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On Generalized Fibonacci Polynomials: Horadam Polynomials

Yüksel Soykan

Department of Mathematics, Science Faculty, Zonguldak Bülent Ecevit University,
67100, Zonguldak, Turkey
e-mail: yuksel_soykan@hotmail.com

Abstract

In this paper, we investigate the generalized Fibonacci (Horadam) polynomials and we deal with, in detail, two special cases which we call them (r,s)-Fibonacci and (r,s)-Lucas polynomials. We present Binet's formulas, generating functions, Simson's formulas, and the summation formulas for these polynomial sequences. Moreover, we give some identities and matrices associated with these sequences. Finally, we present several expressions and combinatorial results of the generalized Fibonacci polynomials.

1 Introduction: Generalized Fibonacci (Horadam) Polynomials

The sequence of Fibonacci numbers $\{F_n\}$ and the sequence of Lucas numbers $\{L_n\}$ are defined by

$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 2, \qquad F_0 = 0, \ F_1 = 1,$$

and

$$L_n = L_{n-1} + L_{n-2}, \quad n \geq 2, \qquad L_0 = 2, \ L_1 = 1$$

respectively. The generalizations of Fibonacci and Lucas sequences lead to several nice and interesting sequences.

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The generalized Fibonacci polynomials (or Horadam polynomials or x-Horadam numbers or generalized $(r(x), s(x))$-polynomials or $(r(x), s(x))$ Horadam polynomials or 2-step Fibonacci polynomials)

$$\{W_n(W_0, W_1; r(x), s(x))\}_{n\geq 0}$$

(or $\{W_n(x)\}_{n\geq 0}$ or shortly $\{W_n\}_{n\geq 0}$) is defined as follows:

$$W_n(x) = r(x)W_{n-1}(x) + s(x)W_{n-2}(x), \qquad W_0(x) = a(x), W_1(x) = b(x), \quad n \geq 2 \tag{1.1}$$

where $W_0(x), W_1(x)$ are arbitrary complex (or real) polynomials with real coefficients and $r(x)$ and $s(x)$ are polynomials with real coefficients with $r(x) \neq 0$, $s(x) \neq 0$.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n}(x) = -\frac{r(x)}{s(x)}W_{-(n-1)}(x) + \frac{1}{s(x)}W_{-(n-2)}(x)$$

for $n = 1, 2, 3, ...$ when $s(x) \neq 0$. Therefore, recurrence (1.1) holds for all integers n. Note that $W_{-n}(x)$ need not to be a polynomial in the ordinary sense. For some references on special cases of Horadam polynomials see [3,4,5,11,12] for papers and [1,2,6,7,8,10] for books.

Binet's formula of generalized Fibonacci (Horadam) polynomials can be calculated using its characteristic equation (the quadratic equation, polynomial) which is given as

$$y^2 - r(x)y - s(x) = 0. \tag{1.2}$$

The roots of characteristic equation are

$$\alpha(x) := \alpha = \frac{r(x) + \sqrt{r^2(x) + 4s(x)}}{2}, \quad \beta(x) := \beta = \frac{r(x) - \sqrt{r^2(x) + 4s(x)}}{2}, \tag{1.3}$$

and the followings hold

$$\begin{aligned}
\alpha+\beta &= r(x), \\
\alpha\beta &= -s(x), \\
(\alpha-\beta)^2 &= (\alpha+\beta)^2-4\alpha\beta = r^2(x)+4s(x), \\
r(x)-2\alpha &= -(\alpha-\beta), \\
r(x)-2\beta &= \alpha-\beta, \\
(r(x)-2\alpha)(r(x)-2\beta) &= -(\alpha-\beta)^2 = -(r^2(x)+4s(x)).
\end{aligned}$$

Note also that

$$\begin{aligned}
(W_1-\alpha W_0)(W_1-\beta W_0) &= W_1^2+\alpha\beta W_0^2-\alpha W_0W_1-\beta W_0W_1 \\
&= W_1^2+\alpha\beta W_0^2-(\alpha+\beta)\,W_1W_0 \\
&= W_1^2-sW_0^2-rW_1W_0 \\
&\Rightarrow \\
(W_1-\alpha W_0)(W_1-\beta W_0) &= W_1^2-sW_0^2-rW_1W_0.
\end{aligned}$$

If the roots α and β of characteristic equation (1.2) are distinct, i.e., $\alpha\neq\beta$, then $r^2(x)+4s(x)\neq 0$ and if the roots α and β of characteristic equation (1.2) are equal, i.e., $\alpha=\beta$, then (1.2) can be written as

$$y^2-r(x)y-s(x)=(y-\alpha)^2=y^2-2\alpha y+\alpha^2=0$$

and, in this case,

$$\begin{aligned}
\alpha &= \frac{r(x)}{2}, \\
r(x) &= 2\alpha, \\
s(x) &= -\alpha^2=-\frac{r^2(x)}{4}, \\
r^2(x)+4s(x) &= 0.
\end{aligned}$$

Now, we define two special cases of the polynomials $W_n(x)$. $(r(x),s(x))$-Fibonacci polynomials $\{G_n(0,1;r(x),s(x))\}_{n\geq 0}$ (or shortly $G_n(x)$)

and $(r(x), s(x))$-Lucas polynomials $\{H_n(2, r(x); r(x), s(x))\}_{n\geq 0}$ (or shortly $H_n(x)$) are defined, respectively, by the second-order recurrence relations

$$G_{n+2}(x) = r(x)G_{n+1} + s(x)G_n(x), \qquad G_0(x) = 0, G_1(x) = 1, \tag{1.4}$$

$$H_{n+2}(x) = r(x)H_{n+1} + s(x)H_n(x), \qquad H_0(x) = 2, H_1(x) = r(x). \tag{1.5}$$

The (sequences of polinomials) $\{G_n(x)\}_{n\geq 0}$ and $\{H_n(x)\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} G_{-n}(x) &= -\frac{r(x)}{s(x)}G_{-(n-1)}(x) + \frac{1}{s(x)}G_{-(n-2)}(x), \\ H_{-n}(x) &= -\frac{r(x)}{s(x)}H_{-(n-1)}(x) + \frac{1}{s(x)}H_{-(n-2)}(x), \end{aligned}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (1.4) and (1.5) hold for all integers n.

NOTE: For the sake of simplicity throughout the rest of the paper we use

$$W_n, r, s, W_0, W_1, \alpha, \beta, G_n, H_n, G_0, G_1, H_0, H_1$$

instead of

$$W_n(x), r(x), s(x), W_0(x), W_1(x), \alpha(x), \beta(x), G_n(x), H_n(x), G_0(x), G_1(x), H_0(x), H_1(x),$$

respectively. For example, we write

$$W_n = rW_{n-1} + W_{n-2}, \qquad W_0 = a, W_1 = b, \quad n \geq 2$$

for the equation (1.1).

Next, we present the first few values of the (r, s)-Fibonacci and (r, s)-Lucas polynomials with positive and negative subscripts:

Table 1: The first few values of the special second-order numbers with positive and negative subscripts.

n	0	1	2	3	4
G_n	0	1	r	$s+r^2$	$r\left(2s+r^2\right)$
G_{-n}		$\frac{1}{s}$	$-\frac{r}{s^2}$	$\frac{1}{s^3}\left(s+r^2\right)$	$-\frac{r}{s^4}\left(2s+r^2\right)$
H_n	2	r	$2s+r^2$	$r\left(3s+r^2\right)$	$4r^2s+r^4+2s^2$
H_{-n}		$-\frac{r}{s}$	$\frac{2s+r^2}{s^2}$	$-r\frac{3s+r^2}{s^3}$	$\frac{4r^2s+r^4+2s^2}{s^4}$

Some special cases of (r,s)-Fibonacci sequence $\{G_n(0,1;r,s)\}_{n\geq 0}$ and (r,s)-Lucas sequence $\{H_n(2,r;r,s)\}_{n\geq 0}$ are as follows:

1. $G_n(0,1;1,1) = F_n$, Fibonacci sequence,
2. $H_n(2,1;1,1) = L_n$, Lucas sequence,
3. $G_n(0,1;2,1) = P_n$, Pell sequence,
4. $H_n(2,2;2,1) = Q_n$, Pell-Lucas sequence,
5. $G_n(0,1;1,2) = J_n$, Jacobsthal sequence,
6. $H_n(2,1;1,2) = j_n$, Jacobsthal-Lucas sequence.
7. $G_n(0,1;3,-2) = M_n$, Mersenne sequence,
8. $H_n(2,3;3,-2) = H_n$, Mersenne-Lucas sequence,
9. $G_n(0,1;6,-1) = B_n$, balancing sequence,
10. $H_n(2,6;6,-1) = H_n$, modified Lucas-balancing sequence,
11. $G_n(0,1;1,-\frac{1}{4}) = G_n$, modified Oresme sequence,
12. $H_n(2,1;1,-\frac{1}{4}) = H_n$, Oresme-Lucas sequence,
13. $G_n(0,1;x,1) = F_n(x)$, Fibonacci polynomials,

14. $H_n(2, x; x, 1) = L_n(x)$, Lucas polynomials,

15. $G_n(0, 1; 2x, 1) = P_n(x)$, Pell polynomials,

16. $H_n(2, 2x; 2x, 1) = Q_n(x)$, Pell-Lucas polynomials,

17. $G_n(0, 1; 1, 2x) = J_n(x)$, Jacobsthal polynomials,

18. $H_n(2, 1; 1, 2x) = j_n(x)$, Jacobsthal-Lucas polynomials.

Using the roots α, β and recurrence relation (1.1), Binet's formula can be given as follows:

Theorem 1.

(a) *(Distinct Roots Case: $\alpha \neq \beta$) Binet's formula of generalized Fibonacci (Horadam) polynomials is*

$$W_n = \frac{p_1\alpha^n}{\alpha - \beta} + \frac{p_2\beta^n}{\beta - \alpha} = \frac{p_1\alpha^n - p_2\beta^n}{\alpha - \beta} \tag{1.6}$$

where

$$p_1 = W_1 - \beta W_0, \; p_2 = W_1 - \alpha W_0.$$

(b) *(Single Root Case: $\alpha = \beta$) Binet's formula of generalized Fibonacci (Horadam) polynomials is*

$$W_n = (D_1 + D_2 n)\alpha^n \tag{1.7}$$

where

$$\begin{aligned} D_1 &= W_0, \\ D_2 &= \frac{1}{\alpha}(W_1 - \alpha W_0). \end{aligned}$$

Proof.

(a) If the roots α, β of (1.2) are distinct, then (the sequences $(\alpha^n)_{n\geq 0}$ and $(\beta^n)_{n\geq 0}$ are both solutions of (1.1)) the general formula of W_n is in the following form:

$$W_n = A_1\alpha^n + A_2\beta^n$$

where the coefficients A_1 and A_2 are determined by the system of linear equations

$$\begin{aligned} W_0 &= A_1 + A_2 \\ W_1 &= A_1\alpha + A_2\beta. \end{aligned}$$

Solving these two simultaneous equations for W_0 and W_1, we obtain

$$A_1 = \frac{W_1 - \beta W_0}{\alpha - \beta}, \ A_2 = \frac{W_1 - \alpha W_0}{\beta - \alpha}.$$

(b) If the roots α, β of (1.2) are equal, i.e., $\alpha = \beta$, then W_n is in the following form:

$$W_n = (D_1 + D_2 \times n)\alpha^n$$

where D_1 and D_2 are the polynomials whose values are determined by the values W_0 and any other known value of the sequence. By using the values W_0 and W_1, we obtain

$$\begin{aligned} W_0 &= (D_1 + D_2 \times 0)\alpha^0 \\ W_1 &= (D_1 + D_2 \times 1)\alpha^1. \end{aligned}$$

Solving these two simultaneous equations for W_0 and W_1, we get

$$D_1 = W_0, D_2 = \frac{1}{\alpha}(W_1 - \alpha W_0)$$

i.e.,

$$W_n = (W_0 + \frac{1}{\alpha}(W_1 - \alpha W_0)\, n)\alpha^n.$$

□

Note that (a) and (b) of the above theorem can be given as follows:

Theorem 2. *The general term of the generalized Fibonacci (Horadam) polynomials W_n can be presented by the following Binet's formula:*

$$W_n = \begin{cases} \frac{W_1-\beta W_0}{\alpha-\beta}\alpha^n - \frac{W_1-\alpha W_0}{\alpha-\beta}\beta^n & , \quad \text{if } \alpha \neq \beta \text{ (Distinct Roots Case)} \\ (nW_1 - \alpha(n-1)W_0)\alpha^{n-1} & , \quad \text{if } \alpha = \beta \text{ (Single Root Case)} \end{cases} \tag{1.8}$$

$$= \begin{cases} \frac{W_1-\beta W_0}{\alpha-\beta}\alpha^n - \frac{W_1-\alpha W_0}{\alpha-\beta}\beta^n & , \quad \text{if } \alpha \neq \beta \text{ (Distinct Roots Case)} \\ (nW_1 - \frac{r}{2}(n-1)W_0)\left(\frac{r}{2}\right)^{n-1} & , \quad \text{if } \alpha = \beta \text{ (Single Root Case)} \end{cases}.$$

(1.6) can be written in the following form:

$$W_n = A_1\alpha^n + A_2\beta^n \tag{1.9}$$

where

$$A_1 = \frac{W_1-\beta W_0}{\alpha-\beta}, \; A_2 = \frac{W_1-\alpha W_0}{\beta-\alpha}.$$

Note that

$$\begin{aligned} A_1A_2 &= \frac{(W_1^2 - sW_0^2 - rW_1W_0)}{-(r^2+4s)}, \\ A_1 + A_2 &= W_0. \end{aligned}$$

Note also that

$$\begin{aligned} D_1D_2 &= \frac{W_0(2W_1 - rW_0)}{r}, \\ D_1 + D_2 &= 2\frac{W_1}{r}. \end{aligned}$$

The Binet's form of a sequence satisfying (1.6) for non-negative integers is valid for all integers n and we have the following formula

$$W_{-n} = \frac{\beta^n p_1 - \alpha^n p_2}{\alpha^n\beta^n(\alpha^n p_1 - \beta^n p_2)}W_n.$$

Note that for all n, we have

$$\begin{aligned} G_{-n} &= \frac{-1}{\alpha^n\beta^n}G_n = \frac{-1}{(-s)^n}G_n, \\ H_{-n} &= \frac{1}{\alpha^n\beta^n}H_n = \frac{1}{(-s)^n}H_n. \end{aligned}$$

If the roots α, β of (1.2) are distinct, then we can also give Binet's formula of the generalized (r, s) polynomials (the generalized Fibonacci polynomials) for the negative subscripts as follows: for $n = 1, 2, 3, \ldots$ we have

$$W_{-n} = \frac{-r+\alpha}{s}\frac{\alpha^{-n+1}p_1}{\alpha-\beta} + \frac{-r+\beta}{s}\frac{\beta^{-n+1}p_2}{\beta-\alpha}.$$

If the roots α, β of (1.2) are distinct, i.e., $\alpha \neq \beta$, then for all integers n, (r, s)-Fibonacci and (r, s)-Lucas polynomials (using initial conditions in (1.6)) can be expressed using Binet's formulas as

$$G_n = \frac{\alpha^n}{(\alpha-\beta)} + \frac{\beta^n}{(\beta-\alpha)} = \frac{\alpha^n - \beta^n}{\alpha-\beta}, \tag{1.10}$$

and

$$H_n = \alpha^n + \beta^n, \tag{1.11}$$

respectively. If the roots α, β of (1.2) are equal, i.e., $\alpha = \beta$, then for all integers n, (r, s)-Fibonacci and (r, s)-Lucas polynomials (using initial conditions in (1.6)) can be expressed using Binet's formulas as

$$G_n = n\alpha^{n-1},$$

and

$$H_n = 2\alpha^n,$$

respectively, i.e.,

$$G_n = \begin{cases} \dfrac{\alpha^n - \beta^n}{\alpha-\beta} &, \text{ if } \alpha \neq \beta \text{ (Distinct Roots Case)} \\ n\alpha^{n-1} &, \text{ if } \alpha = \beta \text{ (Single Root Case)} \end{cases},$$

$$H_n = \begin{cases} \alpha^n + \beta^n &, \text{ if } \alpha \neq \beta \text{ (Distinct Roots Case)} \\ 2\alpha^n &, \text{ if } \alpha = \beta \text{ (Single Root Case)} \end{cases}.$$

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} W_n y^n$ of the sequence $\{W_n\}$.

Lemma 3. *Suppose that* $f_{W_n}(y) = \sum\limits_{n=0}^{\infty} W_n y^n$ *is the ordinary generating function of the generalized Fibonacci (Horadam) polynomials* $\{W_n\}_{n\geq 0}$*. Then,* $\sum\limits_{n=0}^{\infty} W_n y^n$ *is given by*

$$\sum_{n=0}^{\infty} W_n y^n = \frac{W_0 + (W_1 - rW_0)y}{1 - ry - sy^2}. \tag{1.12}$$

Proof. Using the definition of generalized Fibonacci polynomials, and substracting $ry\sum_{n=0}^{\infty} W_n y^n$ and $sy^2\sum_{n=0}^{\infty} W_n y^n$ from $\sum_{n=0}^{\infty} W_n y^n$, we obtain

$$\begin{aligned}
(1 - ry - sy^2)\sum_{n=0}^{\infty} W_n y^n &= \sum_{n=0}^{\infty} W_n y^n - ry\sum_{n=0}^{\infty} W_n y^n - sy^2\sum_{n=0}^{\infty} W_n y^n \\
&= \sum_{n=0}^{\infty} W_n y^n - r\sum_{n=0}^{\infty} W_n y^{n+1} - s\sum_{n=0}^{\infty} W_n y^{n+2} \\
&= \sum_{n=0}^{\infty} W_n y^n - r\sum_{n=1}^{\infty} W_{n-1} y^n - s\sum_{n=2}^{\infty} W_{n-2} y^n \\
&= (W_0 + W_1 y) - rW_0 y + \sum_{n=2}^{\infty}(W_n - rW_{n-1} - sW_{n-2})y^n \\
&= W_0 + (W_1 - rW_0)y.
\end{aligned}$$

Rearranging above equation, we obtain (1.12). □

Lemma 3 gives the following results as particular examples (generating functions of (r,s)-Fibonacci and (r,s)-Lucas polynomials).

