}, \tag{2.2}$$

is called defect spectrum of T.

The two subspectra (2.1) and (2.2) form a (not necessarily disjoint) subdivision

$$\sigma(T) = \sigma_{ap}(T) \cup \sigma_\delta(T) \tag{2.3}$$

of the spectrum. There is another subspectrum,

$$\sigma_{co}(T,X) := \left\{\lambda \in \mathbb{K} : \overline{R(\lambda I - T)} \neq X\right\} \tag{2.4}$$

which is often called compression spectrum.

These three sets make up the non-discrete spectrum of the spectrum. (2.4) gives rise to another (not necessarily disjoint) decomposition

$$\sigma(T) = \sigma_{ap}(T) \cup \sigma_{co}(T) \tag{2.5}$$

of the spectrum. Clearly, $\sigma_p(T) \subseteq \sigma_{ap}(T)$ and $\sigma_{co}(T) \subseteq \sigma_\delta(T)$. Moreover, we note that

$$\sigma_r(T) = \sigma_{co}(T) \backslash \sigma_p(T) \tag{2.6}$$

and

$$\sigma_c(T) = \sigma(T) \backslash [\sigma_p(T) \cup \sigma_{co}(T)]. \tag{2.7}$$

Sometimes it is useful to relate the spectrum of a bounded linear operator to that of its adjoint.

Proposition 2.1. *[4, Proposition 1.3] The spectra and subspectra of an operator $T \in B(X)$ and its adjoint $T^* \in B(X^*)$ are related by the following relations:*
(a) $\sigma(T^*) = \sigma(T)$.
(b) $\sigma_c(T^*) \subseteq \sigma_{ap}(T)$.
(c) $\sigma_{ap}(T^*) = \sigma_\delta(T)$.
(d) $\sigma_\delta(T^*) = \sigma_{ap}(T)$.
(e) $\sigma_p(T^*) = \sigma_{co}(T)$.
(f) $\sigma_{co}(T^*) \supseteq \sigma_p(T)$.
(g) $\sigma(T) = \sigma_{ap}(T) \cup \sigma_p(T^*) = \sigma_p(T) \cup \sigma_{ap}(T^*)$.

We can write the above definition as follows (Table 1)

In order to obtain this non-discrete decomposition for a finite linear operator T, the following template created in the articles [5]-[7], [10] and [11] is used, considering the proposition and Theorems in [4].

Table 2. Non-discrete decomposition of the spectrum

		(1)	(2)	(3)
		R_λ exists and is bounded	R_λ exists and is unbounded	R_λ does not exists
(I)	$\mathcal{R}(T_\lambda) = X$	$\lambda \in \rho(T)$	$-$	$\lambda \in \sigma_{ap}(T)$
(II)	$\mathcal{R}(T_\lambda) \neq X$ $\overline{\mathcal{R}(T_\lambda)} = X$	$\lambda \in \rho(T)$	$\lambda \in \sigma_{ap}(T)$ $\lambda \in \sigma_\delta(T)$	$\lambda \in \sigma_{ap}(T)$ $\lambda \in \sigma_\delta(T)$
(III)	$\overline{\mathcal{R}(T_\lambda)} \neq X$	$\lambda \in \sigma_\delta(T)$ $\lambda \in \sigma_{co}(T)$	$\lambda \in \sigma_{ap}(T)$ $\lambda \in \sigma_\delta(T)$ $\lambda \in \sigma_{co}(T)$	$\lambda \in \sigma_{ap}(T)$ $\lambda \in \sigma_\delta(T)$ $\lambda \in \sigma_{co}(T)$

Various divisions of the spectrum are possible. The non-discrete spectrum (Apporoximate point spectrum, defect spectrum and compression spectrum) can be found in the book entitled "Nonlinear Spectral Theory";, published by J. Appell et al. Using this Table 2, separation of an operator for the first time in the literature was handled in 2011 by Kh. Amirov and Nuh Durna, Mustafa Yıldırım [3]. After, using this separation, the non-discrete spectrum of some bounded linear operators on various spaces has been studied by various authors ([3], [8], [13]-[16]).

3 Boundedness and Various Spectral Decompositions of C_q^α

In this section, we will show that C_q^α is a bounded linear operator on c, and then examine its various spectral decompositions.

In order for the αq- Cesàro operator matrix given by $C_q^\alpha = (c_{nk}^\alpha(q))$ to be in $B(c)$, it must be shown that the conditions of the Kojima-Schur Theorem given in [29, p.166] are met.

Lemma 3.1. *Let $0<q<1$, $0<\alpha<1$. Then $C_q^\alpha \in B(c)$ and $\left\|C_q^\alpha\right\|=1$.*

Proof. i) Since $\alpha<1$, the following inequality is obtained

$$\begin{aligned}\left\|C_q^\alpha\right\| &= \sup_n \sum_{k=0}^{\infty}\left|c_{nk}^\alpha\right| = \sup_n\left\{\tfrac{1}{1+q+\cdots+q^n}\sum_{k=0}^{n}(\alpha q)^{n-k}\right\}\\ &\leq \sup_n \sum_{k=0}^{n}\tfrac{1+q+\cdots+q^n}{1+q+\cdots+q^n}\\ &= 1\end{aligned} \tag{3.1}$$

On the other hand, the following inequality is valid;

$$\begin{aligned}\left\|C_q^\alpha\right\| &= \sup_{x\neq\theta}\frac{\left\|C_q^\alpha(x)\right\|_c}{\|x\|_c}\\ &= \sup_{x\neq\theta}\frac{\left\|\left(x_0, \tfrac{\alpha q}{1+q}x_0+\tfrac{1}{1+q}x_1,\cdots\right)\right\|_c}{\|x\|_c}\\ &\overset{x=e_0}{\geq} \left\|\left(1,\tfrac{\alpha q}{1+q},\tfrac{\alpha^2q^2}{1+q+q^2},\cdots\right)\right\|_c\\ &= \sup\left(1,\tfrac{\alpha q}{1+q},\tfrac{\alpha^2q^2}{1+q+q^2},\cdots\right)=1.\end{aligned} \tag{3.2}$$

From 3.1 and 3.2, $\left\|C_q^\alpha\right\|=1$ is obtained.
ii) For each k

$$\begin{aligned}\lim_{n\to\infty}\sum_{k=p}^{\infty}c_{nk} &= \lim_{n\to\infty}\sum_{k=p}^{n}\tfrac{(\alpha q)^{n-k}}{1+q+\cdots+q^n}\\ &= \lim_{n\to\infty}\tfrac{1}{1+q+\cdots+q^n}\sum_{k=p}^{n}(\alpha q)^{n-k}\\ &= \lim_{n\to\infty}\frac{\frac{1-(\alpha q)^{n-p+1}}{1-\alpha q}}{\frac{1-q^{n+1}}{1-q}}\\ &= \begin{cases}\frac{1-q}{1-\alpha q}\left(\frac{1}{\alpha q}\right)^p & ,p>1\\ \frac{1-q}{1-\alpha q} & ,0<p<1.\end{cases}\end{aligned}$$

Thus, i) and ii) satisfy the conditions of the Kojima-Schur Theorem in [29, p.166]. This proves the Lemma. □

It should be noted that if $0<q<1$ and $0<\alpha<1$, there is always $m\in\mathbb{N}_0$ such that $\alpha<q^m$. There is also $\alpha<q^0=1$.

