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A Class of p-valent Functions in the Punctured Disc

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Abstract

Motivated by Aouf differential operator, a class $F_{\lambda, p}^{n}(\alpha, \beta, \gamma)$ of p-valent functions in the punctured disc $U^* = \{z : 0 < |z| < 1\} = U \backslash \{0\}$ is defined. The coefficient estimates, growth and distortion theorems for the class are obtained.

1. Introduction

Let A denote the class of functions $g(z)$ of the form

$$g(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$.

Let $g(z) \in A$, $n \in N_0$. Then the Ruscheweyh differential operator of the function $g(z)$ is defined as $R^n : A \to A$

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$$R^0 g(z) = g(z)$$

$$R^1 g(z) = zg'(z)$$

$$\vdots$$

$$R^n g(z) = z(R^{n-1} g(z))' + (n-1) R^{n-1}(g(z)), \ z \in \mathbb{U}.$$

Liu and Srivastava [1] and [2] respectively defined linear operators L_p and D^n on a class $\sum_p$ of functions which are analytic and p-valent in the punctured disc $\mathbb{U} := \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\} = \mathbb{U}\backslash\{0\}$. Joshi and Aouf [3] considered a class consisting of regular and p-valent functions in the punctured disc with fixed second coefficients. Some properties on the class were established. Joshi and Srivastava [4] studied a certain family of meromorphically multivalent functions by using the Ruscheweyh derivative. Aouf [5] introduced a class of meromorphic multivalent functions in a punctured disc by using a differential operator to obtain some properties of the class. Logu [6] considered the works of Aouf and others thereby introduced and studied a generalized subclass of meromorphic p-valent functions using a differential operator to obtain some interesting results. Singh and Singh [7] investigated a generalized subclass of multivalent functions related to sigmoid function. Nunokawa and Sokol [8] proved some new sufficient conditions for function to be p-valent or p-valently starlike in the unit disc.

Motivated by the works of Aouf and others, let $\sum_p^*$ denote the class of functions of the form

$$f(z) = z^{-p} + \sum_{k=1}^{\infty} a_k z^{k-p} \quad (a_k \geq 0; \ p \in \mathbb{N} = 1, 2, ...) \tag{1.2}$$

be analytic and p-valent in the punctured disc $U^* = \{z : z \in C$ and $0 < |z| < 1 = U\backslash\{0\}\}$.

For $f(z) \in \sum_p^*$, define

$$D_{\lambda, p}^0 f(z) = f(z)$$

$$D^1_{\lambda,p}f(z) = (1-\lambda)f(z) + \frac{\lambda}{p}zf'(z) + \frac{2\lambda}{z^p}$$

$$\Rightarrow D^1_{\lambda,p}f(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty}\left(\frac{p-2\lambda p+\lambda k}{p}\right)a_k z^{k-p}.$$

Also,

$$D^2_{\lambda,p}f(z) = (1-\lambda)D^1_{\lambda,p}f(z) + \frac{\lambda z}{p}(D^1_{\lambda,p}f(z))' + \frac{2\lambda}{z^p}$$

$$= \frac{1}{z^p} + \sum_{k=1}^{\infty}\left(\frac{p-2\lambda p+\lambda k}{p}\right)^2 a_k z^{k-p}$$

$$\vdots$$

$$D^n_{\lambda,p}f(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty}\left(\frac{p-2\lambda p+\lambda k}{p}\right)^n a_k z^{k-p}. \tag{1.3}$$

2. Main Results

Here, the main results are discussed.

Definition 2.1. A function $f(z)$ of the form (1.2) is in the class $F^n_{\lambda,p}(\alpha, \beta, \gamma)$ if it satisfies

$$\left|\frac{z^{p+1}(D^n_{\lambda,p}f(z))' + p}{(2\gamma-1)z^{p+1}(D^n_{\lambda,p}f(z))' + (2\gamma\alpha - p)}\right| < \beta \tag{2.1}$$

$$\left(z \in U^*;\ \alpha(0 \le \alpha < p);\ \beta(0 < \beta \le 1);\ \gamma\left(\frac{1}{2} < \gamma \le 1\right)\right);\ \lambda \ge 0;\ p \in \mathbb{N};\ n \in \mathbb{N}_0.$$

Theorem 2.1. *Let a function $f(z)$ be in the class $F^n_{\lambda,p}(\alpha, \beta, \gamma)$. Then*

$$\sum_{k=1}^{\infty}(k-p)\left(\frac{p-2\lambda p+\lambda k}{p}\right)^n(1+2\beta\gamma-\beta)a_k \le 2\beta\gamma(p-\alpha) \tag{2.2}$$

for $0 \le \alpha < p;\ 0 < \beta \le 1;\ \frac{1}{2} < \gamma \le 1;\ n \in \mathbb{N}_0;\ p \in \mathbb{N};\ \lambda \ge 0.$

