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Generalized Jordan Derivations of Incidence Algebras

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Abstract

For a given ring $\mathfrak{R}$ and a locally finite pre-ordered set $(X, \leq)$, consider $I(X, \mathfrak{R})$ to be the incidence algebra of X over $\mathfrak{R}$. Motivated by a Xiao's result which states that every Jordan derivation of $I(X, \mathfrak{R})$ is a derivation in the case $\mathfrak{R}$ is 2-torsion free, one proves that each generalized Jordan derivation of $I(X, \mathfrak{R})$ is a generalized derivation provided $\mathfrak{R}$ is 2-torsion free, getting as a consequence the above mentioned result.

1. Introduction

For a given ring $\mathfrak{R}$, recall that a linear map d from $\mathfrak{R}$ into itself is called a derivation if $d(ab) = d(a)b + ad(b)$ for all $a, b \in \mathfrak{R}$; and a Jordan derivation if $d(a^2) = d(a)a + ad(a)$ for each $a \in \mathfrak{R}$. More generally [5], if there is a derivation $\tau : \mathfrak{R} \to \mathfrak{R}$ such that $d(ab) = d(a)b + a\tau(b)$ for all $a, b \in \mathfrak{R}$, then d is called a generalized derivation and τ is the relating derivation; analogously, if there is a Jordan derivation

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$\tau : \mathfrak{R} \to \mathfrak{R}$ such that $d(a^2) = d(a)a + a\tau(a)$ for all $a \in \mathfrak{R}$, then d is called a generalized Jordan derivation and τ is the relating Jordan derivation. The structures of derivations, Jordan derivations, generalized derivations and generalized Jordan derivations were systematically studied. It is obvious that every generalized derivation is a generalized Jordan derivation and every derivation is a Jordan derivation. But the converse is in general not true. Herstein [4] showed that every Jordan derivation from a 2-torsion free prime ring into itself is a derivation. Brešar [2] proved that Herstein's result is true for 2-torsion free semiprime rings. Jing and Lu, motivated by the concept of generalized derivation, introduce this concept of generalized Jordan derivation in [5].

Let us now recall the notion of incidence algebra [7], [12], which we deal in this paper. Let $(X, \leq)$ be a locally finite pre-ordered set. This means $\leq$ is a reflexive and transitive binary relation on the set X, and for any $x \leq y$ in X there are only finitely many elements z satisfying $x \leq z \leq y$. The incidence algebra $I(X, \mathfrak{R})$ of X over $\mathfrak{R}$ is defined as the set

$$I(X, \mathfrak{R}) := \{f : X \times X \to \mathfrak{R} \mid f(x, y) = 0 \text{ if } x \not\leq y\}$$

with algebraic operation given by

$$(f + g)(x, y) = f(x, y) + g(x, y),$$

$$(rf)(x, y) = rf(x, y),$$

$$(fg)(x, y) = \sum_{x \leq z \leq y} f(x, z)g(z, y)$$

for all $f, g \in I(X, \mathfrak{R}), r \in \mathfrak{R}$ and $x, y, z \in X$. The product fg is usually called convolution in function theory. It would be helpful to point out that the full matrix algebra $M_n(\mathfrak{R})$ and the upper (or lower) triangular matrix algebras $T_n(\mathfrak{R})$ are special examples of incidence algebras. The identity element δ of $I(X, \mathfrak{R})$ is given by $\delta(x, y) = \delta_{xy}$ for $x \leq y$, where $\delta_{xy} \in \{0, 1\}$ is the Kronecker delta. For given $x, y \in X$ with $x \leq y$, let e_{xy} be defined by $e_{xy}(u, v) = 1$ if $(u, v) = (x, y)$, and $e_{xy}(u, v) = 0$ otherwise. Then $e_{xy}e_{uv} = \delta_{yu}e_{xv}$ by the definition of convolution. Moreover, the set $B := \{e_{xy} \mid x \leq y\}$ forms an $\mathfrak{R}$-linear basis of $I(X, \mathfrak{R})$. Note that incidence algebras allow infinite summation, and hence the $\mathfrak{R}$-linear map here means a map preserving in finite sum and scalar multiplication.