Theorem 3.2. *Let $0 < q < 1$ and $0 < \alpha \leq 1$. Then, if $\alpha < q^m$, then*

$$\sigma_p\left(C_q^\alpha, c\right) = \left\{ \frac{1}{\sum\limits_{k=0}^{1} q^k}, \frac{1}{\sum\limits_{k=0}^{2} q^k}, \ldots, \frac{1}{\sum\limits_{k=0}^{m} q^k} : \ \alpha < q^m \right\} \cup \{1\}$$

and if $\alpha = 1$, then $\sigma_p\left(C_q^\alpha, c\right) = \{1\}$.

Proof. Let $C_q^\alpha x = \lambda x$. In this case, the following equations exist;

$$\begin{aligned}
x_0 &= \lambda x_0 \\
\frac{1}{\sum\limits_{k=0}^{1} q^k}\left(\alpha q x_0 + x_1\right) &= \lambda x_1 \\
\frac{1}{\sum\limits_{k=0}^{2} q^k}\left((\alpha q)^2 x_0 + \alpha q x_1 + x_2\right) &= \lambda x_2 \\
&\vdots \\
\frac{1}{\sum\limits_{k=0}^{n} q^k}\left((\alpha q)^n x_0 + (\alpha q)^{n-1} x_1 + \cdots + \alpha q x_{n-1} + x_n\right) &= \lambda x_n \\
&\vdots
\end{aligned} \tag{3.3}$$

If $x_0 \neq 0$, then $\lambda = 1$ is obtained from the first line of equation (3.3). If we put $\lambda = 1$ in other equations, we get $x_n = \alpha^n x_0$, $n = 1, 2, \ldots$. Thus, since $0 < \alpha \leq 1$, the eigenvector corresponding to $\lambda = 1$ is $x = (\alpha^n) \in c$.

Same way, let x_m be the first nonzero term of the sequence (x_n). Thus, from the mth row of the equation (3.3)

$$\left(\lambda - \frac{1}{\sum\limits_{k=0}^{m} q^k}\right) x_m = 0$$

is found. Hence, we have

$$\lambda = \frac{1}{\sum\limits_{k=0}^{m} q^k}, \tag{3.4}$$

since $x_m \neq 0$. Since $x_1 = x_2 = x_3 = \cdots = x_{m-1} = 0$, from the equations after the (m+1)th line of system, we get

$$x_{m+n} = \frac{\alpha^n q^n \left(\sum\limits_{k=0}^{m} q^k\right)\left(\sum\limits_{k=0}^{m+1} q^k\right)\cdots\left(\sum\limits_{k=0}^{m+n-1} q^k\right)}{(q^{m+1})^n \left(\sum\limits_{k=0}^{1} q^k\right)\cdots\left(\sum\limits_{k=0}^{n-1} q^k\right)} x_m,\ n = 1, 2, 3, \ldots;$$

that is,

$$x_n = \left(\frac{\alpha q}{q^{m+1}}\right)^{n-m} \frac{\left(\sum\limits_{k=0}^{m} q^k\right)\left(\sum\limits_{k=0}^{m+1} q^k\right)\cdots\left(\sum\limits_{k=0}^{n-1} q^k\right)}{\left(\sum\limits_{k=0}^{1} q^k\right)\left(\sum\limits_{k=0}^{2} q^k\right)\cdots\left(\sum\limits_{k=0}^{n-m-1} q^k\right)} x_m \text{ for } n > m. \tag{3.5}$$

Since $q^m > q^n$,

$$\lim_{n\to\infty} \frac{x_{m+n}}{x_{m+n-1}} = \lim_{n\to\infty} \frac{\alpha}{q^m}\left[1 + q^n \frac{1-q^m}{1-q^n}\right] = \frac{\alpha}{q^m}$$

is obtained. Therefore, for $\alpha < q^m$, we have $\lim_{n\to\infty} x_{m+n} = 0$, that is, the eigenvector corresponding to $\lambda = \frac{1}{\sum\limits_{k=0}^{m} q^k}$ for $\alpha < q^m$ is $x = (0, 0, \ldots, 0, x_m, x_{m+1}, \ldots, x_{m+n}, \ldots) \in c$. Also, for $\alpha = \lambda = 1$, $C_1^\alpha(q)\, x = \lambda x$ with $x = (1, 1, 1, \ldots) \in c$. So the point spectrum is obtained as follows; $\sigma_p\left(C_q^\alpha, c\right) = \left\{\frac{1}{\sum\limits_{k=0}^{1} q^k}, \frac{1}{\sum\limits_{k=0}^{2} q^k}, \ldots, \frac{1}{\sum\limits_{k=0}^{m} q^k} :\ \alpha < q^m\right\} \cup \{1\}$. □

The following Lemma will be very useful for calculating the adjoint of operator C_q^α:

Lemma 3.3. *[42, p.267] If $T : c \to c$ is a linear transformation and $T^* : \ell^1 \to \ell^1$, $T^* g = g \circ T$, $g \in c^* \cong \ell^1$, then T and T^* have matrix representations, also*

$T^*: \ell^1 \to \ell^1$ is given by

$$T^* = A^* = \begin{pmatrix} \chi(\lim A) & (\vartheta_n)_{n=0}^{\infty} \\ (a_k)_{k=0}^{\infty} & A^t \end{pmatrix}$$
$$= \begin{pmatrix} \chi(\lim A) & \vartheta_0 & \vartheta_1 & \vartheta_2 & \cdots \\ a_0 & a_{00} & a_{10} & a_{20} & \cdots \\ a_1 & a_{01} & a_{11} & a_{21} & \cdots \\ a_2 & a_{02} & a_{12} & a_{22} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where

$$\begin{aligned} a_k &= \lim_n a_{nk} \\ \chi(A) &= \lim Ae - \sum_{k=0}^{\infty} \lim Ae_k = \lim_n \sum_k a_{nk} - \sum_k \lim_n a_{nk} \\ \vartheta_n &= \chi(P_n \circ T) = (P_n \circ T)\, e - \sum_k a_{nk}, \\ a_{nk} &= P_n(T(e_k)) = (T(e_k))_n . \end{aligned}$$

If $0 < \alpha < 1$, $0 < q < 1$, then one can get the following result from the above Lemma for $\alpha q-$Cesàro matrix.