Corollary 4. *Generating functions of* (r,s)*-Fibonacci and* (r,s)*-Lucas polynomials are*

$$\begin{aligned}
\sum_{n=0}^{\infty} G_n y^n &= \frac{y}{1 - ry - sy^2}, \\
\sum_{n=0}^{\infty} H_n y^n &= \frac{2 - ry}{1 - ry - sy^2},
\end{aligned}$$

respectively.

Proof. In Lemma 3, take $W_n = G_n$ with $G_0 = 0, G_1 = 1$ and $W_n = H_n$ with $H_0 = 2, H_1 = r$, respectively. □

2 Simson Formulas

There is a well-known Simson Identity (Cassini Identity) for Fibonacci sequence $\{F_n\}$, namely

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

which can be written in the form

$$\begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = (-1)^n.$$

The following theorem gives generalization of this result to the generalized Fibonacci polynomials W_n.

Theorem 5 (Simson Formula of Generalized Fibonacci (Horadam) Polynomials)**.** *For all integers n, we have*

$$\begin{vmatrix} W_{n+1} & W_n \\ W_n & W_{n-1} \end{vmatrix} = (-1)^n s^n \begin{vmatrix} W_1 & W_0 \\ W_0 & W_{-1} \end{vmatrix}. \tag{2.1}$$

Proof. Eq. (2.1) can be proved by mathematical induction. For the proof of the case of generalized Fibonacci (Horadam) numbers, see Soykan [9, Theorem 2.1]. □

The previous theorem gives the following results as particular examples.

Corollary 6. *For all integers n, Simson formula of (r,s)-Fibonacci and (r,s)-Lucas polynomials are given as*

$$\begin{aligned} \begin{vmatrix} G_{n+1} & G_n \\ G_n & G_{n-1} \end{vmatrix} &= (-1)^n s^{n-1}, \\ \begin{vmatrix} H_{n+1} & H_n \\ H_n & H_{n-1} \end{vmatrix} &= (-1)^{n+1} s^{n-1}(r^2+4s), \end{aligned}$$

respectively.

3 Some Identities

In this section, we obtain some identities of generalized Fibonacci (Horadam) polynomials, (r,s)-Fibonacci polynomials and (r,s)-Lucas polynomials. Firstly, we can give a few basic relations between $\{G_n\}$ and $\{W_n\}$.

Lemma 7. *Let $n \in \mathbb{Z}$. Then the following equalities are true:*

$$\begin{aligned}
s^3W_n &= ((s+r^2)W_1 - r(2s+r^2)W_0)G_{n+4} + (-r(2s+r^2)W_1 \\
&\quad +(3r^2s+r^4+s^2)W_0)G_{n+3}, \qquad (3.1)\\
s^2W_n &= (-W_1r+(r^2+s)W_0)G_{n+3} + ((s+r^2)W_1 - r(2s+r^2)W_0)G_{n+2},\\
sW_n &= (W_1 - rW_0)G_{n+2} + (-rW_1 + (s+r^2)W_0)G_{n+1},\\
W_n &= W_0G_{n+1} + (W_1 - rW_0)G_n,\\
W_n &= W_1G_n + sW_0G_{n-1},
\end{aligned}$$

and

$$\begin{aligned}
s^3(-W_1^2+sW_0^2+rW_1W_0)G_n &= -((s+r^2)W_1 + srW_0)W_{n+4} + (r(2s+r^2)W_1 \\
&\quad +s(s+r^2)W_0)W_{n+3},\\
s^2(-W_1^2+sW_0^2+rW_1W_0)G_n &= (rW_1+sW_0)W_{n+3} - ((s+r^2)W_1 + srW_0)W_{n+2},\\
s(-W_1^2+sW_0^2+rW_1W_0)G_n &= -W_1W_{n+2} + (rW_1+sW_0)W_{n+1},\\
(-W_1^2+sW_0^2+rW_1W_0)G_n &= W_0W_{n+1} - W_1W_n,\\
(-W_1^2+sW_0^2+rW_1W_0)G_n &= -(W_1-rW_0)W_n + sW_0W_{n-1}.
\end{aligned}$$

Proof. We prove (3.1). Writing

$$W_n = a \times G_{n+4} + b \times G_{n+3}$$

and solving the system of equations

$$\begin{aligned}
W_0 &= a \times G_4 + b \times G_3\\
W_1 &= a \times G_5 + b \times G_4
\end{aligned}$$

we find that $a = \frac{(s+r^2)W_1 - r(2s+r^2)W_0}{s^3}$, $b = \frac{-r(2s+r^2)W_1 + (3r^2s+r^4+s^2)W_0}{s^3}$. The other equalities can be proved similarly. □

Note that all the identities in the above Lemma can be proved by induction as well.

Secondly, we can give a few basic relations between $\{H_n\}$ and $\{W_n\}$.

Lemma 8. *Let $n \in \mathbb{Z}$. Then the following equalities are true:*

$$\begin{aligned}
s^3(4s+r^2)W_n &= (-r(3s+r^2)W_1 + (4r^2s+r^4+2s^2)W_0)H_{n+4} \\
&\quad +((r^4+2s^2+4r^2s)W_1 - r(5r^2s+r^4+5s^2)W_0)H_{n+3}, \\
s^2(4s+r^2)W_n &= ((2s+r^2)W_1 - r(3s+r^2)W_0)H_{n+3} + (-r(3s+r^2)W_1 \\
&\quad +(r^4+2s^2+4r^2s)W_0)H_{n+2}, \\
s(4s+r^2)W_n &= (-rW_1 + (2s+r^2)W_0)H_{n+2} + ((2s+r^2)W_1 - r(3s+r^2)W_0)H_{n+1}, \\
(4s+r^2)W_n &= (2W_1 - rW_0)H_{n+1} + (-rW_1 + (2s+r^2)W_0)H_n, \\
(4s+r^2)W_n &= (rW_1 + 2sW_0)H_n + s(2W_1 - rW_0)H_{n-1},
\end{aligned}$$

and

$$\begin{aligned}
s^3(-W_1^2 + sW_0^2 + rW_0W_1)H_n &= (r(3s+r^2)W_1 + s(2s+r^2)W_0)W_{n+4} \\
&\quad -((r^4+2s^2+4r^2s)W_1 + rs(3s+r^2)W_0)W_{n+3}, \\
s^2(-W_1^2 + sW_0^2 + rW_0W_1)H_n &= -((2s+r^2)W_1 + rsW_0)W_{n+3} + (r(3s+r^2)W_1 \\
&\quad +s(2s+r^2)W_0)W_{n+2}, \\
s(-W_1^2 + sW_0^2 + rW_0W_1)H_n &= (rW_1 + 2sW_0)W_{n+2} - ((2s+r^2)W_1 + rsW_0)W_{n+1}, \\
(-W_1^2 + sW_0^2 + rW_0W_1)H_n &= (-2W_1 + rW_0)W_{n+1} + (rW_1 + 2sW_0)W_n, \\
(-W_1^2 + sW_0^2 + rW_0W_1)H_n &= (-rW_1 + (2s+r^2)W_0)W_n + s(-2W_1 + rW_0)W_{n-1}.
\end{aligned}$$

Thirdly, we give a few basic relations between $\{G_n\}$ and $\{H_n\}$.

Lemma 9. *Let $n \in \mathbb{Z}$. Then the following equalities are true:*

$$\begin{aligned}
s^3H_n &= -(3rs+r^3)G_{n+4} + (4r^2s+r^4+2s^2)G_{n+3}, \\
s^2H_n &= (2s+r^2)G_{n+3} - (3rs+r^3)G_{n+2}, \\
sH_n &= -rG_{n+2} + (2s+r^2)G_{n+1}, \\
H_n &= 2G_{n+1} - rG_n, \\
H_n &= rG_n + 2sG_{n-1},
\end{aligned}$$

and

$$\begin{aligned}(r^2s^3+4s^4)G_n &= -(3rs+r^3)H_{n+4}+(4r^2s+r^4+2s^2)H_{n+3},\\ (r^2s^2+4s^3)G_n &= (2s+r^2)H_{n+3}-(3rs+r^3)H_{n+2},\\ (r^2s+4s^2)G_n &= -rH_{n+2}+(2s+r^2)H_{n+1},\\ (r^2+4s)G_n &= 2H_{n+1}-rH_n,\\ (r^2+4s)G_n &= rH_n+2sH_{n-1}.\end{aligned}$$

Note that from the last Lemma we have

$$(r^2+4s)G_n = 2H_{n+1}-rH_n \tag{3.2}$$

so if $(r^2+4s)=0$, i.e., $\alpha=\beta$ (in this case $r=2\alpha, s=-\alpha^2$), then

$$2H_{n+1}-rH_n=0$$

and so $2H_{n+1}-2\alpha H_n=0$ and thus, if $(r^2+4s)=0$, i.e., $\alpha=\beta$, then

$$H_{n+1}=\alpha H_n. \tag{3.3}$$

We now give some identities.

Corollary 10. *Let $n,m\in\mathbb{Z}$. Then*

(a)

- **(i)** $H_n = G_{n+1}+sG_{n-1}$.
- **(ii)** $(r^2+4s)G_n = H_{n+1}+sH_{n-1}$.

(b)

- **(i)** $(r^2+4s)G_n+H_n = (H_{n+1}+G_{n+1})+s(H_{n-1}+G_{n-1})$.
- **(ii)** $(r^2+4s)G_n-H_n = (H_{n+1}-G_{n+1})+s(H_{n-1}-G_{n-1})$.

Proof.

(a) We use the identities

$$H_n = 2G_{n+1} - rG_n,$$
$$(r^2+4s)G_n = 2H_{n+1} - rH_n.$$

Note also that

$$G_{n+1} = rG_n + sG_{n-1} \Rightarrow G_{n+1} - rG_n = sG_{n-1},$$
$$H_{n+1} = rH_n + sH_{n-1} \Rightarrow H_{n+1} - rH_n = sH_{n-1}.$$

(i)

$$H_n = 2G_{n+1} - rG_n = G_{n+1} + G_{n+1} - rG_n = G_{n+1} + sG_{n-1}.$$

(ii)

$$(r^2+4s)G_n = 2H_{n+1} - rH_n = H_{n+1} + H_{n+1} - rH_n = H_{n+1} + sH_{n-1}.$$

Note: the proof of (i) and (ii) can also be given by using Binet's formulas of G_n and H_n.

(b)

(i) Use (a).

(ii) Use (a).

□

Note that

$$(r^2+4s)G_nH_n = (2H_{n+1} - rH_n)(2G_{n+1} - rG_n).$$

4 Special Identities

We now present a few special identities for the generalized Fibonacci (Horadam) polynomials $\{W_n\}$, (r,s)-Fibonacci polynomials $\{G_n\}$ and (r,s)-Lucas polynomials $\{H_n\}$.

Theorem 11. *Let n, m and k be any integers. Then the following identities are true:*

(a) *Catalan's identity.*

(i) $W_{n+m}W_{n-m} - W_n^2 = -(-s)^{n-m}(W_1^2 - sW_0^2 - rW_0W_1)G_m^2.$

(ii) $G_{n+m}G_{n-m} - G_n^2 = -(-s)^{n-m}G_m^2.$

(iii) $H_{n+m}H_{n-m} - H_n^2 = (-s)^{n-m}(r^2 + 4s)G_m^2.$

(b) *Cassini's identity.*

(i) $W_{n+1}W_{n-1} - W_n^2 = -(-s)^{n-1}(W_1^2 - sW_0^2 - rW_0W_1).$

(ii) $G_{n+1}G_{n-1} - G_n^2 = -(-s)^{n-1}.$

(iii) $H_{n+1}H_{n-1} - H_n^2 = (-s)^{n-1}(r^2 + 4s).$

(c) *d'Ocagne's identity.*

(i) $W_{m+1}W_n - W_mW_{n+1} = -(W_1^2 - sW_0^2 - rW_0W_1)(-H_mG_n + G_{m+n}).$

(ii) $G_{m+1}G_n - G_mG_{n+1} = -(-H_mG_n + G_{m+n}).$

(iii) $H_{m+1}H_n - H_mH_{n+1} = (r^2 + 4s)(-H_mG_n + G_{m+n}).$

(d) *Gelin-Cesàro's identity.*

(i) $W_{n+2}W_{n+1}W_{n-1}W_{n-2} - W_n^4 = (-s)^{n-3}(W_1^2 - sW_0^2 - rW_0W_1)(s\left(r^2 - s\right)W_n^2 + r^2(-s)^n W_1^2 - r^2 s(-s)^n W_0^2 - r^3(-s)^n W_0W_1).$

(ii) $G_{n+2}G_{n+1}G_{n-1}G_{n-2} - G_n^4 = (-s)^{n-3}\left(r^2(-s)^n - s\left(s - r^2\right)G_n^2\right).$

(iii) $H_{n+2}H_{n+1}H_{n-1}H_{n-2} - H_n^4 = (-s)^{n-3}(r^2 + 4s)\left(-s\left(r^2 - s\right)H_n^2 + r^2(-s)^n(r^2 + 4s)\right).$

(e) *Melham's identity.*

(i) $W_{n+1}W_{n+2}W_{n+6} - W_{n+3}^3 = -(-s)^{n+1}(W_1^2 - sW_0^2 - rW_0W_1)(r^3W_{n+2} - (-s)^2W_{n+1})$.

(ii) $G_{n+1}G_{n+2}G_{n+6} - G_{n+3}^3 = -(-s)^{n+1})(r^3G_{n+2} - (-s)^2G_{n+1})$.

(iii) $H_{n+1}H_{n+2}H_{n+6} - H_{n+3}^3 = (-s)^{n+1}(r^2 + 4s)(r^3H_{n+2} - (-s)^2H_{n+1})$.

(f) *Vajda's identity (Taguiri's identity (a generalization of Catalan's identity)).*

(i) $W_{n+m}W_{n+k} - W_nW_{n+m+k} = (-s)^n(W_1^2 - sW_0^2 - rW_0W_1)G_kG_m$.

(ii) $G_{n+m}G_{n+k} - G_nG_{n+m+k} = (-s)^nG_kG_m$.

(iii) $H_{n+m}H_{n+k} - H_nH_{n+m+k} = -(-s)^n(r^2 + 4s)G_kG_m$.

Proof. To prove (i)'s, use the identities (1.6) and (1.7) in Theorem 1 or the identity (1.8) in Theorem 2. Then, for (ii)'s and (iii)'s, set $W_n = G_n$ and $W_n = H_n$ in (i)'s, respectively. For (b), set $m = 1$ in (a). □

In the last Theorem, all the identities are true without any restriction on α and β i.e., if $\alpha \neq \beta$ (distinct roots case: $r^2 + 4s \neq 0$) or if $\alpha = \beta$ (single root case: $r^2 + 4s = 0$). However, if $\alpha = \beta$ (single root case: $r^2 + 4s = 0$) the results of the above Theorem can be given in the following form.

Remark 12. *Let n, m and k be any integers. If $\alpha = \beta$ (single root case: $r^2+4s = 0$), then the following identities are true:*

(a) *Catalan's identity.*

(i) $W_{n+m}W_{n-m} - W_n^2 = -\frac{1}{4}m^2\left(2W_1 - rW_0\right)^2\left(\frac{r}{2}\right)^{2n-2}$.

(ii) $G_{n+m}G_{n-m} - G_n^2 = -m^2\left(\frac{r}{2}\right)^{2n-2}$.

(iii) $H_{n+m}H_{n-m} - H_n^2 = 0$.

(b) *Cassini's identity.*

(i) $W_{n+1}W_{n-1} - W_n^2 = -\frac{1}{4}\left(2W_1 - rW_0\right)^2\left(\frac{r}{2}\right)^{2n-2}$.

(ii) $G_{n+1}G_{n-1} - G_n^2 = -\left(\frac{r}{2}\right)^{2n-2}$.

(iii) $H_{n+1}H_{n-1} - H_n^2 = 0.$

(c) *d'Ocagne's identity.*

(i) $W_{m+1}W_n - W_mW_{n+1} = -\frac{1}{4}\left(\frac{r}{2}\right)^{m+n-1}(m-n)\left(-2W_1 + rW_0\right)^2.$

(ii) $G_{m+1}G_n - G_mG_{n+1} = -\left(\frac{r}{2}\right)^{m+n-1}(m-n).$

(iii) $H_{m+1}H_n - H_mH_{n+1} = 0.$

(d) *Gelin-Cesàro's identity.*

(i) $W_{n+2}W_{n+1}W_{n-1}W_{n-2} - W_n^4 = -\frac{1}{16}\left(-2W_1 + rW_0\right)^2 (4(5n^2-4)W_1^2 + r^2(5n^2-10n+1)W_0^2 - 4r(5n^2-5n-4)W_0W_1)\left(\frac{r}{2}\right)^{4n-4}.$

(ii) $G_{n+2}G_{n+1}G_{n-1}G_{n-2} - G_n^4 = (4-5n^2)\left(\frac{r}{2}\right)^{4n-4}.$

(iii) $H_{n+2}H_{n+1}H_{n-1}H_{n-2} - H_n^4 = 0.$

(e) *Melham's identity.*

(i) $W_{n+1}W_{n+2}W_{n+6} - W_{n+3}^3 = \frac{1}{8}\left(\frac{r}{2}\right)^{3n+6}(2W_1 - rW_0)^2(-2(7n+15)W_1 + r(7n+8)W_0).$

(ii) $G_{n+1}G_{n+2}G_{n+6} - G_{n+3}^3 = -(7n+15)\left(\frac{r}{2}\right)^{3n+6}.$

(iii) $H_{n+1}H_{n+2}H_{n+6} - H_{n+3}^3 = 0.$

(f) *Vajda's identity.*

(i) $W_{n+m}W_{n+k} - W_nW_{n+m+k} = \frac{1}{8}km\left(2W_1 - rW_0\right)^2 H_{2n+m+k-2}.$

(ii) $G_{n+m}G_{n+k} - G_nG_{n+m+k} = \frac{1}{2}kmH_{2n+m+k-2}.$

(iii) $H_{n+m}H_{n+k} - H_nH_{n+m+k} = 0.$

5 On the Recurrence Properties of Generalized Fibonacci Polynomials

Now, we can propose a problem as follows: Whether and how can the generalized Fibonacci (Horadam) polynomials W_n at negative indices be expressed by the sequence itself at positive indices?