Proof. Suppose $f(z) \in F_{\lambda, p}^{n}(\alpha, \beta, \gamma)$, then by definition

$$\left| \frac{z^{p+1}(D_{\lambda, p}^{n} f(z))' + p}{(2\gamma - 1) z^{p+1}(D_{\lambda, p}^{n} f(z))' + (2\gamma\alpha - p)} \right| < \beta$$

$$(D_{\lambda, p}^{n} f(z))' = \left(\frac{1}{z^{p}} + \sum_{k=1}^{\infty} a_k z^{k-p} \left(\frac{p - 2\lambda p + \lambda k}{p} \right)^{n} \right)'$$

$$= \frac{-p}{z^{p+1}} + \sum_{k=1}^{\infty} (k-p) \left(\frac{p - 2\lambda p + \lambda k}{p} \right)^{n} a_k z^{k-p-1}. \tag{2.3}$$

Substituting (2.3), we have

$$\left| \frac{z^{p+1}\left[\frac{-p}{z^{p+1}} + \sum_{k=1}^{\infty} (k-p) \left(\frac{p - 2\lambda p + \lambda k}{p} \right)^{n} a_k z^{k-p-1} \right] + p}{(2\gamma - 1) z^{p+1}\left[\frac{-p}{z^{p+1}} + \sum_{k=1}^{\infty} (k-p) \left(\frac{p - 2\lambda p + \lambda k}{p} \right)^{n} a_k z^{k-p-1} \right] + (2\gamma\alpha - p)} \right|$$

$$= \left| \frac{-p + \sum_{k=1}^{\infty} (k-p) \left(\frac{p - 2\lambda p + \lambda k}{p} \right)^{n} a_k z^{k} + p}{(2\gamma - 1)\left[-p + \sum_{k=1}^{\infty} (k-p) \left(\frac{p - 2\lambda p + \lambda k}{p} \right)^{n} a_k z^{k} \right] + (2\gamma\alpha - p)} \right|$$

$$= \left| \frac{\sum_{k=1}^{\infty} (k-p) \left(\frac{p - 2\lambda p + \lambda k}{p} \right)^{n} a_k z^{k}}{2\gamma(\alpha - p) + (2\gamma - 1) \sum_{k=1}^{\infty} (k-p) \left(\frac{p - 2\lambda p + \lambda k}{p} \right)^{n} a_k z^{k}} \right|$$

$$< \frac{\left| \sum_{k=1}^{\infty} (k-p) \left(\frac{p - 2\lambda p + \lambda k}{p} \right)^{n} a_k z^{k} \right|}{\left| 2\gamma(\alpha - p) - (2\gamma - 1) \sum_{k=1}^{\infty} (k-p) \left(\frac{p - 2\lambda p + \lambda k}{p} \right)^{n} a_k z^{k} \right|}$$

$$= \left| \sum_{k=1}^{\infty} (k-p) \left(\frac{p - 2\lambda p + \lambda k}{p} \right)^n a_k z^k \right|$$

$$< \beta \left| 2\gamma(\alpha - p) - (2\gamma - 1) \sum_{k=1}^{\infty} (k-p) \left(\frac{p - 2\lambda p + \lambda k}{p} \right)^n a_k z^k \right|$$

$$\leq 2\beta\gamma |(\alpha - p)| + \beta \left| (1 - 2\gamma) \sum_{k=1}^{\infty} (k-p) \left(\frac{p - 2\lambda p + \lambda k}{p} \right)^n a_k z^k \right|$$

$$= 2\beta\gamma |(p - \alpha)| + \beta \left| (1 - 2\gamma) \sum_{k=1}^{\infty} (k-p) \left(\frac{p - 2\lambda p + \lambda k}{p} \right)^n a_k \right| |z^k|.$$

Since $|z| = r < 1$,

$$\sum_{k=1}^{\infty} (k-p) \left(\frac{p - 2\lambda p + \lambda k}{p} \right)^n a_k - \beta(1 - 2\gamma) \sum_{k=1}^{\infty} (k-p) \left(\frac{p - 2\lambda p + \lambda k}{p} \right)^n a_k$$

$$\leq 2\beta\gamma(p - \alpha)$$

$$= \sum_{k=1}^{\infty} (k-p) \left(\frac{p - 2\lambda p + \lambda k}{p} \right)^n (1 + 2\beta\gamma - \beta) a_k$$

$$\leq 2\beta\gamma(p - \alpha).$$

This implies that

$$a_k \leq \frac{2\beta\gamma(p - \alpha)}{(k-p) \left(\frac{p - 2\lambda p + \lambda k}{p} \right)^n (1 + 2\beta\gamma - \beta)}; \quad k \geq 1; \ p \in \mathbb{N}; \ n \in \mathbb{N}.$$

Corollary 1. *A function* $f(z)$ *is in the class* $F_{\lambda, p}^n(\alpha, 1, 1)$ *if*

$$\sum_{k=1}^{\infty} (k-p) \left(\frac{p - 2\lambda p + \lambda k}{p} \right)^n a_k \leq (p - \alpha).$$