Making use of the relations given above Lemma 3.3, we easily arrive at the following Lemma:

Lemma 3.4. *The adjoint of $C_1(q)$ on c is given by*

$$\left(C_q^{\alpha}\right)^* = \begin{pmatrix} \frac{1-q}{1-\alpha q} & 0 & 0 & \cdots \\ 0 & & & \\ 0 & & \left(C_q^{\alpha}\right)^t & \\ \vdots & & & \end{pmatrix}. \tag{3.6}$$

Proof. Since $0 < \alpha < 1$, $0 < q < 1$, we get

$$c_k = \lim_n c_{nk} = \lim_{n\to\infty} \frac{(\alpha q)^{n-k}(1-q)}{1-q^{n+1}} = (\alpha q)^{-k}(1-q) \lim_{n\to\infty} \frac{(\alpha q)^n}{1-q^{n+1}} = 0$$

and

$$\sum_{k=0}^{n} c_{nk} = \tfrac{(1-q)}{1-q^{n+1}} \sum_{k=0}^{n} (\alpha q)^{n-k} = \tfrac{(1-q)}{1-q^{n+1}} \sum_{k=0}^{n} (\alpha q)^{k} = \tfrac{(1-q)}{1-q^{n+1}} \tfrac{1-(\alpha q)^{n+1}}{(1-\alpha q)}.$$

Thus, $\chi\left(C_q^\alpha\right)$ can be calculated as follows;

$$\begin{aligned}\chi\left(C_q^\alpha\right) &= \lim_n \textstyle\sum_k c_{nk} - \sum_k \lim_n c_{nk} \\ &= \lim_n \tfrac{(1-q)}{1-q^{n+1}} \tfrac{1-(\alpha q)^{n+1}}{(1-\alpha q)} - 0 = \tfrac{1-q}{1-\alpha q}.\end{aligned}$$

To calculate ϑ_n, the following expression is required;

$$\left(P_n \circ C_q^\alpha\right) e = \left\{\sum_{k=0}^{n} c_{nk} x_k\right\}_{x=e} = \sum_{k=0}^{n} c_{nk} = \frac{(1-q)}{1-q^{n+1}} \frac{1-(\alpha q)^{n+1}}{(1-\alpha q)}. \tag{3.7}$$

From (3.7), we have

$$\vartheta_n = \left(P_n \circ C_q^\alpha\right) e - \sum_{k=0}^{n} c_{nk} = \frac{(1-q)}{1-q^{n+1}} \frac{1-(\alpha q)^{n+1}}{(1-\alpha q)} - \frac{(1-q)}{1-q^{n+1}} \frac{1-(\alpha q)^{n+1}}{(1-\alpha q)} = 0$$

Hence, by Lemma 3.3, desired is obtained. □

Theorem 3.5. *If* $0 < q < 1$ *and* $0 < \alpha < 1$, *then*

$$\begin{aligned}\sigma_p\left(\left[C_q^\alpha\right]^*, (c)^* \simeq \ell^1\right) &= \left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1-q}{1-\alpha^2 q^2}\right| < \frac{1-q}{1-\alpha^2 q^2}\alpha q\right\} \\ &\cup \left\{1, \frac{1}{\sum\limits_{k=0}^{1} q^k}, \ldots, \frac{1}{\sum\limits_{k=0}^{m} q^k} : \ \alpha < q^m\right\}.\end{aligned}$$

Proof. Let $x \neq 0$ and $\left(C_1^\alpha\right)^*(q)\, x = \lambda x$. Since the matrix $\left(C_1^\alpha\right)^*(q)$ is given by the equation (3.6), the following system of equations is obtained

$$\begin{aligned}
\tfrac{1-q}{1-\alpha q}x_0 &= \lambda x_0\\
x_1+\frac{(\alpha q)}{\sum\limits_{k=0}^{1}q^k}x_2+\frac{(\alpha q)^2}{\sum\limits_{k=0}^{2}q^k}x_3+\frac{(\alpha q)^3}{\sum\limits_{k=0}^{3}q^k}x_4+\cdots &= \lambda x_1\\
\frac{1}{\sum\limits_{k=0}^{1}q^k}x_2+\frac{(\alpha q)}{\sum\limits_{k=0}^{2}q^k}x_3+\frac{(\alpha q)^2}{\sum\limits_{k=0}^{3}q^k}x_4+\cdots &= \lambda x_2\\
\frac{1}{\sum\limits_{k=0}^{2}q^k}x_3+\frac{(\alpha q)}{\sum\limits_{k=0}^{3}q^k}x_4+\cdots &= \lambda x_3\\
&\vdots \qquad .
\end{aligned}$$

Hence

$$x_n=\frac{x_1}{(\alpha q\lambda)^n}\prod_{k=1}^{n}\left(\lambda-\frac{1}{\sum\limits_{v=0}^{k-1}q^v}\right),\ n=2,3,\ldots .$$

where $x_1\neq 0$.

Let's assume that

$$\lambda\in\left\{1,\frac{1}{\sum\limits_{k=0}^{1}q^k},\frac{1}{\sum\limits_{k=0}^{2}q^k},\ldots,\frac{1}{\sum\limits_{k=0}^{n}q^k},\ldots\right\}.$$

If $\lambda=1$, then $(C_1^\alpha)^*(q)\,x=x$ for $x=(x_0,x_1,0,\ldots)\in\ell^1$ and $x\neq\theta$. So we get that $1\in\sigma_p\left((C_1^\alpha)^*(q),\ell^1\right)$.

If $\lambda=\dfrac{1}{1+q}$, then $(C_1^\alpha)^*(q)\,x=\dfrac{1}{1+q}x$ for $x=\left(x_0,x_1,-\frac{x_1}{\alpha},0,0,\ldots\right)\in\ell^1$ and $x\neq\theta$. Hence, we get that $\dfrac{1}{1+q}\in\sigma_p\left((C_1^\alpha)^*(q),\ell^1\right)$.

If $\lambda=\dfrac{1}{1+q+q^2}$, then $(C_1^\alpha)^*(q)\,x=\dfrac{1}{1+q+q^2}x$ for $x=$

$\left(x_0, x_1, -\frac{(1+q)}{\alpha}x_1, \frac{q}{\alpha^2}x_1, 0, 0, \ldots\right) \in \ell^1$ and $x \neq \theta$. So we get that $\dfrac{1}{1+q+q^2} \in \sigma_p\left(\left(C_1^\alpha\right)^*(q), \ell^1\right)$.