We present the following result which completely solves the above problem for the generalized Fibonacci (Horadam) polynomials W_n.

Theorem 13. *For $n \in \mathbb{Z}$, for the generalized Fibonacci (Horadam) polynomials, we have*

$$\begin{aligned} W_{-n} &= (-1)^{-n-1}s^{-n}(W_n - H_nW_0) \\ &= -(-s)^{-n}(W_n - H_nW_0) \\ &= (-1)^{n+1}s^{-n}(W_n - H_nW_0). \end{aligned}$$

Proof. If the roots of characteristic equation (1.2) are distinct, then by using the Binet's formulas of W_n and H_n, we get

$$\begin{aligned} (-1)^{n+1}s^nW_{-n} &= -(-s)^nW_{-n} \\ &= -\alpha^n\beta^n(A_1\alpha^{-n} + A_2\beta^{-n}) \\ &= -(\beta^nA_1 + \alpha^nA_2) \\ &= (A_1\alpha^n + A_2\beta^n) - (A_1 + A_2)(\alpha^n + \beta^n) \\ &= W_n - W_0H_n \\ &\Rightarrow \\ W_{-n} &= (-1)^{-n-1}s^{-n}(W_n - H_nW_0). \end{aligned}$$

and if the roots of characteristic equation (1.2) are equal, then by using the Binet's formulas of W_n and H_n, we obtain

$$\begin{aligned}
(-1)^{n+1}s^nW_{-n} &= -(-s)^nW_{-n} \\
&= -\alpha^{2n}(D_1+D_2\times(-n))\alpha^{-n} \\
&= -\alpha^{2n}(W_0+\frac{1}{\alpha}(W_1-\alpha W_0)\times(-n))\alpha^{-n} \\
&= (nW_1-\alpha(n-1)W_0)\alpha^{n-1}-W_0\times 2\alpha^n \\
&= (D_1+D_2\times n)\alpha^n-W_0\times 2\alpha^n \\
&= W_n-W_0H_n \\
&\Rightarrow \\
W_{-n} &= (-1)^{-n-1}s^{-n}(W_n-H_nW_0).
\end{aligned}$$

This proves the theorem. □

Note that from the definition of H_n, we obtain

$$H_{-n}=(-s)^{-n}H_n$$

i.e., $H_{-n}=(-s)^{-n}H_n$ and so $H_n=(-s)^nH_{-n}$. Note also that

$$(-s)^n=\frac{1}{2}(H_n^2-H_{2n}).$$

If we take $W_n=H_n$ in Theorem 13 and by using $G_0=0$, we obtain

$$G_{-n}=(-1)^{-n-1}s^{-n}(G_n-H_nG_0)=(-1)^{-n-1}s^{-n}G_n=-(-s)^{-n}G_n.$$

Since

$$\begin{aligned}
W_n &= W_0G_{n+1}+(W_1-rW_0)G_n, \\
H_n &= 2G_{n+1}-rG_n,
\end{aligned}$$

and

$$W_{-n}=-(-s)^{-n}(W_n-H_nW_0)$$

we get

$$\begin{aligned} W_{-n} &= -(-s)^{-n}(W_n - H_n W_0) \\ &= -(-s)^{-n}(W_0 G_{n+1} + (W_1 - rW_0)G_n - (2G_{n+1} - rG_n)W_0) \\ &= (-s)^{-n}(W_0 G_{n+1} - W_1 G_n). \end{aligned}$$

By using Lemma 8 and Theorem 13 we obtain the following theorem.

Theorem 14. *For $n \in \mathbb{Z}$, for the generalized Fibonacci polynomials, we have*

$$W_{-n} = \frac{(-1)^{n+1} s^{-n}}{-W_1^2 + sW_0^2 + rW_0 W_1}((2W_1 - rW_0)W_0 W_{n+1} - (W_1^2 + sW_0^2)W_n).$$

6 Sum Formulas

In this section, we present sum formulas of generalized Fibonacci (Horadam) polynomials.

6.1 The Sum Formula $\sum_{k=0}^{n} z^k W_{mk+j}$

The following theorem presents sum formulas of generalized Fibonacci (Horadam) polynomials.

Theorem 15. *Let z be a real (or complex) number. For all integers m and j, for generalized Fibonacci (Horadam) polynomials, we have the following sum formulas:*

(a) *If $(-s)^m z^2 - zH_m + 1 \neq 0$, then*

$$\sum_{k=0}^{n} z^k W_{mk+j} = \frac{((-s)^m z - H_m) z^{n+1} W_{mn+j} + (-s)^m z^{n+1} W_{mn+j-m} + W_j - (-s)^m z W_{j-m}}{(-s)^m z^2 - zH_m + 1}. \tag{6.1}$$

(b) *If $(-s)^m z^2 - zH_m + 1 = u(z-a)(z-b) = 0$ for some $u, a, b \in \mathbb{C}$ with $u \neq 0$ and $a \neq b$, i.e., $z = a$ or $z = b$, then*

$$\sum_{k=0}^{n} z^k W_{mk+j} = \frac{(z(n+2)(-s)^m - (n+1)H_m) z^n W_{j+mn} + (-s)^m (n+1) z^n W_{mn+j-m} - (-s)^m W_{j-m}}{2(-s)^m z - H_m}.$$

(c) *If* $(-s)^m z^2 - zH_m + 1 = u(z-c)^2 = 0$ *for some* $u, c \in \mathbb{C}$ *with* $u \neq 0$, *i.e.,* $z = c$, *then*

$$\sum_{k=0}^{n} z^k W_{mk+j} = \frac{(n+1)\left((-s)^m (n+2)z^n - nz^{n-1}H_m\right)W_{mn+j} + n(n+1)(-s)^m z^{n-1} W_{mn+j-m}}{2(-s)^m}.$$

Proof.

(a) Note that if the roots of characteristic equation (1.2) are distinct, then

$$\begin{aligned}\sum_{k=0}^{n} z^k W_{mk+j} &= z^n W_{mn+j} + \sum_{k=0}^{n-1} z^k W_{mk+j} \\ &= z^n W_{mn+j} + \sum_{k=0}^{n-1} (A_1\alpha^{mk+j} + A_2\beta^{mk+j}) z^k \\ &= z^n W_{mn+j} + A_1\alpha^j \left(\frac{(\alpha^m z)^n - 1}{\alpha^m z - 1}\right) + A_2\beta^j \left(\frac{(\beta^m z)^n - 1}{\beta^m z - 1}\right).\end{aligned}$$

Simplifying the last equalities in the last two expression imply (6.1) as required. If the roots of characteristic equation (1.2) are equal, then the proof is similar.

(b) We use (6.1). For $z = a$ and $z = b$, the right hand side of the above sum formula (6.1)) is an indeterminate form. Now, we can use L'Hospital rule. Then we get (b) by using

$$\begin{aligned}&\sum_{k=0}^{n} a^k W_{mk+j} \\ &= \left.\frac{\frac{d}{dz}\left(((-s)^m z - H_m)z^{n+1} W_{mn+j} + (-s)^m z^{n+1} W_{mn+j-m} + W_j - (-s)^m z W_{j-m}\right)}{\frac{d}{dz}\left((-s)^m z^2 - zH_m + 1\right)}\right|_{z=a} \\ &= \left.\frac{(z(n+2)(-s)^m - (n+1)H_m)z^n W_{j+mn} + (-s)^m (n+1) z^n W_{mn+j-m} - (-s)^m W_{j-m}}{2(-s)^m z - H_m}\right|_{z=a} \\ &= \frac{(a(n+2)(-s)^m - (n+1)H_m)a^n W_{j+mn} + (-s)^m (n+1) a^n W_{mn+j-m} - (-s)^m W_{j-m}}{2(-s)^m a - H_m}\end{aligned}$$

and

$$\sum_{k=0}^{n} b^k W_{mk+j}$$
$$= \left.\frac{\frac{d}{dz}\left(((-s)^m z - H_m)z^{n+1}W_{mn+j} + (-s)^m z^{n+1}W_{mn+j-m} + W_j - (-s)^m zW_{j-m}\right)}{\frac{d}{dz}\left((-s)^m z^2 - zH_m + 1\right)}\right|_{z=b}$$
$$= \left.\frac{(z(n+2)(-s)^m - (n+1)H_m)z^n W_{j+mn} + (-s)^m (n+1)z^n W_{mn+j-m} - (-s)^m W_{j-m}}{2(-s)^m z - H_m}\right|_{z=b}$$
$$= \frac{(b(n+2)(-s)^m - (n+1)H_m)b^n W_{j+mn} + (-s)^m (n+1)b^n W_{mn+j-m} - (-s)^m W_{j-m}}{2(-s)^m b - H_m}.$$

(c) We use (6.1). For $z = c$, the right hand side of the above sum formula (6.1) is an indeterminate form. Now, we can use L'Hospital rule (twice). Then we get (c) by using

$$\sum_{k=0}^{n} c^k W_{mk+j}$$
$$= \left.\frac{\frac{d^2}{dz^2}\left(((-s)^m z - H_m)z^{n+1}W_{mn+j} + (-s)^m z^{n+1}W_{mn+j-m} + W_j - (-s)^m zW_{j-m}\right)}{\frac{d^2}{dz^2}\left((-1)^m z^2 - zH_m + 1\right)}\right|_{z=c}$$
$$= \left.\frac{(n+1)\left((-s)^m (n+2)z^n - nz^{n-1}H_m\right)W_{mn+j} + n(n+1)(-s)^m z^{n-1}W_{mn+j-m}}{2(-s)^m}\right|_{z=c}$$
$$= \frac{(n+1)\left((-s)^m (n+2)c^n - nc^{n-1}H_m\right)W_{mn+j} + n(n+1)(-s)^m c^{n-1}W_{mn+j-m}}{2(-s)^m}.$$

□

Note that (6.1) can be written in the following form

$$\sum_{k=1}^{n} z^k W_{mk+j} = \frac{((-s)^m z - H_m)z^{n+1}W_{mn+j} + (-s)^m z^{n+1}W_{mn+j-m} + z(H_m - (-s)^m z)W_j - (-s)^m zW_{j-m}}{(-s)^m z^2 - zH_m + 1}.$$

6.2 The Sum Formula $\sum_{k=0}^{n} z^k W_{mk+j}^2$

In this subsection, we present sum formulas of generalized Fibonacci (Horadam) polynomials. The following theorem presents sum formulas of generalized Fibonacci (Horadam) polynomials in the case the roots α and β of characteristic equation (1.2) are distinct, i.e., $r^2 + 4s \neq 0$.

Theorem 16 (Distinct Roots Case). *Suppose that the roots α and β of characteristic equation (1.2) are distinct, i.e. $r^2+4s\neq 0$. Let z be a real (or complex) number. For all integers m and j, for generalized Fibonacci (Horadam) polynomials, we have the following sum formulas:*

(a) *If $(1+(-s)^{2m}z^2-zH_{2m})((-s)^m z-1)\neq 0$, then*

$$\sum_{k=0}^{n} z^k W_{mk+j}^2=\frac{\Omega_1}{(r^2+4s)(1+(-s)^{2m}z^2-zH_{2m})((-s)^m z-1)} \tag{6.2}$$

where

$\Omega_1=(r^2+4s)((-s)^m z-1)((-s)^{2m}z-H_{2m})z^{n+1}W_{mn+j}^2+(r^2+4s)(-s)^{2m}((-s)^m z-1)z^{n+1}W_{mn-m+j}^2+(r^2+4s)((-s)^m z-1)W_j^2-(r^2+4s)(-s)^{2m}((-s)^m z-1)zW_{j-m}^2+2(-s)^j(W_1^2-sW_0^2-rW_1W_0)((-s)^{mn}z^n-1)(H_{2m}-2(-s)^m)z.$

(b) *If $(1+(-s)^{2m}z^2-zH_{2m})((-s)^m z-1)=u(z-a)(z-b)(z-c)=0$ for some $u,a,b,c\in\mathbb{C}$ with $u\neq 0$ and $a\neq b\neq c$, i.e., $z=a$ or $z=b$ or $z=c$, then*

$$\sum_{k=0}^{n} z^k W_{mk+j}^2=\frac{\Omega_2}{(r^2+4s)(3(-s)^{3m}z^2-2(-s)^m(H_{2m}+(-s)^m)z+H_{2m}+(-s)^m)}$$

where

$\Omega_2=(r^2+4s)((-s)^m((-s)^{2m}z-H_{2m})z^{n+1}+((-s)^m z-1)((-s)^{2m}(n+2)z-(n+1)H_{2m})z^n)W_{mn+j}^2+(r^2+4s)(-s)^{2m}((-s)^m(n+2)z-(n+1))z^nW_{mn-m+j}^2+(r^2+4s)(-s)^m W_j^2-(r^2+4s)(-s)^{2m}(2(-s)^m z-1)W_{j-m}^2+2(-s)^j(W_1^2-sW_0^2-rW_1W_0)(z^n(-s)^{mn}(n+1)-1)(H_{2m}-2(-s)^m).$

(c) *If $(1+(-s)^{2m}z^2-zH_{2m})((-s)^m z-1)=u(z-a)^2(z-c)=0$ for some $u,a,c\in\mathbb{C}$ with $u\neq 0$ and $a\neq c$, then if $z=c$, then*

$$\sum_{k=0}^{n} z^k W_{mk+j}^2=\frac{\Omega_3}{(r^2+4s)(3(-s)^{3m}z^2-2(-s)^m(H_{2m}+(-s)^m)z+H_{2m}+(-s)^m)}$$

where

$\Omega_3 = (r^2 + 4s)((-s)^m ((-s)^{2m} z - H_{2m})z^{n+1} + ((-s)^m z - 1)((-s)^{2m} (n+2) z - (n+1)H_{2m})z^n)W^2_{mn+j} + (r^2+4s)(-s)^{2m}((-s)^m(n+2)z-(n+1))z^nW^2_{mn-m+j} + (r^2+4s)(-s)^m W_j^2 - (r^2+4s)(-s)^{2m}(2(-s)^m z - 1)W^2_{j-m} + 2(-s)^j (W_1^2 - sW_0^2 - rW_1W_0)(z^n(-s)^{mn}(n+1)-1)(H_{2m} - 2(-s)^m)$

and if $z = a$, *then*

$$\sum_{k=0}^{n} z^k W^2_{mk+j} = \frac{\Omega_4}{2(r^2+4s)(-s)^m (3(-s)^{2m} z - (-s)^m - H_{2m})}$$

where

$\Omega_4 = (r^2+4s)((-s)^{3m}(n+3)(n+2)z^2 - z(-s)^m(n+2)(n+1)(H_{2m}+(-s)^m) + n(n+1)H_{2m})z^{n-1}W^2_{mn+j} + (r^2+4s)(-s)^{2m}(n+1)((2+n)(-s)^m z^n - nz^{n-1})W^2_{mn-m+j} - 2(r^2+4s)(-s)^{3m} W^2_{j-m} + 2n(n+1)(-s)^{mn+j}(W_1^2 - sW_0^2 - rW_1W_0)(H_{2m} - 2(-s)^m) z^{n-1}.$

(d) *If* $(1+(-s)^{2m}z^2 - zH_{2m})((-s)^m z - 1) = u(z-a)^3 = 0$ *for some* $u, a \in \mathbb{C}$ *with* $u \neq 0$, *i.e.,* $z = a$, *then*

$$\sum_{k=0}^{n} z^k W^2_{mk+j} = \frac{\Omega_5}{6(-s)^{3m}(r^2+4s)}$$

where

$\Omega_5 = (r^2+4s)(n+1)((-s)^{3m}(n+3)(n+2)z^2 - n(-s)^m(n+2)(H_{2m}+(-s)^m)z + n(n-1)H_{2m})z^{n-2}W^2_{mn+j} + n(-s)^{2m}(r^2+4s)(n+1)((n+2)(-s)^m z + 1 - n)z^{n-2}W^2_{mn-m+j} + 2(n-1)n(n+1)(-s)^{mn+j}(H_{2m} - 2(-s)^m)(W_1^2 - sW_0^2 - rW_1W_0)z^{n-2}.$

Proof.

(a) *Note that*

$$\sum_{k=0}^{n-1} a^{mk+j} = a^j \left(\frac{(a^m)^n - 1}{a^m - 1}\right)$$

and

$$\begin{aligned}
\sum_{k=0}^{n} z^k W_{mk+j}^2 &= z^n W_{mn+j}^2 + \sum_{k=0}^{n-1} z^k W_{mk+j}^2 \\
&= z^n W_{mn+j}^2 + \sum_{k=0}^{n-1} (A_1\alpha^{mk+j} + A_2\beta^{mk+j})^2 z^k \\
&= z^n W_{mn+j}^2 + A_1^2\alpha^{2j} \sum_{k=0}^{n-1} (\alpha^{2m} z)^k + A_2^2\beta^{2j} \sum_{k=0}^{n-1} (\beta^{2m} z)^k \\
&\quad +2A_1A_2\alpha^j\beta^j \sum_{k=0}^{n-1} (\alpha^m\beta^m z)^k.
\end{aligned}$$

Simplifying the last equalities in the last two expression imply (6.2) as required.