Incidence algebras were first considered by Ward [15] as generalized algebras of arithmetic functions. Rota and Stanley developed incidence algebras as the fundamental structures of enumerative combinatorial theory and allied areas of arithmetic function theory (see [11]). Motivated by the results of Stanley [13], automorphisms and other algebraic mappings of incidence algebras have been extensively studied (see [1], [3], [6], [7], [8], [9], [10], [11] and the references therein). Baclawski [1] studied the automorphisms and derivations of incidence algebras $I(X, \mathfrak{R})$ when X is a locally finite partially ordered set. More specifically, he proved that every derivation of $I(X, \mathfrak{R})$ with X a locally finite partially ordered set can be decomposed as a sum of an inner derivation and a transitive induced derivation. Koppinen [7] has extended these results to the incidence algebras $I(X, \mathfrak{R})$ with X a locally finite pre-ordered set. Xiao [14] proved that every Jordan derivation of $I(X, \mathfrak{R})$ is a derivation provided that $\mathfrak{R}$ is 2-torsion free. Motivated by Xiao's result our main objective is to prove that every generalized Jordan derivation of $I(X, \mathfrak{R})$ is a generalized derivation provided that $\mathfrak{R}$ is 2-torsion free.

2. Results

We first collect some background material to prove our main result. Throughout this section, $\mathfrak{R}$ denotes a 2-torsion free ring. Let $\Xi : I(X, \mathfrak{R}) \to I(X, \mathfrak{R})$ be a generalized Jordan derivation and $\tau : I(X, \mathfrak{R}) \to I(X, \mathfrak{R})$ the relating Jordan derivation.

Lemma 2.1. *For all* $a, b, c \in I(X, \mathfrak{R})$, *the following statements hold*:

(1) $\Xi(ab + ba) = \Xi(a)b + a\tau(b) + \Xi(b)a + b\tau(a)$,

(2) $\Xi(aba) = \Xi(a)ba + a\tau(b)a + ab\tau(a)$,

(3) $\Xi(abc + cba) = \Xi(a)bc + a\tau(b)c + ab\tau(c) + \Xi(c)ba + c\tau(b)a + cb\tau(a)$.

Proof. See [5]. □

According to Lemma 2.1, $\Xi(aba) = \Xi(a)ba + a\tau(b)a + ab\tau(a)$. In the case $ab = ba = 0$, we obtain $a\tau(b)a = 0$. Furthermore, it follows that

$$\Xi(e) = \Xi(e)e + e\tau(e), \tag{1}$$

for any idempotent $e \in I(X, \mathfrak{R})$. In particular, since (1), $e\tau(a)e = 0$, for any $a \in I(X, \mathfrak{R})$ satisfying $ea = ae = 0$, and $\Xi(a)e + a\tau(e) + \Xi(e)a + e\tau(a) = 0$. Multiplying by e on the right yields

$$\Xi(a)e + a\tau(e) = 0 = \Xi(e)a + e\tau(a), \tag{2}$$

for any idempotent e satisfying $ea = ae = 0$.

Now assume that the set $B := \{e_{xy} \mid x \leq y\}$ forms an $\mathfrak{R}$-linear basis of $I(X, \mathfrak{R})$. It is a consequence of (1) that

$$\Xi(e_{ii}) = \Xi(e_{ii})e_{ii} + e_{ii}\tau(e_{ii}) \text{ and } e_{ki}\tau(e_{ii})e_{ij} = 0, \tag{3}$$

for all i and $k \leq i \leq j$. From Lemma 2.1 and the fact that $\Xi(e_{ij}) = \Xi(e_{ii}e_{ij} + e_{ij}e_{ii})$ for all $1 \leq i < j \leq n$, we obtain

$$\Xi(e_{ij}) = \Xi(e_{ii})e_{ij} + e_{ii}\tau(e_{ij}) + \Xi(e_{ij})e_{ii} + e_{ij}\tau(e_{ii}) \tag{4}$$

whenever $i < j$. Furthermore (2) implies that

$$\Xi(e_{kj})e_{ii} + e_{kj}\Xi(e_{ii}) = \Xi(e_{ii})e_{kj} + e_{ii}\tau(e_{kj}) = 0 \tag{5}$$

for all $k, j \neq i$. Define a $\mathfrak{R}$-linear map ϕ from $I(X, \mathfrak{R})$ into itself by letting

$$\phi(e_{ij}) = \Xi(e_{ii})e_{ij} + e_{ii}\tau(e_{ij}), \quad i \leq j. \tag{6}$$

According to (3), $\phi(e_{ii}) = \Xi(e_{ii})$. Xiao proved the following result.