Similarly, we obtain

$$\left\{\frac{1}{\sum\limits_{k=0}^{n} q^k}\right\}_{n=0}^{\infty} \subset \sigma_p\left(\left(C_1^\alpha\right)^*(q), (c)^* \simeq \ell^1\right).$$

Let us now assume that $\lambda \notin \left\{\dfrac{1}{\sum\limits_{k=0}^{n} q^k}\right\}_{n=0}^{\infty}$. If

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{x_{n+1}}{x_n}\right| &= \lim_{n\to\infty}\left|\frac{1}{\alpha q\lambda}\left(\lambda - \frac{1}{\sum\limits_{v=0}^{n} q^v}\right)\right| \\ &= \lim_{n\to\infty}\left|\frac{1}{\alpha q\lambda}\left(\lambda - \frac{1-q}{1-q^{n+1}}\right)\right| \\ &= \left|\frac{1}{\alpha q\lambda}\left(\lambda - \frac{1-q}{\lim_{n\to\infty}\left(1-q^{n+1}\right)}\right)\right| < 1,\end{aligned} \tag{3.8}$$

the series $\sum |x_n|$ is convergent and hence $(x_n) \in \ell^1$. For this,

$$\begin{aligned}&\left|\frac{1}{\alpha q\lambda}\left(\lambda - (1-q)\right)\right| < 1 \Leftrightarrow \left|1 - \frac{1-q}{\lambda}\right| < \alpha q \\ &\overset{\lambda=u+iv}{\Leftrightarrow} \left|1 - \frac{1-q}{u^2+v^2}u + \frac{1-q}{u^2+v^2}vi\right| < \alpha q \\ &\Leftrightarrow \left|\lambda - \frac{(1-q)}{(1-\alpha^2 q^2)}\right| < \frac{(1-q)\,\alpha q}{1-\alpha q}.\end{aligned} \tag{3.9}$$

is realized. (3.9) shows us, if $\left|\lambda - \dfrac{(1-q)}{(1-\alpha^2 q^2)}\right| < \dfrac{(1-q)\,\alpha q}{1-\alpha q}$, then $(x_n) \in \ell^1$. Also, since

$$\frac{1}{\sum\limits_{k=0}^{m+1} q^k}, \frac{1}{\sum\limits_{k=0}^{m+2} q^k}, \ldots \in \left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1-q}{1-\alpha^2 q^2}\right| < \frac{1-q}{1-\alpha^2 q^2}\alpha q\right\}$$

for $\alpha > q^m$, $m = 0, 1, 2, \ldots$, the desired result is obtained. □

We know that c is a Banach space, $\sigma\left(\left[C_q^\alpha\right]^*, \ell^1\right) = \sigma\left(C_q^\alpha, c\right)$ and $\sigma_p\left(C_q^\alpha, c\right) \subset \sigma\left(C_q^\alpha, c\right)$ are valid. Let's determine the spectrum of C_q^α using these.

An infinite matrix defined as

$$\triangle_{a,b} = \begin{pmatrix} a_0 & 0 & 0 & 0 & \dots \\ b_0 & a_1 & 0 & 0 & \dots \\ 0 & b_1 & a_2 & 0 & \ddots \\ 0 & 0 & b_2 & a_3 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

is called a lower triangular double band matrix with a variable sequence, where (a_k) and (b_k) are two nonzero sequences of real numbers with

$$\lim_{n\to\infty} a_n = a \text{ and } \lim_{n\to\infty} b_n = b \neq 0. \tag{3.10}$$

This matrix defines a $\triangle_{a,b} : c \to c$ operator with

$$\triangle_{a,b}x = \triangle_{a,b}(x_k) = (a_k x_k + b_{k-1}x_{k-1})_{k=0}^{\infty} \text{ with } x_{-1} = b_{-1} = 0 \tag{3.11}$$

Corollary 3.6. *The matrix operator $\triangle_{a,b}$ is bounded on c and $\|\triangle_{a,b}\|_{B(c)} = \sup\left(|a_k| + |b_{k-1}|\right)$.*

The matrix $\triangle_{a,b}$ satisfies the conditions of the Kojima-Schur Theorem, so it can be easily seen that it is $\triangle_{a,b} \in B(c)$ and it can also be seen that $\|\triangle_{a,b}\|_{B(c)} = \sup\left(|a_k| + |b_{k-1}|\right)$ as in [19].

Theorem 3.7.

$$\sigma\left(\triangle_{a,b}, c\right) = \{\lambda \in \mathbb{C} : |\lambda - a| \leq |b|\} \cup \{a_k : k \in \mathbb{N},\ \ |a_k - a| > |b|\} \cup \{1\}.$$

The proof of the theorem can be given by the same method as [19, Theorem 2.1].

Let $0 < \alpha < 1$ and $0 < q < 1$. The generalized αq-Cesàro matrix $C_q^\alpha : c \to c$

has an inverse and this inverse matrix is given by:

$$\left[C_q^\alpha\right]^{-1} = \begin{pmatrix} A_0 & 0 & 0 & 0 & \cdots \\ B_0 & A_1 & 0 & 0 & \cdots \\ 0 & B_1 & A_2 & 0 & \ddots \\ 0 & 0 & B_2 & A_3 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix},$$

where

$$A_n = \sum_{k=0}^{n} q^k \text{ and } B_n = -\alpha q a_n \text{ for all } n \in \mathbb{N}_0. \tag{3.12}$$

Therefore,

$$\begin{aligned} \lim_{n\to\infty} A_n &= \lim_{n\to\infty} \left(1+q+q^2+\cdots+q^n\right) = \lim_{n\to\infty} \frac{1-q^{n+1}}{1-q} = \frac{1}{1-q} = a, \\ \lim_{n\to\infty} B_n &= \lim_{n\to\infty} -\alpha q \left(1+q+q^2+\cdots+q^n\right) = -\frac{\alpha q}{1-q} = b \end{aligned} \tag{3.13}$$

is obtained from (3.12).

It is clear that the operators $\left[C_q^\alpha\right]^{-1}$ and C_q^α are bijective. If we take $\triangle_{a,b} = \left[C_q^\alpha\right]^{-1}$ in Theorem 3.7, it is easily seen that the inverse operator $\left[C_q^\alpha\right]^{-1}$ is bounded on the sequence space c. If we take $\triangle_{a,b} = \left[C_q^\alpha\right]^{-1}$ in Corollary 3.6, it is an obvious result that it will be $0 \notin \sigma(\triangle_{a,b}, c)$.

Theorem 3.8. *If* $0 < q < 1$ *and* $0 < \alpha < 1$, *then*

$$\begin{aligned} \sigma\left(C_q^\alpha, c\right) &= \left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1-q}{1-\alpha^2 q^2}\right| \leq \frac{1-q}{1-\alpha^2 q^2}\alpha q\right\} \\ &\cup \left\{\frac{1}{\sum_{k=0}^{1} q^k}, \ldots, \frac{1}{\sum_{k=0}^{m} q^k} : \ \alpha < q^m\right\} \cup \{1\}. \end{aligned}$$

Proof. From Theorem 3.5, the following scope is obtained:

$$\begin{aligned} &\sigma_p\left(\left[C_q^\alpha\right]^*, \ell^1\right) = \left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1-q}{1-\alpha^2 q^2}\right| < \frac{1-q}{1-\alpha^2 q^2}\alpha q\right\} \\ &\cup \left\{\frac{1}{1+q}, \ldots, \frac{1}{1+q+\cdots+q^m} : \ \alpha < q^m\right\} \cup \{1\} \\ &\subset \sigma\left(\left[C_q^\alpha\right]^*, \ell^1\right) = \sigma\left(C_q^\alpha, c\right). \end{aligned}$$

If closure is taken from both sides,

$$\left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1-q}{1-\alpha^2 q^2}\right| \leq \frac{1-q}{1-\alpha^2 q^2}\alpha q\right\} \cup \left\{1, \frac{1}{1+q}, \frac{1}{1+q+q^2}, \ldots\right\} \subset \sigma\left(C_q^\alpha, c\right)$$

is obtained since the spectrum is closed and $1-q \in \left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1-q}{1-\alpha^2 q^2}\right| \leq \frac{1-q}{1-\alpha^2 q^2}\alpha q\right\}$. Also, taking $\lambda = \frac{1}{\sum_{k=0}^{m} q^k}$ and $\alpha = q^m$ turns the inequality $\left|\lambda - \frac{1-q}{1-\alpha^2 q^2}\right| \leq \frac{1-q}{1-\alpha^2 q^2}\alpha q$ into equality.it should be noted that when $\alpha = q^m$ for a $m \in \mathbb{N}$, $\lambda = \frac{1}{\sum_{k=0}^{m} q^k}$ is the point at the right end of the circle and on the x-axis.