(b) Note that we can write $(1+(-s)^{2m}z^2 - zH_{2m})((-s)^m z - 1) = 0$ as

$$(-s)^{3m}(z^2 - z\frac{1}{(-s)^{2m}}H_{2m} + \frac{1}{(-s)^{2m}})(z - \frac{1}{(-s)^m}) = 0.$$

Solving this equation we find that

$$\begin{aligned}
z_1 &= a = \frac{1}{2(-s)^{2m}}\left(H_{2m} + \sqrt{H_{2m}^2 - 4(-s)^{2m}}\right), \\
z_2 &= b = \frac{1}{2(-s)^{2m}}\left(H_{2m} - \sqrt{H_{2m}^2 - 4(-s)^{2m}}\right), \\
z_3 &= c = \frac{1}{(-s)^m}.
\end{aligned}$$

If $H_{2m}^2 - 4(-s)^{2m} \neq 0$, then $a \neq b$. We assume that $b \neq c$. We use (6.2). For $z = a$, the right hand side of the above sum formula 6.2) is an indeterminate form. Now, we can use L'Hospital rule. Then we get (b) by using

$$\sum_{k=0}^{n} a^k W_{mk+j}^2 = \left.\frac{\frac{d}{dz}(\Omega_1)}{\frac{d}{dz}\left((r^2+4s)(1+(-s)^{2m}z^2 - zH_{2m})((-s)^m z - 1)\right)}\right|_{z=a}.$$

The proof for the case $z = b$ and $z = c$ are the same.

(c) If $H_{2m}^2-4(-s)^{2m}=0$, then $a=b=\dfrac{H_{2m}}{2(-s)^{2m}}$. We suppose that $a\neq c=\frac{1}{(-s)^m}$. If $z=c$, then the required result is obtained by (b). Now suppose that $z=a$. We use (6.2). For $z=a$, the right hand side of the above sum formula (6.2) is an indeterminate form. Now, we can use L'Hospital rule (twice). Then we get the required result by using

$$\sum_{k=0}^{n}a^kW_{mk+j}^2=\left.\frac{\frac{d^2}{dz^2}(\Omega_1)}{\frac{d^2}{dz^2}((r^2+4s)(1+(-s)^{2m}z^2-zH_{2m})((-s)^mz-1))}\right|_{z=a}.$$

(d) If $H_{2m}^2-4(-s)^{2m}=0$, then $a=b=\dfrac{H_{2m}}{2(-s)^{2m}}$. We suppose that $a=c=\frac{1}{(-s)^m}$. We use (6.2). For $z=a$, the right hand side of the above sum formula (6.2) is an indeterminate form. Now, we can use L'Hospital rule (three times). Then we get (b) by using

$$\sum_{k=0}^{n}a^kW_{mk+j}^2=\left.\frac{\frac{d^3}{dz^3}(\Omega_1)}{\frac{d^3}{dz^3}((r^2+4s)(1+(-s)^{2m}z^2-zH_{2m})((-s)^mz-1))}\right|_{z=a}$$

□

Note that (6.2) can be written in the following form:

$$\sum_{k=1}^{n}z^kW_{mk+j}^2=\frac{\Omega_6}{(r^2+4s)(1+(-s)^{2m}z^2-zH_{2m})((-s)^mz-1)}$$

where

$\Omega_6=(r^2+4s)((-s)^mz-1)((-s)^{2m}z-H_{2m})z^{n+1}W_{mn+j}^2+(r^2+4s)(-s)^{2m}((-s)^mz-1)z^{n+1}W_{mn-m+j}^2-(r^2+4s)((-s)^mz-1)((-s)^{2m}z-H_{2m})zW_j^2-(r^2+4s)(-s)^{2m}((-s)^mz-1)W_{j-m}^2z+2(-s)^j(W_1^2-sW_0^2-rW_1W_0)((-s)^{mn}z^n-1)(H_{2m}-2(-s)^m)z$.

The following theorem presents sum formulas of generalized Fibonacci polynomials in the case the roots α and β of characteristic equation (1.2) are equal, i.e., $\alpha=\beta$ so that $r^2+4s=0$.

Theorem 17 (Single Root Case)**.** *Assume that the roots α and β of characteristic equation (1.2) are equal, i.e., $\alpha=\beta$ so that $r^2+4s=0$. Let z be a real (or* *complex) number. For all integers m and j, for generalized Fibonacci (Horadam) polynomials, we have the following sum formulas:*

(a) *if $((-s)^m z-1)^3 \neq 0$, i.e., $z \neq (-s)^{-m}$, then*

$$\sum_{k=0}^{n} z^k W_{mk+j}^2 = \frac{\Omega_7}{((-s)^m z-1)^3} \tag{6.3}$$

where

$\Omega_7 = ((-s)^m z-1)((-s)^{2m} z - 2(-s)^m) z^{n+1} W_{mn+j}^2 + (-s)^{2m}((-s)^m z - 1) z^{n+1} W_{mn-m+j}^2 + ((-s)^m z - 1) W_j^2 - (-s)^{2m}((-s)^m z - 1) z W_{j-m}^2 + 2m^2 (-s)^{m+j-1}(W_1^2 - sW_0^2 - rW_1W_0)((-s)^{mn} z^n - 1) z.$

(b) *if $((-s)^m z-1)^3 = 0$, i.e., $z = (-s)^{-m}$, then*

$$\sum_{k=0}^{n} z^k W_{mk+j}^2 = \Omega_8$$

where

$\Omega_8 = \frac{1}{6s}(-s)^{-mn}(n+1)(3ns(-s)^m W_{mn-m+j}^2 - 3s(n-2)W_{mn+j}^2 - 2m^2 n(n-1)(-s)^{j+mn}(W_1^2 - sW_0^2 - rW_1W_0)).$

Proof. Note that

$$H_{2m} = \alpha^{2m} + \beta^{2m} = \left(\frac{r+\sqrt{r^2+4s}}{2}\right)^{2m} + \left(\frac{r-\sqrt{r^2+4s}}{2}\right)^{2m} \tag{6.4}$$

and if $r^2+4s=0$, then $s = -\frac{r^2}{4} = -\alpha^2$ and

$$H_{2m} = 2\alpha^{2m} = 2(-s)^m.$$

By using (6.4), we see that

$$\lim_{s\to -\frac{r^2}{4}} \frac{(H_{2m} - 2(-s)^m)}{(r^2+4s)} = m^2 (-s)^{m-1}. \tag{6.5}$$

(a) Use (6.5) and (6.2) (which is given in Theorem 16 (a)).

(b) We use (6.3). For $z=(-s)^{-m}$, the right hand side of the above sum formula (6.3) is an indeterminate form. Now, we can use L'Hospital rule (three times). Then we get (b) by using

$$\sum_{k=0}^{n} z^k W_{mk+j}^2 = \left.\frac{\frac{d^3}{dz^3}(\Omega_7)}{\frac{d^3}{dz^3}\left(((-s)^m z-1)^3\right)}\right|_{z=(-s)^{-m}} = \Omega_8.$$

□

Note that (6.3) can be written in the following form:

$$\sum_{k=1}^{n} z^k W_{mk+j}^2 = \frac{\Omega_9}{((-s)^m z-1)^3}$$

where

$\Omega_9 = ((-s)^m z-1)((-s)^{2m} z - 2(-s)^m) z^{n+1} W_{mn+j}^2 + (-s)^{2m}((-s)^m z - 1) z^{n+1} W_{mn-m+j}^2 - z(-s)^m((-s)^m z-2)((-s)^m z-1) W_j^2 - (-s)^{2m}((-s)^m z - 1) z W_{j-m}^2 + 2m^2(-s)^{m+j-1}(W_1^2 - sW_0^2 - rW_1W_0)((-s)^{mn} z^n - 1) z.$

6.3 The Sum Formula $\sum_{k=0}^{n} z^k W_{mk+j}^3$

In this section, we present sum formulas of generalized Fibonacci polynomials.

The following theorem presents sum formulas of generalized Fibonacci polynomials in the case the roots α and β of characteristic equation (1.2) are distinct, i.e., $r^2+4s\neq 0$.

Theorem 18 (Distinct Roots Case)**.** *Suppose that the roots α and β of characteristic equation (1.2) are distinct, i.e., $r^2+4s\neq 0$. Let z be a real (or complex) number. For all integers m and j, for generalized Fibonacci (Horadam) polynomials, we have the following sum formulas:*

(a) *If $((-s)^{3m}z^2 - zH_{3m}+1)((-s)^{3m}z^2 - z(-s)^m H_m + 1)\neq 0$, then*

$$\sum_{k=0}^{n} z^k W_{mk+j}^3 = \frac{\Psi_1}{(r^2+4s)((-s)^{3m}z^2 - zH_{3m}+1)((-s)^{3m}z^2 - z(-s)^m H_m+1)} \tag{6.6}$$

where

$\Psi_1 = (r^2+4s)z^{n+1}(-s)^{3m}((-s)^{3m}z^2-(-s)^m zH_m+1)W^3_{mn-m+j} + (r^2+4s)z^{n+1}((-s)^{3m}z-H_{3m})((-s)^{3m}z^2-(-s)^m zH_m+1)W^3_{mn+j} - (r^2+4s)(-s)^{3m}z((-s)^{3m}z^2-(-s)^m zH_m+1)W^3_{j-m} + (r^2+4s)((-s)^{3m}z^2-(-s)^m zH_m+1)W^3_j + 3z^n(-s)^{mn+m+j}z((-s)^{3m}z^2-zH_{3m}+1)(W_1^2-sW_0^2-rW_0W_1)W_{mn+m+j} + 3z^n(-s)^{mn+2m+j}z((-s)^{3m}z^2-(-s)^m zH_m+1)(W_1^2-sW_0^2-rW_0W_1)W_{mn-m+j} - 3z^n(-s)^{mn+j}z((-s)^{4m}z^2H_m-((-s)^m zH_m-1)H_{3m})(W_1^2-sW_0^2-rW_0W_1)W_{mn+j} - 3z(-s)^{m+j}((-s)^{3m}z^2-zH_{3m}+1)(W_1^2-sW_0^2-rW_0W_1)W_{m+j} - 3z(-s)^{2m+j}((-s)^{3m}z^2-(-s)^m zH_m+1)(W_1^2-sW_0^2-rW_0W_1)W_{j-m} + 3z(-s)^j((-s)^{4m}z^2H_m-H_{3m}((-s)^m zH_m-1))(W_1^2-sW_0^2-rW_0W_1)W_j.$

(b) *If* $((-s)^{3m}z^2-zH_{3m}+1)((-s)^{3m}z^2-z(-s)^m H_m+1)=u(z-a)(z-b)(z-c)(z-d)=0$ *for some* $u,a,b,c,d\in\mathbb{C}$ *and* $u\neq 0$ *and* $a\neq b\neq c\neq d$, *i.e.,* $z=a$ *or* $z=b$ *or* $z=c$ *or* $z=d$, *then*

$$\sum_{k=0}^{n} z^k W^3_{mk+j} = \frac{\Psi_2}{\Lambda_1}$$

where

$\Psi_2 = (r^2+4s)(-s)^{3m}z^n((-s)^{3m}z^2(n+3)-z(-s)^m(n+2)H_m+n+1)W^3_{mn-m+j} + (r^2+4s)((-s)^{6m}(n+4)z^3-(-s)^{3m}((-s)^m H_m+H_{3m})(n+3)z^2+(-s)^m(H_mH_{3m}+(-s)^{2m})(n+2)z-(n+1)H_{3m})z^nW^3_{mn+j} + (r^2+4s)(-s)^{3m}(-3(-s)^{3m}z^2+2(-s)^m zH_m-1)W^3_{j-m} + (r^2+4s)(2(-s)^{3m}z-(-s)^m H_m)W^3_j + 3(-s)^{mn+m+j}((-s)^{3m}(n+3)z^2-z(n+2)H_{3m}+n+1)(W_1^2-sW_0^2-rW_0W_1)z^nW_{mn+m+j} + 3(-s)^{mn+2m+j}((-s)^{3m}(n+3)z^2-z(-s)^m(n+2)H_m+n+1)z^n(W_1^2-sW_0^2-rW_0W_1)W_{mn-m+j} + 3(-s)^{mn+j}(-(-s)^{4m}(n+3)z^2H_m+z(-s)^m(n+2)H_{3m}H_m-(n+1)H_{3m})z^n(W_1^2-sW_0^2-rW_0W_1)W_{mn+j} + 3(-s)^{m+j}(-3(-s)^{3m}z^2+2zH_{3m}-1)(W_1^2-sW_0^2-rW_0W_1)W_{m+j} + 3(-s)^{2m+j}(-3(-s)^{3m}z^2+2(-s)^m zH_m-1)(W_1^2-sW_0^2-rW_0W_1)W_{j-m} + 3(-s)^j(3(-s)^{4m}z^2H_m-2(-s)^m zH_mH_{3m}+H_{3m})(W_1^2-sW_0^2-rW_0W_1)W_j$

and

$\Lambda_1 = (r^2 + 4s)(4(-s)^{6m}z^3 - 3(-s)^{3m}((-s)^m H_m + H_{3m})z^2 + 2(-s)^m(2(-s)^{2m} + H_m H_{3m})z - ((-s)^m H_m + H_{3m}))$.

(c) *If* $((-s)^{3m}z^2 - zH_{3m} + 1)((-s)^{3m}z^2 - z(-s)^m H_m + 1) = u(z-a)^2(z-b)(z-c) = 0$ *for some* $u, a, b, c \in \mathbb{C}$ *and* $u \neq 0$ *and* $a \neq b \neq c$, *i.e.,* $z = a$ *or* $z = b$ *or* $z = c$, *then if* $z = b$ *or* $z = c$, *then*

$$\sum_{k=0}^{n} z^k W_{mk+j}^3 = \frac{\Psi_3}{\Lambda_2}$$

where

$\Psi_3 = (r^2 + 4s)(-s)^{3m}z^n((-s)^{3m}z^2(n+3) - z(-s)^m(n+2)H_m + n + 1) W_{mn-m+j}^3 + (r^2 + 4s)((-s)^{6m}(n+4)z^3 - (-s)^{3m}((-s)^m H_m + H_{3m})(n+3)z^2 + (-s)^m(H_m H_{3m} + (-s)^{2m})(n+2)z - (n+1)H_{3m})z^n W_{mn+j}^3 + (r^2 + 4s)(-s)^{3m}(-3(-s)^{3m}z^2 + 2(-s)^m z H_m - 1)W_{j-m}^3 + (r^2 + 4s)(2(-s)^{3m}z - (-s)^m H_m)W_j^3 + 3(-s)^{mn+m+j}((-s)^{3m}(n+3)z^2 - z(n+2)H_{3m} + n + 1)(W_1^2 - sW_0^2 - rW_0W_1)z^n W_{mn+m+j} + 3(-s)^{mn+2m+j}((-s)^{3m}(n+3)z^2 - z(-s)^m(n+2)H_m + n + 1)z^n(W_1^2 - sW_0^2 - rW_0W_1)W_{mn-m+j} + 3(-s)^{mn+j}(-(-s)^{4m}(n+3)z^2 H_m + z(-s)^m(n+2)H_{3m}H_m - (n+1)H_{3m})z^n(W_1^2 - sW_0^2 - rW_0W_1) W_{mn+j} + 3(-s)^{m+j}(-3(-s)^{3m}z^2 + 2zH_{3m} - 1)(W_1^2 - sW_0^2 - rW_0W_1)W_{m+j} + 3(-s)^{2m+j}(-3(-s)^{3m}z^2 + 2(-s)^m z H_m - 1)(W_1^2 - sW_0^2 - rW_0W_1)W_{j-m} + 3(-s)^j(3(-s)^{4m}z^2 H_m - 2(-s)^m z H_m H_{3m} + H_{3m})(W_1^2 - sW_0^2 - rW_0W_1)W_j$

and

$\Lambda_2 = (r^2 + 4s)(4(-s)^{6m}z^3 - 3(-s)^{3m}((-s)^m H_m + H_{3m})z^2 + 2(-s)^m(2(-s)^{2m} + H_m H_{3m})z - ((-s)^m H_m + H_{3m}))$

and if $z = a$, *then*

$$\sum_{k=0}^{n} z^k W_{mk+j}^3 = \frac{\Psi_4}{2(r^2 + 4s)(-s)^m(6(-s)^{5m}z^2 - 3z(-s)^{2m}((-s)^m H_m + H_{3m}) + 2(-s)^{2m} + H_m H_{3m})}$$

where

$\Psi_4 = (r^2 + 4s)(-s)^{3m}((-s)^{3m}(n+3)(n+2)z^2 - (-s)^m z(n+2)(n+1)H_m + n(n+1))z^{n-1}W_{mn-m+j}^3 + (r^2 + 4s)((-s)^{6m}(n+4)(n+3)z^3$

$-\ (-s)^{3m}(n+3)(n+2)((-s)^m H_m + H_{3m})z^2 + z(-s)^m(n+2)(n+1)(H_m H_{3m} + (-s)^{2m}) - n(n+1)H_{3m})z^{n-1}W^3_{mn+j} + 2(r^2+4s)(-s)^{4m}(H_m - 3(-s)^{2m}z)W^3_{j-m} + 2(r^2+4s)(-s)^{3m}W^3_j + 3(-s)^{mn+m+j}((-s)^{3m}(n+3)(n+2)z^2 - z(n+2)(n+1)H_{3m} + n(n+1))(W_1^2 - sW_0^2 - rW_0W_1)z^{n-1}W_{mn+m+j} + 3z^{n-1}(-s)^{mn+2m+j}((-s)^{3m}(n+3)(n+2)z^2 - z(-s)^m(n+2)(n+1)H_m + n(n+1))(W_1^2 - sW_0^2 - rW_0W_1)W_{mn-m+j} + 3z^{n-1}(-s)^{mn+j}(-z^2(-s)^{4m}(n+3)(n+2)H_m + z(-s)^m(n+2)(n+1)H_{3m}H_m - n(n+1)H_{3m})(W_1^2 - sW_0^2 - rW_0W_1)W_{mn+j} + 6(-s)^{m+j}(H_{3m} - 3(-s)^{3m}z)(W_1^2 - sW_0^2 - rW_0W_1)W_{m+j} + 6(-s)^{3m+j}(H_m - 3(-s)^{2m}z)(W_1^2 - sW_0^2 - rW_0W_1)W_{j-m} + 6(-s)^{m+j}(3(-s)^{3m}z - H_{3m})H_m(W_1^2 - sW_0^2 - rW_0W_1)W_j.$