Lemma 2.2 (Lemma 3.2 [14]). *Let $\tau : I(X, \mathfrak{R}) \to I(X, \mathfrak{R})$ be a Jordan derivation. Then*

$$\tau(e_{ij}) = \sum_{x \in L_i} C_{xi}^{ii} e_{xj} + C_{ij}^{ij} e_{ij} + \sum_{y \in R_j} C_{jy}^{jj} e_{iy} + C_{ji}^{ij} e_{ji}$$

for all $e_{ij} \in B$, where the coefficients C_{xy}^{ij} are subject to the following relations

$$C_{jk}^{jj} + C_{jk}^{kk} = 0, \text{ if } j \leq k;$$

$$C_{ij}^{ij} + C_{jk}^{jk} = C_{ik}^{ik}, \text{ if } i \leq j, \ j \leq k.$$

Lemma 2.3. ϕ *is a generalized derivation.*

Proof. Let us consider $d(e_{ij}) = \sum_{x \in L_i} C_{xi}^{ii} e_{xj} + C_{ij}^{ij} e_{ij} + \sum_{y \in R_j} C_{jy}^{jj} e_{iy}$ for all $e_{ij} \in B$, where the coefficients C_{xy}^{ij} are subject to the following relations

$$C_{jk}^{jj} + C_{jk}^{kk} = 0, \quad \text{if } j \leq k;$$

$$C_{ij}^{ij} + C_{jk}^{jk} = C_{ik}^{ik}, \quad \text{if } i \leq j,\ j \leq k.$$

By [14, Theorem 2.2] d is a derivation. First we check that

$$\phi(e_{ij} e_{kl}) = \phi(e_{ij}) e_{kl} + e_{ij} d(e_{kl}), \tag{7}$$

for all $e_{ij}, e_{kl} \in B$. We split the argument into two cases.

Case 1: $j \neq k$. Since $\phi(e_{ij} e_{kl}) = 0$, it suffices to prove that $\phi(e_{ij}) e_{kl} + e_{ij} d(e_{kl}) = 0$. By (6) we get

$$\begin{aligned} \phi(e_{ij}) e_{kl} + e_{ij} d(e_{kl}) &= (\Xi(e_{ii}) e_{ij} + e_{ii} \tau(e_{ij})) e_{kl} + e_{ij} d(e_{kl}) \\ &= e_{ii} \tau(e_{ij}) e_{kl} + e_{ij} d(e_{kl}). \end{aligned}$$

If $i \neq k$, then

$$\begin{aligned} e_{ii} \tau(e_{ij}) e_{kl} + e_{ij} d(e_{kl}) &= e_{ii} \tau(e_{ij}) e_{kl} + e_{ij} d(e_{kk}) e_{kl} \\ &= e_{ii} (\tau(e_{ij}) e_{kk} + e_{ij} d(e_{kk})) e_{kl} \\ &= e_{ii} 0 e_{kl} \\ &= 0, \end{aligned}$$

by Lemma 2.2 and $\tau(e_{ij}) e_{kk} = \tau(e_{ij} e_{kk}) - e_{ij} \tau(e_{kk})$. Finally, if $i = k$, then

$$\begin{aligned} e_{ii} \tau(e_{ij}) e_{il} + e_{ij} d(e_{il}) &= e_{ii} \tau(e_{ij}) e_{il} + e_{ij} d(e_{ii} e_{il}) \\ &= e_{ii} \tau(e_{ij}) e_{il} + e_{ij} d(e_{ii}) e_{il} \end{aligned}$$

$$= (e_{ii}\tau(e_{ij}) + e_{ij}d(e_{ii})e_{il}$$

$$= (\tau(e_{ij}) - \tau(e_{ii})e_{ij}$$

$$- \tau(e_{ij})e_{ii} - e_{ij}\tau(e_{ii}) + e_{ij}d(e_{ii}))e_{il}$$

$$= e_{ij}(d(e_{ii}) - \tau(e_{ii}))e_{il} = 0.$$

Case 2: $j = k$. We must prove that

$$\phi(e_{il}) = \phi(e_{ij})e_{jl} + e_{ij}d(e_{jl}).$$

Assume $i < j < l$. As a consequence of (6),

$$\phi(e_{ij})e_{jl} + e_{ij}d(e_{jl}) = (\Xi(e_{ii})e_{ij} + e_{ii}\tau(e_{ij}))e_{jl} + e_{ij}d(e_{jl})$$

$$= \phi(e_{il}) - e_{ii}(\tau(e_{il}) - \tau(e_{ij})e_{jl} - e_{ij}d(e_{jl}))$$

$$= \phi(e_{il}) - e_{ii}(e_{ij}\tau(e_{jl}) + \tau(e_{jl})e_{ij} + e_{jl}\tau(e_{ij}) - e_{ij}d(e_{jl}))$$