From the explanation above the $\left[C_q^\alpha\right]^{-1}$ is invertible and bounded on c. From Theorem 3.7, it is known that

$$\begin{aligned} \sigma\left(\left[C_q^\alpha\right]^{-1}, c\right) &= \sigma\left(\triangle_{a,b}, c\right) \\ &= \{\lambda \in \mathbb{C} : |\lambda - a| \leq |b|\} \\ &\cup \{a_k : k \in \mathbb{N},\ |a_k - a| > |b|\} \cup \{1\} \end{aligned} \tag{3.14}$$

With $a = \frac{1}{1-q}$, $b = -\frac{\alpha q}{1-q}$, $a_m = \sum_{k=0}^{m} q^k$, the 1st set forming the union on the second side of equation (3.7) is equal to

$$\left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1}{1-q}\right| \leq \frac{\alpha q}{1-q}\right\}$$

and the 2nd set is equal to

$$\{a_k : k \in \mathbb{N},\ |a_k - a| > |b|\} = \{a_0, a_1, \ldots, a_m :\ \alpha < q^m\}.$$

Since

$$\mu = \frac{1}{\lambda} = x + iy \Leftrightarrow \lambda = \frac{1}{\mu} = \frac{1}{x+iy} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2} = u + iv,$$

we have

$$\begin{aligned}
\left|\lambda-\frac{1}{1-q}\right|\leq\frac{\alpha q}{1-q} \quad & \Leftrightarrow\left|\tfrac{1}{\mu}-\frac{1}{1-q}\right|\leq\frac{\alpha q}{1-q}\\
& \Leftrightarrow(u-(1-q))^2+v^2\leq(\alpha q)^2\left(u^2+v^2\right)\\
& \Leftrightarrow\left|\mu-\frac{1-q}{1-\alpha^2q^2}\right|\leq\alpha q\frac{1-q}{1-\alpha^2q^2}.
\end{aligned}$$

Therefore, the following expressions can be written from the spectral maping Theorem;

$$\begin{aligned}
\sigma\left(C_q^\alpha,c\right) &=\left\{\tfrac{1}{\lambda}\in\mathbb{C}:\lambda\in\sigma\left(\left[C_q^\alpha\right]^{-1},c\right)\right\}\\
&=\left\{\tfrac{1}{\lambda}\in\mathbb{C}:|\lambda-a|\leq|b|\right\}\cup\left\{\tfrac{1}{a_m}:m\in\mathbb{N},|a_m-a|>|b|\right\}\cup\{1\}\\
&=\left\{\tfrac{1}{\lambda}\in\mathbb{C}:\left|\lambda-\frac{1}{1-q}\right|\leq\frac{\alpha q}{1-q}\right\}\cup\left\{\tfrac{1}{a_m}:m\in\mathbb{N},\alpha<q^m\right\}\cup\{1\}\\
&=\left\{\lambda\in\mathbb{C}:\left|\lambda-\tfrac{1-q}{1-(\alpha q)^2}\right|\leq\tfrac{1-q}{1-(\alpha q)^2}\alpha q\right\}\\
&\cup\left\{\frac{1}{\sum\limits_{k=0}^{0}q^k},\frac{1}{\sum\limits_{k=0}^{1}q^k},\ldots,\frac{1}{\sum\limits_{k=0}^{m}q^k}:\ \alpha<q^m\right\}\cup\{1\}.
\end{aligned}$$

□

Remark 3.9. In Theorem 3.8, since $\left\{\frac{1}{\sum\limits_{k=0}^{1}q^k},\frac{1}{\sum\limits_{k=0}^{2}q^k},\ldots,\frac{1}{\sum\limits_{k=0}^{m}q^k}:\ \alpha<q^m\right\}\subset\left\{\lambda\in\mathbb{C}:\left|\lambda-\tfrac{1}{1+q}\right|\leq\tfrac{q}{1+q}\right\}$ as $\alpha\to1^-$, we get

$$\begin{aligned}
\sigma\left(C_1(q),c\right) &=\left\{\lambda\in\mathbb{C}:\left|\lambda-\tfrac{1-q}{1-q^2}\right|\leq\tfrac{1-q}{1-\alpha q^2}q\right\}\\
&\quad\cup\left\{\frac{1}{\sum\limits_{k=0}^{1}q^k},\frac{1}{\sum\limits_{k=0}^{2}q^k},\ldots,\frac{1}{\sum\limits_{k=0}^{m}q^k}:\ \alpha<q^m\right\}\cup\{1\}\\
&=\left\{\lambda\in\mathbb{C}:\left|\lambda-\tfrac{1}{1+q}\right|\leq\tfrac{q}{1+q}\right\}.
\end{aligned}$$

This shows that [17, Theorem 2.6] is still valid when as $\alpha\to1^-$.

Theorem 3.10. *If* $0<q<1$ *and* $0<\alpha<1$*, then*

$$\sigma_r\left(C_q^\alpha,c\right)=\left\{\lambda\in\mathbb{C}:\left|\lambda-\frac{1-q}{1-\alpha^2q^2}\right|<\frac{1-q}{1-\alpha^2q^2}\alpha q\right\}\cup\{1\}.$$

Proof. We know from Theorem 3.2 and 3.5 that

$$\sigma_p\left(C_q^{\alpha}, c\right) = \left\{\frac{1}{1+q}, \ldots, \frac{1}{1+q+\cdots+q^m} : \alpha < q^m\right\} \cup \{1\},$$

$$\begin{aligned}\sigma_p\left(\left[C_q^{\alpha}\right]^*, (c)^* \simeq \ell^1\right) &= \left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1-q}{1-\alpha^2 q^2}\right| < \frac{1-q}{1-\alpha^2 q^2}\alpha q\right\} \\ &\cup \left\{\frac{1}{\sum\limits_{k=0}^{1} q^k}, \ldots, \frac{1}{\sum\limits_{k=0}^{m} q^k} : \ \alpha < q^m\right\} \cup \{1\}.\end{aligned}$$

Therefore we get

$$\sigma_r\left(C_q^{\alpha}, c\right) = \left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1-q}{1-\alpha^2 q^2}\right| < \frac{1-q}{1-\alpha^2 q^2}\alpha q\right\} \cup \{1\},$$

because $\sigma_r\left(C_q^{\alpha}, c\right) = \sigma_p\left(C_1^*(q), \ell^1\right) \backslash \sigma_p\left(C_q^{\alpha}, c\right)$. □