(d) *If* $((-s)^{3m}z^2 - zH_{3m}+1)((-s)^{3m}z^2 - z(-s)^m H_m + 1) = u(z-a)^3(z-b) = 0$ *for some* $u, a, b \in \mathbb{C}$ *and* $u \neq 0$ *and* $a \neq b$, *i.e.,* $z = a$ *or* $z = b$, *then if* $z = b$, *then*

$$\sum_{k=0}^{n} z^k W^3_{mk+j} = \frac{\Psi_5}{\Lambda_3}$$

where

$\Psi_5 = (r^2+4s)(-s)^{3m}z^n((-s)^{3m}z^2(n+3) - z(-s)^m(n+2)H_m + n+1) W^3_{mn-m+j} + (r^2+4s)((-s)^{6m}(n+4)z^3 - (-s)^{3m}((-s)^m H_m + H_{3m})(n+3)z^2 + (-s)^m(H_mH_{3m} + (-s)^{2m})(n+2)z - (n+1)H_{3m})z^nW^3_{mn+j} + (r^2+4s)(-s)^{3m}(-3(-s)^{3m}z^2 + 2(-s)^m zH_m - 1)W^3_{j-m} + (r^2+4s)(2(-s)^{3m}z - (-s)^m H_m)W^3_j + 3(-s)^{mn+m+j}((-s)^{3m}(n+3)z^2 - z(n+2)H_{3m} + n+1)(W_1^2 - sW_0^2 - rW_0W_1)z^nW_{mn+m+j} + 3(-s)^{mn+2m+j}((-s)^{3m}(n+3)z^2 - z(-s)^m(n+2)H_m + n+1)z^n(W_1^2 - sW_0^2 - rW_0W_1)W_{mn-m+j} + 3(-s)^{mn+j}(-(-s)^{4m}(n+3)z^2H_m + z(-s)^m(n+2)H_{3m}H_m - (n+1)H_{3m})z^n(W_1^2 - sW_0^2 - rW_0W_1) W_{mn+j} + 3(-s)^{m+j}(-3(-s)^{3m}z^2 + 2zH_{3m} - 1)(W_1^2 - sW_0^2 - rW_0W_1)W_{m+j} + 3(-s)^{2m+j}(-3(-s)^{3m}z^2 + 2(-s)^m zH_m - 1)(W_1^2 - sW_0^2 - rW_0W_1)W_{j-m} + 3(-s)^j(3(-s)^{4m}z^2H_m - 2(-s)^m zH_mH_{3m} + H_{3m})(W_1^2 - sW_0^2 - rW_0W_1)W_j$

and

$\Lambda_3 = (r^2+4s)(4(-s)^{6m}z^3 - 3(-s)^{3m}((-s)^m H_m + H_{3m})z^2 + 2(-s)^m(2(-s)^{2m} + H_mH_{3m})z - ((-s)^m H_m + H_{3m}))$

and if $z=a$, then

$$\sum_{k=0}^{n} z^k W_{mk+j}^3 = \frac{\Psi_6}{6(r^2+4s)\left(-s\right)^{3m}\left(4\left(-s\right)^{3m}z-\left(-s\right)^{m}H_m-H_{3m}\right)}$$

where

$\Psi_6 = (r^2+4s)(-s)^{3m}(n+1)((-s)^{3m}(n+3)(n+2)z^2 - z(-s)^m n(n+2)H_m + n(n-1))z^{n-2}W_{mn-m+j}^3 + (r^2+4s)((-s)^{6m}(n+3)(n+2)(n+4)z^3 - (-s)^{3m}(n+3)(n+2)(n+1)((-s)^m H_m + H_{3m})z^2 + (-s)^m n(n+2)(n+1)(H_m H_{3m} + (-s)^{2m})z - n(n-1)(n+1)H_{3m})z^{n-2}W_{mn+j}^3 - 6(r^2+4s)(-s)^{6m}W_{j-m}^3 + 3(-s)^{mn+m+j}(n+1)((-s)^{3m}(n+3)(n+2)z^2 - zn(n+2)H_{3m} + n(n-1))(W_1^2 - sW_0^2 - rW_0W_1)z^{n-2}W_{mn+m+j} + 3(-s)^{mn+2m+j}(n+1)((-s)^{3m}(n+3)(n+2)z^2 - z(-s)^m n(n+2)H_m + n(n-1))(W_1^2 - sW_0^2 - rW_0W_1)z^{n-2}W_{mn-m+j} + 3(-s)^{mn+j}(n+1)(-z^2(-s)^{4m}(n+3)(n+2)H_m + z(-s)^m n(n+2)H_{3m}H_m - n(n-1)H_{3m})(W_1^2 - sW_0^2 - rW_0W_1) z^{n-2}W_{mn+j} - 18(-s)^{4m+j}(W_1^2 - sW_0^2 - rW_0W_1)W_{m+j} - 18(-s)^{5m+j}(W_1^2 - sW_0^2 - rW_0W_1)W_{j-m} + 18(-s)^{4m+j}H_m(W_1^2 - sW_0^2 - rW_0W_1)W_j.$

(e) *If $((-s)^{3m}z^2 - zH_{3m}+1)((-s)^{3m}z^2 - z(-s)^m H_m + 1) = u(z-a)^4 = 0$ for some $u, a \in \mathbb{C}, u \neq 0$ i.e., $z=a$, then*

$$\sum_{k=0}^{n} z^k W_{mk+j}^3 = \frac{\Psi_7}{24(r^2+4s)\left(-s\right)^{6m}}$$

where

$\Psi_7 = (r^2+4s)(-s)^{3m}n(n+1)((-s)^{3m}(n+3)(n+2)z^2 - z(-s)^m(n-1)(n+2)H_m + (n-1)(n-2))z^{n-3}W_{mn-m+j}^3 + (r^2+4s)(n+1)(z^3(-s)^{6m}(n+4)(n+3)(n+2) - z^2(-s)^{3m}n(n+3)(n+2)((-s)^m H_m + H_{3m}) + z(-s)^m n(n-1)(n+2)(H_m H_{3m} + (-s)^{2m}) - n(n-1)(n-2)H_{3m})z^{n-3}W_{mn+j}^3 + 3(-s)^{mn+m+j}n(n+1)(z^2(-s)^{3m}(n+3)(n+2) - z(n+2)(n-1)H_{3m} + (n-1)(n-2))(W_1^2 - sW_0^2 - rW_0W_1)z^{n-3}W_{mn+m+j} + 3(-s)^{mn+2m+j}n(n+1)(z^2(-s)^{3m}(n+3)(n+2) - z(-s)^m(n+2)(n-1)H_m + (n-1)(n-2))(W_1^2 - sW_0^2 - rW_0W_1)z^{n-3}W_{mn-m+j} + 3(-s)^{mn+j}n(n+1)(-z^2(-s)^{4m}(n+3)(n+2)H_m +$

$z(-s)^m(n+2)(n-1)H_{3m}H_m-(n-1)(n-2)H_{3m})(W_1^2-sW_0^2-rW_0W_1)$
$z^{n-3}W_{mn+j}$.

Proof.

(a) *Note that*

$$\begin{aligned}
&\sum_{k=0}^{n-1} z^k W_{mk+j}^3 \\
=& \sum_{k=0}^{n-1}(A_1\alpha^{mk+j}+A_2\beta^{mk+j})^3 z^k \\
=& A_1^3\alpha^{3j}\sum_{k=0}^{n-1}\alpha^{3mk}z^k + A_2^3\beta^{3j}\sum_{k=0}^{n-1}\beta^{3mk}z^k \\
&+3A_1A_2^2\alpha^j\beta^{2j}\sum_{k=0}^{n-1}\alpha^{mk}\beta^{2mk}z^k+3A_1^2A_2\alpha^{2j}\beta^j\sum_{k=0}^{n-1}\alpha^{2mk}\beta^{mk}z^k \\
=& A_1^3\alpha^{3j}\left(\frac{\alpha^{3mn}z^n-1}{\alpha^{3m}z-1}\right)+A_2^3\beta^{3j}\left(\frac{\beta^{3mn}z^n-1}{\beta^{3m}z-1}\right) \\
&+3A_1A_2^2(-s)^j\beta^j\frac{(-s)^{mn}\beta^{mn}z^n-1}{(-s)^m\beta^m z-1}+3A_1^2A_2(-s)^j\alpha^j\frac{(-s)^{mn}\alpha^{mn}z^n-1}{(-s)^m\alpha^m z-1}.
\end{aligned}$$

Simplifying the last equalities in the last two expression imply (6.6) as required.

(b) We use (6.6). For $z=a$, the right hand side of the above sum formula (6.6) is an indeterminate form. Now, we can use L'Hospital rule. Then we get (b) by using

$$\begin{aligned}
&\sum_{k=0}^{n} a^k W_{mk+j}^3 \\
=& \left.\frac{\frac{d}{dz}(\Psi_1)}{\frac{d}{dz}(r^2+4s)((-s)^{3m}z^2-zH_{3m}+1)(((-s)^{3m}z^2-z(-s)^m H_m+1)}\right|_{z=a} \\
=& \frac{\Psi_2}{\Psi}.
\end{aligned}$$

The proof for the case $z=b$, $z=c$ and $z=d$ are the same.

(c) If $z=b$ or $z=c$, then the required result is obtained by (b). Now suppose that $z=a$. We use (6.6). For $z=a$, the right hand side of the above sum formula (6.6) is an indeterminate form. Now, we can use L'Hospital rule (twice). Then we get the required result by using

$$\begin{aligned}
&\sum_{k=0}^{n} a^k W_{mk+j}^3 \\
&= \left.\frac{\frac{d^2}{dz^2}(\Psi_1)}{\frac{d^2}{dz^2}(r^2+4s)((-s)^{3m}z^2 - zH_{3m}+1)(\left(-s\right)^{3m}z^2 - z\left(-s\right)^m H_m + 1)}\right|_{z=a} \\
&= \frac{\Psi_3}{2(r^2+4s)(-s)^m(6(-s)^{5m}z^2 - 3z(-s)^{2m}\left((-s)^m H_m + H_{3m}\right) + 2(-s)^{2m} + H_m H_{3m}\big)}.
\end{aligned}$$

(d) If $z=b$, then the required result is obtained by (b). Now suppose that $z=a$. We use (6.6). For $z=a$, the right hand side of the above sum formula (6.6) is an indeterminate form. Now, we can use L'Hospital rule (three times). Then we get the required result by using

$$\begin{aligned}
&\sum_{k=0}^{n} a^k W_{mk+j}^3 \\
&= \left.\frac{\frac{d^3}{dz^3}(\Psi_1)}{\frac{d^3}{dz^3}(r^2+4s)((-s)^{3m}z^2 - zH_{3m}+1)(\left(-s\right)^{3m}z^2 - z\left(-s\right)^m H_m + 1)}\right|_{z=a} \\
&= \frac{\Psi_4}{6(r^2+4s)\left(-s\right)^{3m}\left(4\left(-s\right)^{3m}z - \left(-s\right)^m H_m - H_{3m}\right)}.
\end{aligned}$$

(e) We use (6.6). For $z=a$, the right hand side of the above sum formula (6.6) is an indeterminate form. Now, we can use L'Hospital rule (four times). Then we get the required result by using

$$\begin{aligned}
&\sum_{k=0}^{n} a^k W_{mk+j}^3 \\
&= \left.\frac{\frac{d^4}{dz^4}(\Psi_1)}{\frac{d^4}{dz^4}(r^2+4s)((-s)^{3m}z^2 - zH_{3m}+1)(\left(-s\right)^{3m}z^2 - z\left(-s\right)^m H_m + 1)}\right|_{z=a} \\
&= \frac{\Psi_7}{24(r^2+4s)\left(-s\right)^{6m}}.
\end{aligned}$$

□

Note that (6.6) can be written in the following form:

$$\sum_{k=1}^{n} z^k W_{mk+j}^3 = \frac{\Psi_8}{(r^2+4s)((-s)^{3m}z^2 - zH_{3m}+1)((-s)^{3m}z^2 - z(-s)^m H_m + 1)}$$

where

$\Psi_8 = (r^2+4s)z^{n+1}(-s)^{3m}((-s)^{3m}z^2 - (-s)^m zH_m + 1)W_{mn-m+j}^3 + (r^2+4s) z^{n+1}((-s)^{3m}z - H_{3m})((-s)^{3m}z^2 - (-s)^m zH_m + 1)W_{mn+j}^3 - (r^2+4s)(-s)^{3m}z((-s)^{3m}z^2 - (-s)^m zH_m + 1)W_{j-m}^3 + (4s+r^2)(H_{3m} - (-s)^{3m}z)((-s)^{3m}z^2 - (-s)^m zH_m + 1)zW_j^3 + 3z^n(-s)^{mn+m+j}z((-s)^{3m}z^2 - zH_{3m} + 1)(W_1^2 - sW_0^2 - rW_0W_1)W_{mn+m+j} + 3z^n(-s)^{mn+2m+j} z((-s)^{3m}z^2 - (-s)^m zH_m + 1)(W_1^2 - sW_0^2 - rW_0W_1)W_{mn-m+j} - 3z^n(-s)^{mn+j}z((-s)^{4m}z^2H_m - ((-s)^m zH_m - 1)H_{3m})(W_1^2 - sW_0^2 - rW_0W_1) W_{mn+j} - 3z(-s)^{m+j}((-s)^{3m}z^2 - zH_{3m} + 1)(W_1^2 - sW_0^2 - rW_0W_1)W_{m+j} - 3z(-s)^{2m+j}((-s)^{3m}z^2 - (-s)^m zH_m + 1)(W_1^2 - sW_0^2 - rW_0W_1)W_{j-m} + 3z(-s)^j((-s)^{4m}z^2H_m - H_{3m}((-s)^m zH_m - 1))(W_1^2 - sW_0^2 - rW_0W_1)W_j.$

The following theorem presents sum formulas of generalized Fibonacci polynomials in the case the roots α and β of characteristic equation (1.2) are equal, i.e., $\alpha = \beta$ so that $r^2 + 4s = 0$.

Theorem 19 (Single Root Case)**.** *Assume that the roots α and β of characteristic equation (1.2) are equal, i.e., $\alpha = \beta$ so that $r^2 + 4s = 0$. Let z be a real (or complex) number. For all integers m and j, for generalized Fibonacci (Horadam) polynomials, we have the following sum formulas:if $(z\left(\frac{r}{2}\right)^{3m} - 1)^4 \neq 0$, i.e., $z \neq \left(\frac{r}{2}\right)^{-3m}$, then*

$$\sum_{k=0}^{n} z^k W_{mk+j}^3 = \frac{\Psi_9}{(z\left(\frac{r}{2}\right)^{3m} - 1)^4} \tag{6.7}$$

where

$\Psi_9 = ((z\left(\frac{r}{2}\right)^{3m} - 1)^4((D_1 + D_2(mn+j))\left(\frac{r}{2}\right)^{mn+j})^3 z^n + (D_1^3(z\left(\frac{r}{2}\right)^{3m} - 1)^3(z^n\left(\frac{r}{2}\right)^{3mn+3j} - \left(\frac{r}{2}\right)^{3j}) + D_2^3(j^3\left(\frac{r}{2}\right)^{3j} - z\left(\frac{r}{2}\right)^{3m+3j}(z^2\left(\frac{r}{2}\right)^{6m}(j - m)^3 - z\left(\frac{r}{2}\right)^{3m}(-6j^2m + 3j^3 + 4m^3) - 3jm^2 - 3j^2m + 3j^3 - m^3) + z^n\left(\frac{r}{2}\right)^{3mn+3j}(z\left(\frac{r}{2}\right)^{3m}(-3jm^2 - 3j^2m - 3m^3n - 3m^3n^2 + 3m^3n^3 + 3j^3 -$

$m^3 - 6jm^2n + 9j^2mn + 9jm^2n^2) - z^2\left(\frac{r}{2}\right)^{6m}(-6j^2m - 6m^3n^2 + 3m^3n^3 + 3j^3 + 4m^3 - 12jm^2n + 9j^2mn + 9jm^2n^2) + z^3\left(\frac{r}{2}\right)^{9m}(j - m + mn)^3 - (j + mn)^3))) + 3D_1D_2(z\left(\frac{r}{2}\right)^{3m} - 1)(D_1(z\left(\frac{r}{2}\right)^{3m} - 1)(j\left(\frac{r}{2}\right)^{3j} - \left(\frac{r}{2}\right)^{3m+3j}(j-m)z + \left(\frac{r}{2}\right)^{3mn+3j}(-j - mn + z\left(\frac{r}{2}\right)^{3m}(j - m + mn))z^n) + D_2(-j^2\left(\frac{r}{2}\right)^{3j} + \left(\frac{r}{2}\right)^{3m+3j}(-2jm + 2j^2 - m^2)z - \left(\frac{r}{2}\right)^{6m+3j}(j-m)^2z^2 + \left(\frac{r}{2}\right)^{3mn+3j}(m^2n^2 + j^2 + 2jmn - z\left(\frac{r}{2}\right)^{3m}(-2m^2n + 2m^2n^2 - 2jm + 2j^2 - m^2 + 4jmn) + z^2\left(\frac{r}{2}\right)^{6m}(j - m + mn)^2)z^n))).$

Proof. Note that

$$\begin{aligned}\sum_{k=0}^{n-1} z^k W_{mk+j}^3 &= \sum_{k=0}^{n-1}((D_1 + D_2(mk+j))\alpha^{mk+j})^3 z^k \\ &= \sum_{k=0}^{n-1}\left(D_1^3\alpha^{3mk+3j}z^k + D_2^3(mk+j)^3\alpha^{3mk+3j}z^k\right) \\ &\quad + \sum_{k=0}^{n-1} 3D_1D_2(mk+j)\left(D_1 + D_2(mk+j)\right)\alpha^{3mk+3j}z^k.\end{aligned}$$

Simplifying the last equalities in the last two expression imply (6.7) as required. □

6.4 The Sum Formulas $\sum_{k=0}^{n} z^k W_k$ and $\sum_{k=0}^{n} kz^k W_k$ via Generating Functions

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} z^n W_n u^n$ of the sequence $\{z^n W_n\}$.