$$= \phi(e_{il}) - e_{ij}(\tau(e_{jl}) - d(e_{jl})) = \phi(e_{il}).$$

If $i = j < l$, then

$$\phi(e_{ii})e_{il} + e_{ii}d(e_{il}) = \Xi(e_{ii})e_{il} + e_{ii}\tau(e_{il}) + e_{ii}d(e_{il}) - e_{ii}\tau(e_{il})$$

$$= \Xi(e_{ii})e_{il} + e_{ii}\tau(e_{il}) = \phi(e_{il}).$$

If $i < j = l$, then

$$\phi(e_{ij})e_{jj} + e_{ij}d(e_{jj}) = (\Xi(e_{ii})e_{ij} + e_{ii}\tau(e_{ij}))e_{jj} + e_{ij}d(e_{jj})$$

$$= \Xi(e_{ii})e_{ij} + e_{ii}\tau(e_{ij}) + e_{ii}\tau(e_{ij})e_{jj}$$

$$- e_{ii}\tau(e_{ij}) + e_{ij}d(e_{jj}).$$

Since $e_{ii}\tau(e_{ij})e_{jj} = C_{ij}^{ij}e_{jj}$,

$$e_{ii}\tau(e_{ij}) = C_{ij}^{ij}e_{ij} + \sum_{y \in R_j} C_{jy}^{jj}e_{iy}$$

and

$$e_{ij}d(e_{jj}) = C_{jj}^{ij}e_{ij} + \sum_{y \in R_j} C_{jy}^{jj}e_{iy}$$

it follows that $e_{ii}\tau(e_{ij})e_{jj} - e_{ii}\tau(e_{ij}) + e_{ij}d(e_{jj}) = 0$. Hence $\phi(e_{ij})e_{jj} + e_{ij}d(e_{jj}) = \Xi(e_{ii})e_{ij} + e_{ii}\tau(e_{ij}) = \phi(e_{ij})$. If $i = j = l$, by (3) we obtain $\phi(e_{ii}) = \Xi(e_{ii}) = \Xi(e_{ii})e_{ii} + e_{ii}\tau(e_{ii}) = \phi(e_{ii})e_{ii} + e_{ii}d(e_{ii})$. Thus, for all $e_{ij}, e_{kl} \in B$, we get $\phi(e_{ij}e_{kl}) = \phi(e_{ij})e_{kl} + e_{ij}d(e_{kl})$. Finally, linearity of ϕ yields $\phi(ab) = \phi(a)b + ad(b)$ for all $a, b \in I(X, \mathfrak{R})$, which proves that ϕ is a generalized derivation. □

We are now in a position to prove the main result of this paper.

Theorem 2.1. *Let $\mathfrak{R}$ be a 2-torsion free commutative ring with identity. Then any generalized Jordan derivation of the incidence algebra $I(X, \mathfrak{R})$ is a generalized derivation.*

Proof. Put $\Psi = \Xi - \phi$, then $\Psi(e_{ij}) = \Xi(e_{ij}) - \phi(e_{ij})$ and $\Psi(e_{ii}) = \Xi(e_{ii}) - \phi(e_{ii}) = 0$ for all $e_{ii} \in B$. Since Ψ is a generalized Jordan derivation, $\Psi(e_{ij}) = \Psi(e_{ij}e_{jj} + e_{jj}e_{ij}) = \Psi(e_{ij})e_{jj} + \Psi(e_{jj})e_{ij} = \Psi(e_{ij})e_{jj}$. According to (4) and (6), if $i < j$ we have

$$\begin{aligned}\Psi(e_{ij}) &= \Xi(e_{ij})e_{ii} + e_{ij}\tau(e_{ii}) \\ &= (\phi(e_{ij}) + \Psi(e_{ij}))e_{ii} + e_{ij}\tau(e_{ii}) \\ &= \phi(e_{ij})e_{ii} + e_{ij}\tau(e_{ii}) + \Psi(e_{ij})e_{ii} \\ &= \phi(e_{ij}e_{ii}) + \Psi(e_{ij})e_{ii} \\ &= \Psi(e_{ij})e_{ii}.\end{aligned}$$

Thus $\Psi(e_{ij}) = \Psi(e_{ij})e_{jj} = 0$. Therefore $\Psi = \Xi - \phi = 0$ and Ξ is a generalized derivation.

As a consequence of our Theorem we have the following result.

Corollary 2.1 (Theorem 3.3 [14]). *Let $\mathfrak{R}$ be a 2-torsion free commutative ring with identity. Then every Jordan derivation of the incidence algebra $I(X, \mathfrak{R})$ is a derivation.*

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