Theorem 3.11. *If* $0 < q < 1$ *and* $0 < \alpha < 1$ *and* $\alpha = q^m$, *then*

$$\begin{aligned}\sigma_c\left(C_q^{\alpha}, c\right) &= \left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1-q}{1-\alpha^2 q^2}\right| = \frac{1-q}{1-\alpha^2 q^2}\alpha q\right\} \\ &\backslash \left\{1, \tfrac{1}{1+q}, \ldots, \tfrac{1}{1+q+\cdots+q^m} : \alpha < q^m\right\}\end{aligned}$$

and if $\alpha \neq q^m$, *then*

$$\sigma_c\left(C_q^{\alpha}, c\right) = \left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1-q}{1-\alpha^2 q^2}\right| = \frac{1-q}{1-\alpha^2 q^2}\alpha q\right\}.$$

Proof. Since

$$\sigma_c\left(C_q^{\alpha}, c\right) = \sigma\left(C_1(q), c\right) \backslash \left\{\left\{\sigma_p\left(C_1(q), c\right) \cup \sigma_r\left(C_q^{\alpha}, c\right)\right\}\right\},$$

the result can be seen immediately from Theorems 3.2, 3.8, 3.10 and Table 2. □

Now let's do the Goldberg classification for the spectrum of operator C_q^{α}.

Lemma 3.12. *[24, p.60]A linear operator* T *has a bounded inverse if and only if* T^* *is onto.*

Theorem 3.13. *If $0 < q < 1$ and $0 < \alpha < 1$, then*

$$\mathcal{III}_2\sigma\left(C_q^\alpha, c\right) = \left\{ \frac{1}{\sum\limits_{k=0}^{m+1} q^k}, \frac{1}{\sum\limits_{k=0}^{m+2} q^k}, \ldots \right\}$$

where $\alpha < q^m$.

Proof. Let $\alpha < q^m$. Hence $\frac{1}{\sum_{k=0}^{m+1} q^k}, \frac{1}{\sum_{k=0}^{m+2} q^k}, \ldots \in \left\{ \lambda \in \mathbb{C} : \left| \lambda - \frac{1-q}{1-\alpha^2 q^2} \right| < \frac{1-q}{1-\alpha^2 q^2} \alpha q \right\}$. Therefore $\left\{ \frac{1}{\sum_{k=0}^{m+1} q^k}, \frac{1}{\sum_{k=0}^{m+2} q^k}, \ldots \right\} \subseteq \sigma_r\left(C_q^\alpha, c\right) = \mathcal{III}_1\sigma\left(C_q^\alpha, c\right) \cup \mathcal{III}_2\sigma\left(C_q^\alpha, c\right)$ from Theorem 3.10. Is it $\left\{ \frac{1}{\sum_{k=0}^{m+1} q^k}, \frac{1}{\sum_{k=0}^{m+2} q^k}, \ldots \right\} \subset \mathcal{III}_2\sigma\left(C_q^\alpha, c\right)$? It is sufficient to show that operator $\left(C_q^\alpha - \lambda I\right)^*$ from Lemma 3.12 is surjective. If $\left(C_q^\alpha - \lambda I\right)^* x = y$, then we get

$$\begin{aligned}
(1-\lambda) x_0 &= y_0 \\
(1-\lambda) x_1 + \frac{\alpha q}{\sum\limits_{k=0}^{1} q^k} x_2 + \frac{(\alpha q)^2}{\sum\limits_{k=0}^{2} q^k} x_3 + \frac{(\alpha q)^3}{\sum\limits_{k=0}^{3} q^k} x_4 + \cdots &= y_1 \\
\left(\frac{1}{\sum\limits_{k=0}^{1} q^k} - \lambda \right) x_2 + \frac{(\alpha q)}{\sum\limits_{k=0}^{2} q^k} x_3 + \frac{(\alpha q)^2}{\sum\limits_{k=0}^{3} q^k} x_4 + \cdots &= y_2 \\
&\vdots
\end{aligned} \tag{3.15}$$

Thus we get

$$\begin{aligned}
x_n &= \frac{x_0}{(\alpha q \lambda)^{n-1}} \prod_{i=0}^{n-1} \left(\lambda - \frac{1}{\sum\limits_{k=0}^{i} q^k} \right) + \frac{y_0}{(\lambda q \alpha)^{n-1}} \prod_{i=1}^{n-1} \left(\lambda - \frac{1}{\sum\limits_{k=0}^{i} q^k} \right) \\
&\quad + \sum_{i=1}^{n-1} \frac{1}{\lambda \sum\limits_{k=0}^{i} q^k} \frac{y_i}{(q\alpha)^{n-i}} \prod_{v=i+1}^{n-1} \left(\lambda - \frac{1}{\sum\limits_{k=0}^{v} q^k} \right) - \frac{1}{\lambda} y_{n+1}.
\end{aligned} \tag{3.16}$$

from (3.15). Since $\lambda \in \left\{ \frac{1}{\sum\limits_{k=0}^{m+1} q^k}, \frac{1}{\sum\limits_{k=0}^{m+2} q^k}, \ldots, \frac{1}{\sum\limits_{k=0}^{n} q^k} \right\}$ ve $(y_n) \in \ell^1$, we have $x = (x_n) \in \ell^1$ where $x_n = -\frac{1}{\lambda} y_n$. Hence the operator $\left(C_q^\alpha - \lambda I\right)^*$ is surjective. For these λ's, the $C_q^\alpha - \lambda I$ operator has bounded inverse. $\frac{1}{\sum_{k=0}^{s} q^k} \in \mathcal{III}_2\sigma\left(C_q^\alpha, c\right)$ for $s > m$.

Let us now assume that $\lambda \notin \left\{ \frac{1}{\sum_{k=0}^{m+1} q^k}, \frac{1}{\sum_{k=0}^{m+2} q^k}, \ldots \right\}$. Since $\frac{1}{\sum_{k=0}^{0} q^k}$, $\frac{1}{\sum_{k=0}^{1} q^k}$, $\ldots$, $\frac{1}{\sum_{k=0}^{m} q^k} \notin \mathcal{III}$ from Table 2, it is $\lambda \neq \frac{1}{\sum_{k=0}^{n} q^k}$ for each $n \in \mathbb{N}$. From here, since $y \in \ell^1$ is in equation (3.16), "sequence of (x_n) is convergent if and only if $\prod_{v=0}^{\infty} \left(\lambda - \dfrac{1}{\sum_{k=0}^{v} q^k} \right)$ must be convergent. Thus, the limit of the general term of the sequence in the infinite product must be 1. From here, we get $\lambda = 2 - q$ since

$$\lim_{v \to \infty} \left(\lambda - \frac{1}{\sum\limits_{k=0}^{v} q^k} \right) = \lim_{v \to \infty} \lambda - \frac{1-q}{1-q^{v+1}} = \lambda - (1-q) = 1.$$