Lemma 20. *Suppose that* $f_{z^n W_n}(u) = \sum_{n=0}^{\infty} z^n W_n u^n$ *is the ordinary generating function of the sequence* $\{z^n W_n\}_{n\geq 0}$. *Then,* $\sum_{n=0}^{\infty} z^n W_n u^n$ *is given by*

$$\sum_{n=0}^{\infty} z^n W_n u^n = \frac{W_0 + z(W_1 - rW_0)u}{1 - rzu - sz^2u^2}. \tag{6.8}$$

Proof. Note that

$$z^n W_n = z^n(rW_{n-1} + sW_{n-2}).$$

Using the definition of generalized Fibonacci numbers, and substracting $rzu\sum_{n=0}^{\infty} z^n W_n u^n$ and $sz^2u^2\sum_{n=0}^{\infty} z^n W_n u^n$ from $\sum_{n=0}^{\infty} z^n W_n u^n$, we obtain

$$\begin{aligned}
&(1-rzu-sz^2u^2)\sum_{n=0}^{\infty} z^n W_n u^n \\
=\;& \sum_{n=0}^{\infty} z^n W_n u^n - rzu\sum_{n=0}^{\infty} z^n W_n u^n - sz^2u^2\sum_{n=0}^{\infty} z^n W_n u^n \\
=\;& \sum_{n=0}^{\infty} z^n W_n u^n - r\sum_{n=0}^{\infty} z^{n+1} W_n u^{n+1} - s\sum_{n=0}^{\infty} z^{n+2} W_n u^{n+2} \\
=\;& \sum_{n=0}^{\infty} z^n W_n u^n - r\sum_{n=1}^{\infty} z^n W_{n-1} u^n - s\sum_{n=2}^{\infty} z^n W_{n-2} u^n \\
=\;& (W_0+zW_1u) - rzW_0u + \sum_{n=2}^{\infty} z^n (W_n - rW_{n-1} - sW_{n-2})u^n \\
=\;& W_0 + z(W_1 - rW_0)u.
\end{aligned}$$

Rearranging above equation, we obtain (6.8). □

Lemma 20 gives the following results as particular examples.

Corollary 21. *Generating functions* $\sum_{n=0}^{\infty} z^n G_n u^n$ *and* $\sum_{n=0}^{\infty} z^n H_n u^n$ *are*

$$\begin{aligned}
\sum_{n=0}^{\infty} z^n G_n u^n &= \frac{zu}{1-rzu-sz^2u^2}, \\
\sum_{n=0}^{\infty} z^n H_n u^n &= \frac{2-rzu}{1-rzu-sz^2u^2},
\end{aligned}$$

respectively.

The following theorem presents some sum formulas of generalized Fibonacci (Horadam) polynomials with positive subscripts.

Theorem 22. *Let z be a nonzero complex (or real) number. Let*

$$S_n = \sum_{k=0}^{n} z^k W_k.$$

Then we have the following properties.

(a) *If* $1 - rz - sz^2 \neq 0$, *then*

$$\begin{aligned} S_n &= \frac{W_0 + z(W_1 - rW_0)}{1 - rz - sz^2} - \frac{2W_1sz^2 + 2W_0sz + W_1rz - W_0rsz^2}{2(1 - rz - sz^2)} z^n G_n \\ &\quad - \frac{W_1z + W_0sz^2}{2(1 - rz - sz^2)} z^n H_n \\ &= \frac{\Theta(z)}{2(1 - rz - sz^2)} \end{aligned} \tag{6.9}$$

where

$\Theta(z) = 2(W_0 + (W_1 - rW_0)z) - (s(2W_1 - rW_0)z + (rW_1 + 2sW_0))z^{n+1}G_n - (W_1 + szW_0)z^{n+1}H_n$.

(b) *If* $1 - rz - sz^2 = u(z - a)(z - b) = 0$ *for some* $u, a, b \in \mathbb{C}$ *with* $u \neq 0$ *and* $a \neq b$, *i.e.*, $z = a$ *or* $z = b$, *then*

$$S_n = \frac{\Theta_1(z)}{-2\,(r + 2sz)}$$

where

$\Theta_1(z) = 2(W_1 - rW_0) + (-(r + nr + 4sz + 2nsz)W_1 + s(-2n + 2rz + nrz - 2)W_0)z^nG_n - z^n((n + 1)W_1 + sz(n + 2)W_0)H_n$.

(c) *If* $1 - rz - sz^2 = u(z - c)^2 = 0$ *for some* $u, c \in \mathbb{C}$ *with* $u \neq 0$, *i.e.*, $z = c$, *then*

$$S_n = \frac{\Theta_2(z)}{4s}$$

where

$\Theta_2(z) = (n+1)((nr + 4sz + 2nsz)W_1 + s(2n - 2rz - nrz)W_0)z^{n-1}G_n + (n + 1)(nz^{n-1}W_1 + sz^n(n + 2)W_0)H_n$.

Proof.

(a) Note that using generating functions, we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} S_n u^n \\
&= \frac{1}{1-u}\frac{W_0+z(W_1-rW_0)u}{1-rzu-sz^2u^2} \\
&= \frac{W_0+z(W_1-rW_0)}{1-rz-sz^2}\frac{1}{1-u} \\
&\quad -\frac{2W_1sz^2+2W_0sz+W_1rz-W_0rsz^2}{2(1-rz-sz^2)}\frac{zu}{1-rzu-sz^2u^2} \\
&\quad -\frac{W_1z+W_0sz^2}{2(1-rz-sz^2)}\frac{2-rzu}{1-rzu-sz^2u^2} \\
&= \frac{W_0+z(W_1-rW_0)}{1-rz-sz^2}\sum_{n=0}^{\infty} u^n \\
&\quad -\frac{2W_1sz^2+2W_0sz+W_1rz-W_0rsz^2}{2(1-rz-sz^2)}\sum_{n=0}^{\infty} z^n G_n u^n \\
&\quad -\frac{W_1z+W_0sz^2}{2(1-rz-sz^2)}\sum_{n=0}^{\infty} z^n H_n u^n \\
&= \sum_{n=0}^{\infty}\left(\frac{W_0+z(W_1-rW_0)}{1-rz-sz^2}-\frac{2W_1sz^2+2W_0sz+W_1rz-W_0rsz^2}{2(1-rz-sz^2)}z^nG_n\right. \\
&\quad \left.-\frac{W_1z+W_0sz^2}{2(1-rz-sz^2)}z^nH_n\right)u^n.
\end{aligned}$$

Comparing on both sides leads to (6.9).

(b) We use (6.9). For $z=a$ and $z=b$, the right hand side of the above sum formula (6.9) is an indeterminate form. Now, we can use L'Hospital rule. Then we get (b) by using

$$\sum_{k=0}^{n} a^k W_k = \left.\frac{\frac{d}{dz}\Theta(z)}{\frac{d}{dz}(2(1-rz-sz^2))}\right|_{z=a}$$

and similarly

$$\sum_{k=0}^{n} b^k W_k = \left.\frac{\frac{d}{dz}\Theta(z)}{\frac{d}{dz}(2(1-rz-sz^2))}\right|_{z=b}.$$

(c) We use (6.9). For $z = c$, the right hand side of the above sum formula (6.9) is an indeterminate form. Now, we can use L'Hospital rule (twice). Then we get (c) by using

$$\sum_{k=0}^{n} c^k W_k = \left. \frac{\frac{d^2}{dz^2}\Theta(z)}{\frac{d^2}{dz^2}(2(1-rz-sz^2))} \right|_{z=c}.$$

□

The following theorem presents some sum formulas of generalized Fibonacci (Horadam) polynomials with positive subscripts.

Theorem 23. *Let z be a non-zero complex (or real) number. Let*

$$Y_n = \sum_{k=0}^{n} k z^k W_k.$$

Then

(a) *If $1 - rz - sz^2 \neq 0$, then*

$$\begin{aligned} Y_n &= \frac{z(1-rz-sz^2)\Theta'(z) + z(r+2sz)\Theta(z)}{2(1-rz-sz^2)^2} \qquad (6.10) \\ &= \frac{\Delta_1}{2(1-rz-sz^2)^2} \end{aligned}$$

where $\Theta(z)$ is as in Theorem 22 (a) and $\Theta'(z)$ denotes the derivative of $\Theta(z)$ with respect to z, and

$\Delta_1 = 2z((sz^2+1)W_1 - sz(rz-2)W_0) + ((2ns^2z^3 + rsz^2 + 3nrsz^2 + nr^2z - 2nsz - 4sz - r - nr)W_1 - s(nrsz^3 + 2sz^2 + r^2z^2 - 2nsz^2 + nr^2z^2 - 2rz - 3nrz + 2n + 2)W_0)z^{n+1}G_n + ((nsz^2 - sz^2 + nrz - n - 1)W_1 + sz(nsz^2 + nrz - n + rz - 2)W_0)z^{n+1}H_n.$

(b) *If $1 - rz - sz^2 = u(z-a)(z-b) = 0$ for some $u, a, b \in \mathbb{C}$ with $u \neq 0$ and $a \neq b$, i.e., $z = a$ or $z = b$, then*

$$\begin{aligned} Y_n &= \frac{4s\Theta(z) + 2sz\Theta'(z) - \left(4sz^2 + 3rz - 2\right)\Theta''(z) + z(1-rz-sz^2)\Theta'''(z)}{4\left(-2s + 6s^2z^2 + r^2 + 6rsz\right)} \\ &= \frac{\Delta_2}{4\left(-2s + 6s^2z^2 + r^2 + 6rsz\right)} \end{aligned}$$

where $\Theta^{''}(z)$ *and* $\Theta^{'''}(z)$ *denote the second and third derivatives of* $\Theta(z)$, *respectively, with respect to* z, *and*

$\Delta_2 = 12szW_1 - 4s(3rz-2)W_0 + ((-nr(n+1)^2 + z((n+2)(n+1)(nr^2-2ns-4s)+sz(n+3)(2n^2sz+3n^2r+8nsz+7nr+2r)))W_1 + s(-2n(n+1)^2 + z(r(3n+2)(n+2)(n+1) - z(n+3)(n^2rsz+3nr^2-2n^2s+n^2r^2+4nrsz-2ns+2r^2+4s)))W_0)z^{n-1}G_n + ((-n(n+1)^2 + z(n+2)(n^2r+n^2sz+2nsz+nr-3sz))W_1 + sz(-(n+1)(n+2)^2 + z(n+3)(n^2sz+n^2r+4nsz+2r+3nr))W_0)z^{n-1}H_n.$

(c) *If* $1-rz-sz^2 = u(z-c)^2 = 0$ *for some* $u, c \in \mathbb{C}$ *with* $u \neq 0$, *i.e.,* $z = c$, *then*

$$\begin{aligned} Y_n &= \frac{-4\,(2r+5sz)\,\Theta^{'''}(z) - (10sz^2+7rz-4)\Theta^{''''}(z) + z((1-rz-sz^2)\Theta^{'''''}(z)}{48s^2} \\ &= \frac{\Delta_3}{12s^2} \end{aligned}$$

where $\Theta^{''''}(z)$ *and* $\Theta^{'''''}(z)$ *denote the fourth and fifth derivatives of* $\Theta(z)$, *respectively, with respect to* z, *and*

$\Delta_3 = z^{-2}((((nz^n(2r+5sz)(n+1)(-r+nr+4sz+2nsz) + 546z^{12}(630s^2z^3+187r^2z+760rsz^2-450sz-154r))W_1 + s(nz^n(n+1)(2n-2rz-nrz-2)(2r+5sz) - 546z^{12}(315rsz^3-440sz^2+270r^2z^2-599rz+308))W_0)G_n + ((nz^n(n-1)(n+1)(2r+5sz) + 6006z^{12}(20sz^2+17rz-14))W_1 + sz(nz^n(n+2)(n+1)(2r+5sz) + 24570z^{12}(7sz^2+6rz-5))W_0)H_n).$

Proof.

(a) We know from Theorem 22 that

$$S_n = \sum_{k=0}^{n} z^k W_k = \frac{\Theta(z)}{2(1-rz-sz^2)}$$

where

$$\begin{aligned} \Theta(z) &= 2(W_0 + (W_1 - rW_0)z) - (s(2W_1 - rW_0)z + (rW_1 + 2sW_0))z^{n+1}G_n \\ &\quad -(W_1 + szW_0)z^{n+1}H_n. \end{aligned}$$

By taking the derivative of the both sides of the above formulas with respect to z, we get

$$\sum_{k=0}^{n} kz^{k-1}W_k = \frac{(1-rz-sz^2)\Theta'(z)+(r+2sz)\Theta(z)}{2(1-rz-sz^2)^2}$$

i.e.

$$\begin{aligned} Y_n &= \sum_{k=0}^{n} kz^{k}W_k = \frac{z(1-rz-sz^2)\Theta'(z)+z(r+2sz)\Theta(z)}{2(1-rz-sz^2)^2} \\ &= \frac{\Delta_1}{2(1-rz-sz^2)^2}. \end{aligned}$$

(b) We use (a). For $z=a$ and $z=b$, the right hand side of the above sum formula 6.10) is an indeterminate form. Now, we can use L'Hospital rule (twice). Then we get (b) by using

$$\begin{aligned} Y_n &= \sum_{k=0}^{n} ka^{k}W_k = \left.\frac{\frac{d^2}{dz^2}(z(1-rz-sz^2)\Theta'(z)+z(r+2sz)\Theta(z))}{\frac{d^2}{dz^2}(2(1-rz-sz^2)^2)}\right|_{z=a} \\ &= \frac{4s\Theta(a)+2sa\Theta'(a)-\left(4sa^2+3ra-2\right)\Theta''(a)+a(1-ra-sa^2)\Theta'''(a)}{4\left(-2s+6s^2a^2+r^2+6rsa\right)} \\ &= \frac{\Delta_2}{4\left(-2s+6s^2a^2+r^2+6rsa\right)} \end{aligned}$$

and similarly

$$\begin{aligned} Y_n &= \sum_{k=0}^{n} ka^{k}W_k \\ &= \frac{4s\Theta(b)+2sb\Theta'(b)-\left(4sb^2+3rb-2\right)\Theta''(b)+b(1-rb-sb^2)\Theta'''(b)}{4\left(-2s+6s^2b^2+r^2+6rsb\right)}. \end{aligned}$$

(c) We use (a). For $z=c$, the right hand side of the above sum formula 6.10) is an indeterminate form. Now, we can use L'Hospital rule (four times). Then we get (c) by using

$$\begin{aligned} Y_n &= \sum_{k=0}^{n} ka^k W_k = \left.\frac{\frac{d^4}{dz^4}(z(1-rz-sz^2)\Theta^{'}(z)+z(r+2sz)\Theta(z))}{\frac{d^4}{dz^4}(2(1-rz-sz^2)^2)}\right|_{z=c} \\ &= \frac{-4\left(2r+5sz\right)\Theta^{'''}(z)-(10sz^2+7rz-4)\Theta^{''''}(z)+z((1-rz-sz^2)\Theta^{'''''}(z)}{48s^2} \\ &= \frac{\Delta_3}{12s^2}. \end{aligned}$$

□

7 Matrices associated with Generalized Fibonacci Polynomials

Matrix formulation of W_n can be given as

$$\begin{pmatrix} W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} r & s \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_1 \\ W_0 \end{pmatrix}. \tag{7.1}$$

We define the square matrix A of order 2 as:

$$A = \begin{pmatrix} r & s \\ 1 & 0 \end{pmatrix}$$

such that $\det A = -s$. From (1.1) we have

$$\begin{pmatrix} W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} r & s \\ 1 & 0 \end{pmatrix} \begin{pmatrix} W_n \\ W_{n-1} \end{pmatrix} \tag{7.2}$$

and from (7.1) (or using (7.2) and induction) we have

$$\begin{pmatrix} W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} r & s \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_1 \\ W_0 \end{pmatrix}.$$

Note that (7.2) can be written in the following form

$$\begin{pmatrix} W_{n+1} & W_n \end{pmatrix}^T = \begin{pmatrix} r & s \\ 1 & 0 \end{pmatrix} \begin{pmatrix} W_n & W_{n-1} \end{pmatrix}^T$$

where T stands for the transpose of the vector.

If we take $W_n = G_n$ in (7.2) we have

$$\begin{pmatrix} G_{n+1} \\ G_n \end{pmatrix} = \begin{pmatrix} r & s \\ 1 & 0 \end{pmatrix} \begin{pmatrix} G_n \\ G_{n-1} \end{pmatrix}. \tag{7.3}$$

We also define

$$B_n = \begin{pmatrix} G_{n+1} & sG_n \\ G_n & sG_{n-1} \end{pmatrix},$$

and

$$C_n = \begin{pmatrix} W_{n+1} & sW_n \\ W_n & sW_{n-1} \end{pmatrix}.$$

In the next theorem, we find the Binet's formulas of W_n, G_n and H_n by matrix method with other results.