Corollary 2. *A function* $f(z)$ *is in the class* $F_{0, p}^0(\alpha, 1, 1)$ *if*

$$\sum_{k=1}^{\infty}(k-p)a_k \le (p-\alpha).$$

Corollary 3. *A function* $f(z)$ *is in the class* $F_{0,\,p}^{0}(\alpha, \beta, 1)$ *if*

$$\sum_{k=1}^{\infty}(k-p)(1+\beta)a_k \le (p-\alpha).$$

Theorem 2.2. *If the function* $f(z)$ *belongs to the class* $F_{\lambda,\,p}^{n}(\alpha, \beta, \gamma)$, *then for* $0 < |f(z)| = r < 1$, *we have*

$$\frac{1}{r^p} - \frac{2\beta\gamma(p-\alpha)}{(1-p)\left(\frac{p-2\lambda p+\lambda}{p}\right)^n(1+2\beta\gamma-\beta)}r^{1-p}$$

$$\le |f(z)|$$

$$\le \frac{1}{r^p} + \frac{2\beta\gamma(p-\alpha)}{(1-p)\left(\frac{p-2\lambda p+\lambda}{p}\right)^n(1+2\beta\gamma-\beta)}r^{1-p}. \tag{2.4}$$

Proof. From Theorem 2.1,

$$(1-p)\left(\frac{p-2\lambda p+\lambda}{p}\right)^n(1+2\beta\gamma-\beta)\sum_{k=1}^{\infty}a_k$$

$$\le \sum_{k=1}^{\infty}(k-p)\left(\frac{p-2\lambda p+\lambda k}{p}\right)^n(1+2\beta\gamma-\beta)a_k$$

$$\le 2\beta\gamma(p-\alpha).$$

That is,

$$\sum_{k=1}^{\infty}a_k \le \frac{2\beta\gamma(p-\alpha)}{(1-p)\left(\frac{p-2\lambda p+\lambda}{p}\right)^n(1+2\beta\gamma-\beta)}.$$

For $0 < |z| = r < 1$,

$$|f(z)| \leq \left| \frac{1}{z^p} + \sum_{k=1}^{\infty} a_k z^{k-p} \right|$$

$$= \frac{1}{r^p} + \sum_{k=1}^{\infty} a_k r^{k-p}$$

$$= \frac{1}{r^p} + r^{1-p} \sum_{k=1}^{\infty} a_k$$

$$|f(z)| \leq \frac{1}{r^p} + \frac{2\beta\gamma(p-\alpha)}{(1-p)\left(\frac{p-2\lambda p+\lambda}{p}\right)^n (1+2\beta\gamma-\beta)} r^{1-p} \tag{2.5}$$

and

$$|f(z)| \leq \frac{1}{r^p} - \frac{2\beta\gamma(p-\alpha)}{(1-p)\left(\frac{p-2\lambda p+\lambda}{p}\right)^n (1+2\beta\gamma-\beta)} r^{1-p}. \tag{2.6}$$

Combining (2.5) and (2.6), gives the desired result.

Theorem 2.3. *Let* $f(z)$ *belong to the class* $F_{\lambda, p}^n(\alpha, \beta, \gamma)$. *Then for* $0 < |f(z)| = r < 1$, *we have*

$$\frac{p}{r^{p+1}} - \frac{2\beta\gamma(p-\alpha)}{\left(\frac{p-2\lambda p+\lambda}{p}\right)^n (1+2\beta\gamma-\beta)} r^{-p}$$

$$\leq |f'(z)|$$

$$\leq \frac{p}{r^{p+1}} + \frac{2\beta\gamma(p-\alpha)}{\left(\frac{p-2\lambda p+\lambda}{p}\right)^n (1+2\beta\gamma-\beta)} r^{-p}. \tag{2.7}$$

Proof. In view of Theorem 2.1,

$$\sum_{k=1}^{\infty} a_k \leq \frac{2\beta\gamma(p-\alpha)}{(1-p)\left(\frac{p-2\lambda p+\lambda}{p}\right)^n (1+2\beta\gamma-\beta)}.$$

Thus,

$$|f'(z)| \leq \left|\frac{-p}{z^{p+1}}\right| + \left|\sum_{k=1}^{\infty} a_k z^{-p}\right|$$

$$|f'(z)| \leq \frac{p}{r^{p+1}} + \frac{2\beta\gamma(p-\alpha)}{\left(\frac{p-2\lambda p+\lambda}{p}\right)^n (1+2\beta\gamma-\beta)} r^{-p}. \tag{2.8}$$

Similarly,

$$|f'(z)| \geq \frac{p}{r^{p+1}} - \frac{2\beta\gamma(p-\alpha)}{\left(\frac{p-2\lambda p+\lambda}{p}\right)^n (1+2\beta\gamma-\beta)} r^{-p}. \tag{2.9}$$

From (2.8) and (2.9), the result follows.

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