Therefore, since the infinite product for $\lambda \neq 2 - q$ will be divergent, it is $x \notin \ell^1$. Thus, while

$$\lambda \notin \left\{ \frac{1}{\sum\limits_{k=0}^{m+1} q^k}, \frac{1}{\sum\limits_{k=0}^{m+2} q^k}, \ldots \right\} \cup \{2-q\}, \tag{3.17}$$

the $\left(C_q^\alpha - \alpha I\right)^*$ operator is not surjective. From here (3.17) for λ's, operator $C_q^\alpha - \lambda I$ has no bounded inverse. So

$$\left\{ \frac{1}{\sum\limits_{k=0}^{m+1} q^k}, \frac{1}{\sum\limits_{k=0}^{m+2} q^k}, \ldots \right\} \subseteq \mathcal{III}_2\sigma\left(C_q^\alpha, c\right) \subseteq \left\{ \frac{1}{\sum\limits_{k=0}^{m+1} q^k}, \frac{1}{\sum\limits_{k=0}^{m+2} q^k}, \ldots \right\} \cup \{2-q\}$$

is valid. If we take $\lambda = 2 - q$, the first component of (x_n) in (3.16) becomes

$$\frac{x_0}{\left(\alpha q \left(2-q\right)\right)^n} \prod_{v=0}^{n-1} \left(\lambda - \frac{1}{\sum\limits_{k=0}^{v} q^k} \right). \tag{3.18}$$

Since $0<\alpha<1$ and $q<1$, $0<\alpha q\left(2-q\right)<1$ is valid. Hence $\dfrac{1}{\left(\alpha q\left(2-q\right)\right)^{n}}\to\infty$. Thus, even if the infinite product in (3.18) is finite, $x\notin c$ is. From here, $x\notin\ell^{1}$ is obtained. Thus, if $\lambda=2-q$, operator $\left(C_{q}^{\alpha}-\lambda I\right)^{*}$ is not surjective; i.e, operator $C_{q}^{\alpha}-\lambda I$ from Lemma 3.12 does not have bounded inverse. As a result,

$$\mathcal{III}_{2}\sigma\left(C_{q}^{\alpha},c\right)=\left\{\frac{1}{\sum\limits_{k=0}^{m+1}q^{k}},\frac{1}{\sum\limits_{k=0}^{m+2}q^{k}},\ldots\right\}$$

is obtained. □

Corollary 3.14. *Let $0<q<1$ and $0<\alpha<1$. Then*

$$\mathcal{III}_{1}\sigma\left(C_{q}^{\alpha},c\right)=\left\{\lambda\in\mathbb{C}:\left|\lambda-\frac{1-q}{1-\alpha^{2}q^{2}}\right|<\frac{1-q}{1-\alpha^{2}q^{2}}\alpha q\right\}\setminus\left\{\frac{1}{\sum\limits_{k=0}^{m+1}q^{k}},\frac{1}{\sum\limits_{k=0}^{m+2}q^{k}},\ldots\right\}$$

where $\alpha<q^{m}$.

Proof. Since $\sigma_{r}(C_{q}^{\alpha},c)=\mathcal{III}_{1}\sigma\left(C_{q}^{\alpha},c\right)\cup\mathcal{III}_{2}\sigma\left(C_{q}^{\alpha},c\right)$, the proof from Theorem 3.10 and Theorem 3.13 is clear. □

The following Corollary is immediately seen from Theorem 3.11.

Corollary 3.15. *Let $0<q<1$ and $0<\alpha<1$. If $\alpha=q^{m}$, then*

$$\mathcal{II}_{2}\sigma\left(C_{q}^{\alpha},c\right)=\left\{\lambda\in\mathbb{C}:\left|\lambda-\frac{1-q}{1-\alpha^{2}q^{2}}\right|=\frac{1-q}{1-\alpha^{2}q^{2}}\alpha q\right\}\setminus\left\{\frac{1}{\sum\limits_{k=0}^{m}q^{k}}\right\}$$

and if $\alpha\neq q^{m}$, then

$$\mathcal{II}_{2}\sigma\left(C_{q}^{\alpha},c\right)=\left\{\lambda\in\mathbb{C}:\left|\lambda-\frac{1-q}{1-\alpha^{2}q^{2}}\right|=\frac{1-q}{1-\alpha^{2}q^{2}}\alpha q\right\}.$$

Let us give the following Lemma, which we will use in the proof of the next Theorem.

Lemma 3.16. *[24, Theorem II 3.7] A linear operator T has a dense range if and only if the adjoint operator T^* is one to one.*

Theorem 3.17. *Let $0 < q < 1$ and $0 < \alpha < 1$. Then*

$$\mathcal{III}_3\sigma\left(C_q^\alpha, c\right) = \left\{\frac{1}{\sum\limits_{k=0}^{1} q^k}, \frac{1}{\sum\limits_{k=0}^{2} q^k}, \ldots, \frac{1}{\sum\limits_{k=0}^{m} q^k}\right\} \cup \{1\}$$

where $\alpha < q^m$.

Proof. We know from Table 2 and Theorem 3.2 that

$$\begin{aligned}\sigma_p\left(C_q^\alpha, c\right) &= \mathcal{I}_3\sigma\left(C_q^\alpha, c\right) \cup \mathcal{II}_3\sigma\left(C_q^\alpha, c\right) \cup \mathcal{III}_3\sigma\left(C_q^\alpha, c\right) \\ &= \left\{\frac{1}{\sum\limits_{k=0}^{1} q^k}, \frac{1}{\sum\limits_{k=0}^{2} q^k}, \ldots, \frac{1}{\sum\limits_{k=0}^{m} q^k}\right\} \cup \{1\}\end{aligned}$$

for $\alpha < q^m$. Let $\left(C_q^\alpha - I\right)^* x = \theta$ and $x_0 = 1$. Thus, we have

$$\begin{aligned}x_1 &= \frac{1}{\lambda\alpha q}\left(\lambda - \frac{1}{\sum\limits_{k=0}^{0} q^k}\right) \\ x_2 &= \frac{1}{(\lambda\alpha q)^2}\left(\lambda - \frac{1}{\sum\limits_{k=0}^{0} q^k}\right)\left(\lambda - \frac{1}{\sum\limits_{k=0}^{1} q^k}\right) \\ &\vdots \\ x_m &= \frac{1}{(\lambda\alpha q)^m}\prod_{k=0}^{m}\left(\lambda - \frac{1}{\sum\limits_{k=0}^{k} q^k}\right).\end{aligned}$$

From these equations the following facts are obtained. Since $x^0 = (1, 0, 0, \ldots) \neq \theta$ and $x^0 \in Ker\left(C_q^\alpha - I\right)$, the $\left(C_q^\alpha - I\right)^*$ operator is not one-to-one. So it is $1 \in \mathcal{III}_3\sigma\left(C_q^\alpha, c\right)$. Since $x^1 = \left(1, \frac{1+q}{\alpha q}\left(\frac{1}{1+q} - 1\right), 0, 0, \ldots\right) \neq \theta$ and