Theorem 24. *For all integers m, n, we have*

(a)

(i) $B_n = A^n$, *i.e.*,

$$\begin{pmatrix} r & s \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} G_{n+1} & sG_n \\ G_n & sG_{n-1} \end{pmatrix}.$$

(ii) *if* $r^2 + 4s \neq 0$, *i.e.*, $\alpha \neq \beta$, *then*

$$\begin{pmatrix} r & 2s \\ 2 & -r \end{pmatrix} \begin{pmatrix} r & s \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} H_{n+1} & sH_n \\ H_n & sH_{n-1} \end{pmatrix},$$

and if $r^2 + 4s = 0$, *i.e.*, $\alpha = \beta$, *then*

$$\begin{pmatrix} r & s \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} \frac{1}{2\alpha}(n+1) H_{n+1} & -\frac{1}{2}\alpha n H_n \\ \frac{1}{2}\frac{n}{\alpha} H_n & -\frac{1}{2}\alpha(n-1)H_{n-1} \end{pmatrix}.$$

(iii)

$$(-W_1^2+sW_0^2+rW_1W_0)\begin{pmatrix} r & s \\ 1 & 0 \end{pmatrix}^n$$
$$= \begin{pmatrix} (-W_1+rW_0)W_{n+1}+sW_0W_n & s(W_0W_{n+1}-W_1W_n) \\ W_0W_{n+1}-W_1W_n & -W_1W_{n+1}+(rW_1+sW_0)W_n \end{pmatrix}$$

where

$$-W_1^2+sW_0^2+rW_0W_1=\begin{vmatrix} W_0 & W_1 \\ W_1 & rW_1+sW_0 \end{vmatrix}=\begin{vmatrix} W_0 & W_1 \\ W_1 & W_2 \end{vmatrix}.$$

(b) $C_1A^n=A^nC_1$.

(c) $C_{n+m}=C_nB_m=B_mC_n$.

(d)

(i) *The Binet's formula of W_n is*

$$W_n=\begin{cases} \dfrac{W_1-\beta W_0}{\alpha-\beta}\alpha^n-\dfrac{W_1-\alpha W_0}{\alpha-\beta}\beta^n & , \text{ if } \alpha\neq\beta \text{ (Distinct Roots Case)} \\ (nW_1-\alpha(n-1)W_0)\alpha^{n-1} & , \text{ if } \alpha=\beta \text{ (Single Root Case)} \end{cases}.$$

(ii) *The Binet's formula of G_n is*

$$G_n=\begin{cases} \dfrac{\alpha^n-\beta^n}{\alpha-\beta} & , \text{ if } \alpha\neq\beta \text{ (Distinct Roots Case)} \\ n\alpha^{n-1} & , \text{ if } \alpha=\beta \text{ (Single Root Case)} \end{cases}.$$

(iii) *The Binet's formula of H_n is*

$$H_n=\begin{cases} \alpha^n+\beta^n & , \text{ if } \alpha\neq\beta \text{ (Distinct Roots Case)} \\ 2\alpha^n & , \text{ if } \alpha=\beta \text{ (Single Root Case)} \end{cases}.$$

Proof.

(a)

(i) We use induction on n. First we assume that $n \geq 0$. For $n = 0$, we get

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} G_1 & sG_0 \\ G_0 & sG_{-1} \end{pmatrix}$$

which is true because $G_0 = 0, G_1 = 1, G_{-1} = \frac{1}{s}$. Suppose that the relation holds for all k with $0 \leq k \leq n$. Then

$$\begin{aligned} \begin{pmatrix} r & s \\ 1 & 0 \end{pmatrix}^{n+1} &= \begin{pmatrix} r & s \\ 1 & 0 \end{pmatrix}^{n} \begin{pmatrix} r & s \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} G_{n+1} & sG_n \\ G_n & sG_{n-1} \end{pmatrix} \begin{pmatrix} r & s \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} rG_{n+1} + sG_n & sG_{n+1} \\ rG_n + sG_{n-1} & sG_n \end{pmatrix} \\ &= \begin{pmatrix} G_{n+2} & sG_{n+1} \\ G_{n+1} & sG_n \end{pmatrix}. \end{aligned}$$

For $n \leq 0$, we use induction on $v = |n| = -n$. For $v = 0$, the relation already been verified. Assume now that it holds for all v with $0 \leq v \leq |n|$. Then

$$\begin{aligned} \begin{pmatrix} r & s \\ 1 & 0 \end{pmatrix}^{-(n+1)} &= \begin{pmatrix} r & s \\ 1 & 0 \end{pmatrix}^{-n-1} = \begin{pmatrix} r & s \\ 1 & 0 \end{pmatrix}^{-n} \begin{pmatrix} r & s \\ 1 & 0 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} G_{-n+1} & sG_{-n} \\ G_{-n} & sG_{-n-1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \frac{1}{s} & -\frac{r}{s} \end{pmatrix} \\ &= \begin{pmatrix} G_{-n} & G_{-n+1} - rG_{-n} \\ G_{-n-1} & G_{-n} - rG_{-n-1} \end{pmatrix} \\ &= \begin{pmatrix} G_{-n} & sG_{-n-1} \\ G_{-n-1} & sG_{-n-2} \end{pmatrix} \\ &= \begin{pmatrix} G_{-(n+1)+1} & sG_{-(n+1)} \\ G_{-(n+1)} & sG_{-(n+1)-1} \end{pmatrix} \end{aligned}$$

which completes the proof by using induction.

Note that proof of the case $n \geq 0$ can also be given as follows.

By expanding the vectors on the both sides of (7.3) to 2-columns and multiplying the obtained on the right-hand side by A, we get

$$B_n = AB_{n-1}.$$

By induction argument, from the last equation, we obtain

$$B_n = A^{n-1}B_1.$$

But $B_1 = A$. It follows that $B_n = A^n$.

(ii) First we consider the case $r^2 + 4s \neq 0$, i.e., $\alpha \neq \beta$. Note that since $(4s + r^2)G_n = 2H_{n+1} - rH_n$, we have

$$\begin{aligned}
(4s + r^2)G_n &= 2H_{n+1} - rH_n, \\
(4s + r^2)G_{n+1} &= 2H_{n+2} - rH_{n+1} = rH_{n+1} + 2sH_n, \\
(4s + r^2)G_{n-1} &= 2H_n - rH_{n-1} = 2H_n - \frac{r}{s}(H_{n+1} - rH_n), \\
H_{n+1} &= rH_n + sH_{n-1} \Rightarrow sH_{n-1} = H_{n+1} - rH_n \\
&\Rightarrow \\
H_{n-1} &= \frac{1}{s}(H_{n+1} - rH_n),
\end{aligned}$$

so from (a) we get

$$\begin{aligned}
&\begin{pmatrix} r & s \\ 1 & 0 \end{pmatrix}^n \\
&= \frac{1}{r^2 + 4s}\begin{pmatrix} rH_{n+1} + 2sH_n & s(2H_{n+1} - rH_n) \\ 2H_{n+1} - rH_n & s(2H_n - \frac{r}{s}(H_{n+1} - rH_n)) \end{pmatrix} \\
&= \frac{1}{r^2 + 4s}\begin{pmatrix} rH_{n+1} + 2sH_n & s(2H_{n+1} - rH_n) \\ 2H_{n+1} - rH_n & s(2H_n - \frac{r}{s}(H_{n+1} - rH_n)) \end{pmatrix} \\
&\quad \times \begin{pmatrix} r & 2s \\ 2 & -r \end{pmatrix}\begin{pmatrix} r & 2s \\ 2 & -r \end{pmatrix}^{-1} \\
&= \begin{pmatrix} H_{n+1} & sH_n \\ H_n & H_{n+1} - rH_n \end{pmatrix}\begin{pmatrix} r & 2s \\ 2 & -r \end{pmatrix}^{-1} \\
&= \begin{pmatrix} H_{n+1} & sH_n \\ H_n & sH_{n-1} \end{pmatrix}\begin{pmatrix} r & 2s \\ 2 & -r \end{pmatrix}^{-1}
\end{aligned}$$

and so

$$\begin{pmatrix} r & 2s \\ 2 & -r \end{pmatrix} \begin{pmatrix} r & s \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} H_{n+1} & sH_n \\ H_n & sH_{n-1} \end{pmatrix}.$$

Note that the proof of the case $r^2+4s \neq 0$ can also given by induction.

Now we assume that $r^2+4s=0$, i.e., $\alpha=\beta$. In this case we have

$$\begin{aligned} \alpha &= \frac{r}{2}, \\ s &= -\alpha^2 \end{aligned}$$

and from (3.3) we get $H_{n+1} = \alpha H_n$, i.e., $H_n = \frac{1}{\alpha} H_{n+1} = \frac{1}{\alpha^2} H_{n+2}$ and so $H_{n-1} = \frac{1}{\alpha} H_n = \frac{1}{\alpha^2} H_{n+1}$. We also have

$$\begin{aligned} H_{-n+1} &= \alpha^2 H_{-n-1}, \\ H_{-n} &= \alpha H_{-n-1}, \\ H_{-n-1} &= \alpha H_{-n-2}, \\ H_{-n} &= \alpha^2 H_{-n-2}. \end{aligned}$$

We prove by induction that

$$\begin{pmatrix} r & s \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} \frac{1}{2\alpha}(n+1) H_{n+1} & -\frac{1}{2}\alpha n H_n \\ \frac{1}{2}\frac{n}{\alpha} H_n & -\frac{1}{2}\alpha(n-1)H_{n-1} \end{pmatrix}.$$

Firstly, we suppose that $n \geq 0$. For $n=0$, we get

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2\alpha} H_1 & 0 \\ 0 & \frac{1}{2}\alpha H_{-1} \end{pmatrix}$$

which is true because $H_1 = r = 2\alpha, H_{-1} = -\frac{r}{s} = -\frac{2\alpha}{-\alpha^2} = \frac{2}{\alpha}$. Suppose that the relation holds for all k with $0 \leq k \leq n$. Then

$$\begin{aligned}
&\begin{pmatrix} r & s \\ 1 & 0 \end{pmatrix}^{n+1} \\
&= \begin{pmatrix} r & s \\ 1 & 0 \end{pmatrix}^{n} \begin{pmatrix} r & s \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2\alpha}(n+1)H_{n+1} & -\frac{1}{2}\alpha n H_n \\ \frac{1}{2}\frac{n}{\alpha}H_n & -\frac{1}{2}\alpha(n-1)H_{n-1} \end{pmatrix} \begin{pmatrix} 2\alpha & -\alpha^2 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} (n+1)H_{n+1} - \frac{1}{2}n\alpha H_n & -\frac{1}{2}\alpha(n+1)H_{n+1} \\ nH_n - \frac{1}{2}\alpha(n-1)H_{n-1} & -\frac{1}{2}n\alpha H_n \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2\alpha}((n+1)+1)H_{(n+1)+1} & -\frac{1}{2}\alpha(n+1)H_{(n+1)} \\ \frac{1}{2}\frac{(n+1)}{\alpha}H_{(n+1)} & -\frac{1}{2}\alpha((n+1)-1)H_{(n+1)-1} \end{pmatrix}.
\end{aligned}$$

which completes the proof by using induction in the case $n \geq 0$. Now, for $n \leq 0$, we use induction on $v = |n| = -n$. For $v = 0$, the relation already been verified. Assume now that it holds for all v with $0 \leq v \leq |n|$. Then

$$\begin{aligned}
&\begin{pmatrix} r & s \\ 1 & 0 \end{pmatrix}^{-(n+1)} \\
&= \begin{pmatrix} \frac{1}{2\alpha}(n+1)H_{n+1} & -\frac{1}{2}\alpha n H_n \\ \frac{1}{2}\frac{n}{\alpha}H_n & -\frac{1}{2}\alpha(n-1)H_{n-1} \end{pmatrix}^{-n} \begin{pmatrix} r & s \\ 1 & 0 \end{pmatrix}^{-1} \\
&= \begin{pmatrix} \frac{1}{2\alpha}(-n+1)H_{-n+1} & -\frac{1}{2}\alpha(-n)H_{-n} \\ \frac{1}{2}\frac{-n}{\alpha}H_{-n} & -\frac{1}{2}\alpha(-n-1)H_{-n-1} \end{pmatrix} \begin{pmatrix} 2\alpha & -\alpha^2 \\ 1 & 0 \end{pmatrix}^{-1} \\
&= \begin{pmatrix} \frac{1}{2\alpha}(-(n+1)+1)H_{-(n+1)+1} & -\frac{1}{2}\alpha(-(n+1))H_{-(n+1)} \\ \frac{1}{2}\frac{-(n+1)}{\alpha}H_{-(n+1)} & -\frac{1}{2}\alpha(-(n+1)-1)H_{-(n+1)-1} \end{pmatrix}
\end{aligned}$$

which completes the proof by using induction.

(iii) Note that since $(-W_1^2 + sW_0^2 + rW_1W_0)G_n = W_0W_{n+1} - W_1W_n$ and

$W_{n+1} = rW_n + sW_{n-1}$ we have

$$\begin{aligned}
(-W_1^2 + sW_0^2 + rW_1W_0)G_{n+1} &= (-W_1 + rW_0)W_{n+1} + sW_0W_n, \\
(-W_1^2 + sW_0^2 + rW_1W_0)G_{n-1} &= \frac{1}{s}(-W_1W_{n+1} + (rW_1 + sW_0)W_n), \\
W_{n+1} &= rW_n + sW_{n-1} \\
&\Rightarrow \\
sW_{n-1} &= W_{n+1} - rW_n \\
&\Rightarrow \\
W_{n-1} &= \frac{1}{s}(W_{n+1} - rW_n),
\end{aligned}$$

so from (a) we obtain, if $(-W_1^2 + sW_0^2 + rW_1W_0) \neq 0$,

$$\begin{aligned}
&\begin{pmatrix} r & s \\ 1 & 0 \end{pmatrix}^n \\
&= \begin{pmatrix} G_{n+1} & sG_n \\ G_n & sG_{n-1} \end{pmatrix} \\
&= \frac{1}{(-W_1^2 + sW_0^2 + rW_1W_0)} \\
&\quad \times \begin{pmatrix} (-W_1 + rW_0)W_{n+1} + sW_0W_n & s(W_0W_{n+1} - W_1W_n) \\ W_0W_{n+1} - W_1W_n & -W_1W_{n+1} + (rW_1 + sW_0)W_n \end{pmatrix}
\end{aligned}$$

Note that if $-W_1^2 + sW_0^2 + rW_0W_1 = 0$, then the claim in (iii) is true for all n because in this case we have

$$\begin{aligned}
-W_1^2 + sW_0^2 + rW_0W_1 &= \begin{vmatrix} W_0 & W_1 \\ W_1 & rW_1 + sW_0 \end{vmatrix} = \begin{vmatrix} W_0 & W_1 \\ W_1 & W_2 \end{vmatrix} = 0, \\
(-W_1 + rW_0)W_{n+1} + sW_0W_n &= \begin{vmatrix} W_0 & W_1 \\ W_{n+1} & sW_n + rW_{n+1} \end{vmatrix} = 0, \\
s(W_0W_{n+1} - W_1W_n) &= s\begin{vmatrix} W_0 & W_1 \\ W_n & W_{n+1} \end{vmatrix} = 0, \\
W_0W_{n+1} - W_1W_n &= \begin{vmatrix} W_0 & W_1 \\ W_n & W_{n+1} \end{vmatrix} = 0, \\
-W_1W_{n+1} + (rW_1 + sW_0)W_n &= \begin{vmatrix} W_0 & W_1 \\ W_{n+1} - rW_n & sW_n \end{vmatrix} = 0.
\end{aligned}$$

So there is no need to restriction on $-W_1^2 + sW_0^2 + rW_0W_1$ whether it is equal zero or not. This completes the proof.

Note that proof can also given by induction.

(b) Using (a) and definition of C_1, (b) follows.