$x^1 \in Ker\left(C_q^\alpha - \frac{1}{1+q}I\right)$, the $\left(C_q^\alpha - \frac{1}{1+q}I\right)^*$ operator is not one-to-one. So it is $\frac{1}{1+q} \in \mathcal{III}_3\sigma\left(C_q^\alpha, c\right)$. Continuing this way, since

$$\begin{aligned} x^m &= \left(1, \frac{1+q+\cdots+q^m}{\alpha q}\left(\frac{1}{\sum\limits_{k=0}^{m} q^k} - 1\right), \frac{(1+q+\cdots+q^m)^2}{(\alpha q)^2}\left(\frac{1}{\sum\limits_{k=0}^{m} q^k} - 1\right)\left(\frac{1}{\sum\limits_{k=0}^{m} q^k} - \frac{1}{\sum\limits_{k=0}^{1} q^k}\right),\right. \\ &\quad \left. \ldots, \frac{(1+q+\cdots+q^m)^2}{(\alpha q)^2}\prod_{k=0}^{m}\left(\frac{1}{\sum\limits_{k=0}^{m} q^k} - \frac{1}{\sum\limits_{k=0}^{k} q^k}\right), 0, \ldots\right) \\ &\quad \neq \theta, \end{aligned}$$

and $x^m \in Ker\left(C_q^\alpha - \frac{1}{1+q}I\right)$, the $\left(C_q^\alpha - \frac{1}{\sum_{k=0}^m q^k}I\right)^*$ operator is not one-to-one. So it is $\frac{1}{\sum_{k=0}^m q^k} \in \mathcal{III}_3\sigma\left(C_q^\alpha, c\right)$. Consequently, we have

$$\mathcal{III}_3\sigma\left(C_q^\alpha, c\right) = \left\{1, \frac{1}{\sum\limits_{k=0}^{1} q^k}, \frac{1}{\sum\limits_{k=0}^{2} q^k}, \ldots, \frac{1}{\sum\limits_{k=0}^{m} q^k}\right\}$$

for $\alpha < q^m$. □

Corollary 3.18. *Let* $0 < q < 1$ *and* $0 < \alpha < 1$. $\mathcal{I}_3\sigma\left(C_q^\alpha, c\right) = \mathcal{II}_3\sigma\left(C_q^\alpha, c\right) = \emptyset$.

Proof. The proof is clear from Table 2, Theorem 3.2 and Theorem 3.17. □

Now, let's determine the defect spectrum, the approximate point spectrum, the compression spectrum of the operator C_q^α.

Theorem 3.19. *Let* $0 < q < 1$ *and* $0 < \alpha < 1$. *The following expressions are hold;*

(a)

$$\begin{aligned} \sigma_{ap}\left(C_q^\alpha, c\right) &= \left\{\lambda \in \mathbb{C} : \left|\lambda - \frac{1-q}{1-\alpha^2 q^2}\right| = \frac{(1-q)\,\alpha q}{1-\alpha^2 q^2}\right\} \\ &\quad \cup \left\{1, \tfrac{1}{1+q}, \tfrac{1}{1+q+q^2}, \ldots\right\}, \ \textit{for } \alpha < q^m. \end{aligned}$$

(b)

$$\begin{aligned}\sigma_{\delta}\left(C_q^{\alpha},c\right) &= \left\{\lambda\in\mathbb{C}:\left|\lambda-\frac{1-q}{1-\alpha^2q^2}\right|\leq\frac{(1-q)\,\alpha q}{1-\alpha^2q^2}\right\}\\ &\cup\left\{1,\frac{1}{\sum\limits_{k=0}^{1}q^k},\ldots,\frac{1}{\sum\limits_{k=0}^{m}q^k}\right\},\ \textit{for } \alpha<q^m.\end{aligned}$$

(c)

$$\begin{aligned}\sigma_{co}\left(C_q^{\alpha},c\right) &= \left\{\lambda\in\mathbb{C}:\left|\lambda-\frac{1-q}{1-\alpha^2q^2}\right|<\frac{(1-q)\,\alpha q}{1-\alpha^2q^2}\right\}\\ &\cup\left\{\frac{1}{\sum\limits_{k=0}^{0}q^k},\frac{1}{\sum\limits_{k=0}^{1}q^k},\ldots,\frac{1}{\sum\limits_{k=0}^{m}q^k}\right\},\ \textit{for } \alpha<q^m.\end{aligned}$$

Proof. (a) The desired result is obtained by using Table 2, Theorem 3.8, Corollary 3.14 and the expression $\sigma_{ap}\left(C_q^{\alpha},c\right)=\sigma\left(C_q^{\alpha},c\right)\backslash\mathcal{III}_1\sigma\left(C_q^{\alpha},c\right)$.

(b) The desired result is obtained by using Theorem 3.13, Table 2, Theorem 3.8, Corollary 3.18 and the expression $\sigma_{\delta}\left(C_q^{\alpha},c\right)=\sigma\left(C_q^{\alpha},c\right)\backslash\mathcal{I}_3\sigma\left(C_q^{\alpha},c\right)$.

(c) The desired result is obtained by using Theorem 3.13, Table 2, Theorem 3.17, Corollary 3.14 and the expression $\sigma_{co}\left(C_q^{\alpha},c\right)=\mathcal{III}_1\sigma\left(C_q^{\alpha},c\right)\cup\mathcal{III}_2\sigma\left(C_q^{\alpha},c\right)\cup\mathcal{III}_3\sigma\left(C_q^{\alpha},c\right)$. □

Corollary 3.20. *Let* $0<q<1$ *and* $0<\alpha<1$.

(a)

$$\begin{aligned}\sigma_{ap}\left(\left[C_q^{\alpha}\right]^*,\ell^1\right) &= \left\{\lambda\in\mathbb{C}:\left|\lambda-\frac{1-q}{1-\alpha^2q^2}\right|\leq\frac{(1-q)\,\alpha q}{1-\alpha^2q^2}\right\}\\ &\cup\left\{1,\frac{1}{\sum\limits_{k=0}^{1}q^k},\ldots,\frac{1}{\sum\limits_{k=0}^{m}q^k}\right\}\ \textit{for } \alpha<q^m,\end{aligned}$$

(b)

$$\begin{aligned}\sigma_{\delta}\left(C_1^{\alpha*}(q),\ell^1\right) &= \left\{\lambda\in\mathbb{C}:\left|\lambda-\frac{1-q}{1-\alpha^2q^2}\right|=\frac{(1-q)\,\alpha q}{1-\alpha^2q^2}\right\}\\ &\cup\left\{1,\tfrac{1}{1+q},\tfrac{1}{1+q+q^2},\ldots\right\}\ \textit{for } \alpha<q^m.\end{aligned}$$

Proof. The expressions $\sigma_{ap}(\left[C_q^{\alpha}\right]^*,\ell^1)=\sigma_{\delta}(C_q^{\alpha},c)$ and $\sigma_{\delta}(\left[C_q^{\alpha}\right]^*,\ell^1)=\sigma_{ap}(C_q^{\alpha},c)$ are known from [4]. Thus, the desired result is obtained from Theorem 3.19. □

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