(c) We have

$$\begin{aligned} AC_{n-1} &= \begin{pmatrix} r & s \\ 1 & 0 \end{pmatrix} \begin{pmatrix} W_n & sW_{n-1} \\ W_{n-1} & sW_{n-2} \end{pmatrix} \\ &= \begin{pmatrix} rW_n + sW_{n-1} & W_{n-2}s^2 + rW_{n-1}s \\ W_n & sW_{n-1} \end{pmatrix} = C_n \end{aligned}$$

i.e. $C_n = AC_{n-1}$. From the last equation, using induction we obtain $C_n = A^{n-1}C_1$. Now

$$C_{n+m} = A^{n+m-1}C_1 = A^{n-1}A^mC_1 = A^{n-1}C_1A^m = C_nB_m$$

and similarly

$$C_{n+m} = B_mC_n.$$

(d) First, we give some remarks. If $a, b, c, d \in \mathbb{R}$ and $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then we have the following results:

- Eigenvalues of the matrix M are $\lambda_1 = \frac{1}{2}a + \frac{1}{2}d + \frac{1}{2}\sqrt{(a-d)^2 + 4bc}$ and $\lambda_2 = \frac{1}{2}a + \frac{1}{2}d - \frac{1}{2}\sqrt{(a-d)^2 + 4bc}$.
- If $(a-d)^2 + 4bc \neq 0$, i.e., $\lambda_1 \neq \lambda_2$, then

$$\begin{aligned} M^n &= \begin{pmatrix} \lambda_1^n + (a-\lambda_1)\frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2} & b\frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2} \\ c\frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2} & \lambda_1^n + (d-\lambda_1)\frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2} \end{pmatrix} \\ &= \frac{1}{\lambda_1 - \lambda_2} \begin{pmatrix} a(\lambda_1^n - \lambda_2^n) - \lambda_1\lambda_2(\lambda_1^{n-1} - \lambda_2^{n-1}) & b(\lambda_1^n - \lambda_2^n) \\ c(\lambda_1^n - \lambda_2^n) & d(\lambda_1^n - \lambda_2^n) - \lambda_1\lambda_2(\lambda_1^{n-1} - \lambda_2^{n-1}) \end{pmatrix}. \end{aligned}$$

- If $(a-d)^2 + 4bc = 0$, i.e., $\lambda_1 = \lambda_2$, then

$$M^n = \begin{pmatrix} \lambda_1^n + n\lambda_1^{n-1}(a-\lambda_1) & bn\lambda_1^{n-1} \\ cn\lambda_1^{n-1} & \lambda_1^n + n\lambda_1^{n-1}(d-\lambda_1) \end{pmatrix}.$$

If we modify this results to our case, we get

- If $r^2+4s\neq 0$, i.e., $\alpha\neq\beta$, then

$$A^n=\begin{pmatrix} r & s \\ 1 & 0 \end{pmatrix}^n=\frac{1}{\alpha-\beta}\begin{pmatrix} \alpha^{n+1}-\beta^{n+1} & -\alpha\beta(\alpha^n-\beta^n) \\ \alpha^n-\beta^n & -\alpha\beta(\alpha^{n-1}-\beta^{n-1}) \end{pmatrix}. \tag{7.4}$$

- If $r^2+4s=0$, i.e., $\alpha=\beta$, then

$$A^n=\begin{pmatrix} r & s \\ 1 & 0 \end{pmatrix}^n=\begin{pmatrix} (n+1)\,\alpha^n & -n\alpha^{n+1} \\ n\alpha^{n-1} & -(n-1)\,\alpha^n \end{pmatrix}. \tag{7.5}$$

(i) Suppose first that $r^2+4s\neq 0$, i.e., $\alpha\neq\beta$. Then, by comparing (7.4) and (a) (iii), we obtain

$$\begin{aligned}(-W_1+rW_0)W_{n+1}+sW_0W_n &= \frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}(-W_1^2+sW_0^2+rW_1W_0),\\ W_0W_{n+1}-W_1W_n &= \frac{\alpha^n-\beta^n}{\alpha-\beta}(-W_1^2+sW_0^2+rW_1W_0).\end{aligned}$$

Solving the above simultaneous equations with respect to W_n and W_{n+1}, we get

$$\begin{aligned}W_n &= \frac{1}{\alpha-\beta}(\alpha^nW_1-\beta^nW_1+\alpha\alpha^nW_0-\beta\beta^nW_0-r\alpha^nW_0+r\beta^nW_0),\\ W_{n+1} &= \frac{1}{\alpha-\beta}(\alpha\alpha^nW_1-\beta\beta^nW_1+s\alpha^nW_0-s\beta^nW_0).\end{aligned}$$

Using the fact

$$\alpha+\beta=r\Rightarrow\alpha=r-\beta,\quad \beta=-r+\alpha,$$

we get

$$W_n=\frac{W_1-\beta W_0}{\alpha-\beta}\alpha^n-\frac{W_1-\alpha W_0}{\alpha-\beta}\beta^n.$$

Suppose now that $r^2+4s=0$, i.e., $\alpha=\beta$. Then, by comparing (7.5) and (a) (iii), we obtain

$$\begin{aligned}(-W_1+rW_0)W_{n+1}+sW_0W_n &= (n+1)\,\alpha^n(-W_1^2+sW_0^2+rW_1W_0),\\ W_0W_{n+1}-W_1W_n &= n\alpha^{n-1}(-W_1^2+sW_0^2+rW_1W_0).\end{aligned}$$

Solving the above simultaneus equations with respect to W_n and W_{n+1}, we get

$$\begin{aligned} W_n &= \frac{1}{\alpha}\left(\alpha\alpha^n W_0 + n\alpha^n W_1 + n\alpha\alpha^n W_0 - nr\alpha^n W_0\right), \\ W_{n+1} &= \frac{1}{\alpha}\left(\alpha\alpha^n W_1 + n\alpha\alpha^n W_1 + ns\alpha^n W_0\right). \end{aligned}$$

Then, using $r = 2\alpha$, we get

$$W_n = (nW_1 - \alpha(n-1)W_0)\alpha^{n-1}.$$

(ii) Just compare the matrices in (7.4) and (7.5) with matrix in (a) (i) or take $W_n = G_n$ in (i).

(iii) Just compare the matrices in (7.4) and (7.5) with matrices in (a) (ii) or take $W_n = H_n$ in (i).

□

Some properties of matrix A^n can be given as

$$A^n = rA^{n-1} + sA^{n-2}$$

and

$$A^{n+m} = A^n A^m = A^m A^n$$

and

$$\det(A^n) = (-s)^n$$

for all integers m and n.

Theorem 25. *For all integers m, n we have*

$$W_{n+m} = W_n G_{m+1} + sW_{n-1}G_m. \tag{7.6}$$

Proof. From the equation $C_{n+m} = C_n B_m = B_m C_n$ we see that an element of C_{n+m} is the product of row C_n and a column B_m. From the last equation we say that an element of C_{n+m} is the product of a row C_n and column B_m. We just compare the linear combination of the 2nd row and 1st column entries of the matrices C_{n+m} and C_nB_m. This completes the proof.

Proof can also be given by using induction as follows. For $m \geq 1$ and $m \leq 0$, we proceed by induction on m. First we assume that $m \geq 1$. For $m = 1$, (7.6) is true because we have, by definition of W_n and the values $G_1 = 1, G_2 = r$,

$$W_{n+1} = rW_n + sW_{n-1} = W_nG_2 + sW_{n-1}G_1.$$

For $m = 2$, (7.6) is true because, we get again by definition of W_n and the values $G_2 = r, G_3 = s + r^2$,

$$\begin{aligned} W_{n+2} &= rW_{n+1} + sW_n = r(rW_n + sW_{n-1}) + sW_n = (s + r^2)W_n + rsW_{n-1} \\ &= W_nG_3 + sW_{n-1}G_2. \end{aligned}$$

Suppose now that (7.6) holds for all m with $1 \leq m \leq k+1$. Then, by assumption, for $m = k$ and $m = k + 1$, we have, respectively,

$$sW_{n+k} = s(W_nG_{k+1} + sW_{n-1}G_k),$$

and

$$rW_{n+k+1} = r(W_nG_{k+2} + sW_{n-1}G_{k+1}).$$

By adding up these two equations, we get

$$rW_{n+k+1} + sW_{n+k} = r(W_nG_{k+2} + sW_{n-1}G_{k+1}) + s(W_nG_{k+1} + sW_{n-1}G_k),$$

i.e.,

$$\begin{aligned} W_{n+k+2} &= W_n\left(rG_{k+2} + sG_{k+1}\right) + sW_{n-1}\left(rG_{k+1} + sG_k\right) \\ &= W_nG_{k+3} + sW_{n-1}G_{k+2} \end{aligned}$$

which yields the (7.6) for $m = k + 2$.

Now, if $m \leq 0$, then we proceed by induction on $|m| = -m = v$. For $v = 0$, that is $m = 0$, (7.6) is true because

$$W_n = W_nG_1 + sW_{n-1}G_0$$

where $G_0 = 0$ and $G_1 = 1$. For $v = 1$, that is, $m = -1$, (7.6) is true because

$$W_{n-1} = W_n G_0 + sW_{n-1}G_{-1}$$

where $G_0 = 0$ and $G_{-1} = \frac{1}{s}$. Suppose now that (7.6) holds for all $v = |m| = -m$ with $1 \leq v \leq k+1$. Then, by assumption, for $v = k$ and $v = k+1$, we have, respectively,

$$\frac{1}{s}G_{n-k} = \frac{1}{s}(W_n G_{-k+1} + sW_{n-1}G_{-k})$$

and

$$\frac{-r}{s}W_{n-k-1} = \frac{-r}{s}(W_n G_{-k} + sW_{n-1}G_{-k-1}).$$

By adding up these two equations, we get

$$\frac{-r}{s}W_{n-k-1} + \frac{1}{s}G_{n-k} = \frac{-r}{s}(W_n G_{-k} + sW_{n-1}G_{-k-1}) + \frac{1}{s}(W_n G_{-k+1} + sW_{n-1}G_{-k}),$$

that is,

$$\begin{aligned} W_{n-k-2} &= W_n(-\frac{r}{s}G_{-k} + \frac{1}{s}G_{-k+1}) + sW_{n-1}(-\frac{r}{s}G_{-k-1} + \frac{1}{s}G_{-k}) \\ &= W_n G_{-k-1} + sW_{n-1}G_{-k-2}; \end{aligned}$$

thus we get (7.6) for $v = |m| = k+2$.

□

By Lemma 7, we know that

$$(-W_1^2 + sW_0^2 + rW_1W_0)G_m = W_0W_{m+1} - W_1W_m,$$

so (7.6) can be written in the following form

$$\begin{aligned} (-W_1^2 + sW_0^2 + rW_1W_0)W_{n+m} &= (W_0W_{m+2} - W_1W_{m+1})W_n \\ &\quad + s(W_0W_{m+1} - W_1W_m)W_{n-1} \\ &= ((-W_1 + rW_0)\, W_{m+1} + sW_0W_m)W_n \\ &\quad + s(W_0W_{m+1} - W_1W_m)W_{n-1}. \end{aligned} \tag{7.7}$$

Corollary 26. *For all integers m, n, we have*

$$\begin{aligned} G_{n+m} &= G_n G_{m+1} + s G_{n-1} G_m, \\ H_{n+m} &= H_n G_{m+1} + s H_{n-1} G_m, \end{aligned}$$

and

$$\begin{aligned} (r^2+4s)H_{n+m} &= (2H_{m+2} - rH_{m+1})H_n + s(2H_{m+1} - rH_m)H_{n-1} \\ &= (2(rH_{m+1} + sH_m) - rH_{m+1})H_n + s(2H_{m+1} - rH_m)H_{n-1} \\ &= (rH_{m+1} + 2sH_m)H_n + s(2H_{m+1} - rH_m)H_{n-1} \\ &= 2H_{m+1}H_{n+1} + (r^2+2s)H_m H_n - r(H_{m+1}H_n + H_m H_{n+1}), \end{aligned}$$

where

$$\begin{aligned} H_{m+2} &= rH_{m+1} + sH_m, \\ H_{n-1} &= \frac{1}{s}(H_{n+1} - rH_n). \end{aligned}$$

If we replace $m = n$, $m = n+1$ and $m = 2n$, respectively, in the last corollory and use the identities (given in Lemma 9)

$$\begin{aligned} G_{n+1} &= rG_n + sG_{n-1}, \\ (r^2+4s)G_n &= rH_n + 2sH_{n-1}, \\ H_n &= rG_n + 2sG_{n-1}, \\ H_n &= 2G_{n+1} - rG_n, \end{aligned}$$

we get the following result.

Corollary 27. *For all integers m, n, we have*

$$\begin{aligned} G_{2n} &= G_n\left(G_{n+1} + sG_{n-1}\right) = G_n H_n, \\ H_{2n} &= H_n G_{n+1} + sH_{n-1}G_n, \\ 2H_{2n} &= (r^2+4s)G_n^2 + H_n^2, \\ (r^2+4s)H_{2n} &= (2H_{n+2} - rH_{n+1})H_n + s(2H_{n+1} - rH_n)H_{n-1}, \end{aligned}$$

and

$$\begin{aligned} G_{2n+1} &= G_n G_{n+2} + sG_{n-1}G_{n+1}, \\ H_{2n+1} &= H_n G_{n+2} + sH_{n-1}G_{n+1}, \\ (r^2+4s)H_{2n+1} &= (2H_{n+3} - rH_{n+2})H_n + s(2H_{n+2} - rH_{n+1})H_{n-1}, \end{aligned}$$

and

$$\begin{aligned} G_{3n} &= G_n G_{2n+1} + sG_{n-1}G_{2n}, \\ H_{3n} &= H_n G_{2n+1} + sH_{n-1}G_{2n}, \\ (r^2+4s)H_{3n} &= (2H_{2n+2} - rH_{2n+1})H_n + s(2H_{2n+1} - rH_{2n})H_{n-1}. \end{aligned}$$

Next, we present some identities on W_n, G_n and H_n.

Corollary 28. *For all integers m, n, j, we have the following identities:*

(a)

(i) $W_{n+m} = W_{n+1}G_m + sW_nG_{m-1}$.

(ii) $W_{n+m} = W_mG_{n+1} + sW_{m-1}G_n$.

(iii) $G_{n+m} = G_{n+1}G_m + sG_nG_{m-1}$.

(iv)

- $G_{n+m+1} = G_{n+1}G_{m+1} + sG_nG_m$
- $G_n = G_{n-m+1}G_m + sG_{n-m}G_{m-1}$
- $G_{n+m-1} = G_nG_m + sG_{n-1}G_{m-1}$
- $G_{2n+1} = G_{n+1}^2 + sG_n^2$
- $G_n = G_mG_{n-m+1} + sG_{m-1}G_{n-m}$

(v) $H_{n+m} = H_{n+1}G_m + sH_nG_{m-1}$.

(vi)

- $H_{n+m+1} = H_{n+1}G_{m+1} + sH_nG_m$
- $H_n = H_{n-m+1}G_m + sH_{n-m}G_{m-1}$

- $H_{n+m-1} = H_nG_m + sH_{n-1}G_{m-1}$
- $H_{2n+1} = H_{n+1}G_{n+1} + sH_nG_n$
- $H_n = H_mG_{n-m+1} + sH_{m-1}G_{n-m}$

(b) $W_{n+m} = W_{m-j}G_{n+j+1} + sW_{m-j-1}G_{n+j}$.

(c) $W_{n+m} = W_{n+j}G_{m-j+1} + sW_{n+j-1}G_{m-j}$.

Proof.

(a)

(i) Take $n \to n+1$ and $m \to m-1$ in Theorem 25.

(ii) Take $n \to m$ and $m \to n$ in Theorem 25.

(iii) Set $W_n = G_n$ in (i).

(iv) Take $n \to n, m \to m-1$ and $n \to n-m, m \to m$ and $n \to n-1, m \to m$ and $n \to n, m \to n+1$ and $n \to m-1 \;\; m \to n-m+1$ in (iii), respectively.

(v) Set $W_n = H_n$ in (i).

(vi) Take $n \to n, m \to m-1$ and $n \to n-m, m \to m$ and $n \to n-1, m \to m$ and $n \to n, m \to n+1$ and $n \to m-1 \;\; m \to n-m+1$ in (v), respectively.

(b) Take $n \to m-j$ and $m \to n+j$ in Theorem 25.

(c) Take $n \to n+j$ and $m \to m-j$ in Theorem 25.

□

Now, we give some identities by using Theorem 25.

Corollary 29. *For all integers n, we have the following identities:*

(a) $W_{2n} = \frac{1}{r}(W_{n+1}G_{n+1} - s^2W_{n-1}G_{n-1})$.

(b) $W_{2n+1} = W_{n+1}G_{n+1} + sW_nG_n$.

(c) $W_{2n-1} = W_nG_n + sW_{n-1}G_{n-1}$.

(d) $W_{3n} = \frac{1}{r}(W_{n+1}G_{n+1}^2 + rsW_nG_n^2 - s^3W_{n-1}G_{n-1}^2)$.

(e) $W_{3n+1} = \frac{1}{r}(W_{n+1}G_{n+1}G_{n+2} + rsW_nG_nG_{n+1} - s^3W_{n-1}G_nG_{n-1})$.

(f) $W_{3n-1} = \frac{1}{r}(W_{n+1}G_nG_{n+1} + rsW_nG_nG_{n-1} - s^3W_{n-1}G_{n-1}G_{n-2})$.

(g) $W_{4n} = W_n(G_{n+1}^3 + s^2G_n^2G_{n-1} + 2sG_n^2G_{n+1}) + sW_{n-1}G_n(\left(s + r^2\right)G_n^2 + 3sG_{n-1}G_{n+1})$.

Proof. **(a)** Take $m = n$ in Theorem 25 and use definition of W_n and G_n.

(b) Take $n \to n + 1$ and $m = n$ in Theorem 25.

(c) Take $n \to n - 1$ in (b).

(d) Take $n \to 2n$ and $m = n$ in Theorem 25 and use (a) and (c).

(e) Take $n \to 2n + 1$ and $m = n$ in Theorem 25 and use (a) and (b).

(f) Take $n \to 2n$ and $m = n - 1$ in Theorem 25 and use (a) and (c).

(g) Take $n \to 3n$ and $m = n$ in Theorem 25 and use (d) and (f).

□

Taking $W_n = G_n$ in the previous corollary, we get

Corollary 30. *For all integers n, we have the following identities:*

(a) $G_{2n} = \frac{1}{r}(G_{n+1}^2 - s^2G_{n-1}^2)$.

(b) $G_{2n+1} = G_{n+1}^2 + sG_n^2$.

(c) $G_{2n-1} = G_n^2 + sG_{n-1}^2$.

(d) $G_{3n} = \frac{1}{r}(G_{n+1}^3 + rsG_n^3 - s^3G_{n-1}^3)$.

(e) $G_{3n+1} = \frac{1}{r}(G_{n+1}^2 G_{n+2} + rsG_n^2 G_{n+1} - s^3 G_n G_{n-1}^2)$.

(f) $G_{3n-1} = \frac{1}{r}(G_n G_{n+1}^2 + rsG_n^2 G_{n-1} - s^3 G_{n-1}^2 G_{n-2})$.

(g) $G_{4n} = G_n(G_{n+1}^3 + sG_n^2(2G_{n+1} + r^2 G_{n-1} + 2sG_{n-1}) + 3s^2 G_{n-1}^2 G_{n+1})$.

Replacing $W_n = H_n$ in Corollary 29, we obtain

Corollary 31. *For all integers m, n, we have the following identities:*

(a) $H_{2n} = \frac{1}{r}(H_{n+1}G_{n+1} - s^2 H_{n-1}G_{n-1})$.

(b) $H_{2n+1} = H_{n+1}G_{n+1} + sH_n G_n$.

(c) $H_{2n-1} = H_n G_n + sH_{n-1}G_{n-1}$.

(d) $H_{3n} = \frac{1}{r}(H_{n+1}G_{n+1}^2 + rsH_n G_n^2 - s^3 H_{n-1}G_{n-1}^2)$.

(e) $H_{3n+1} = \frac{1}{r}(H_{n+1}G_{n+1}G_{n+2} + rsH_n G_n G_{n+1} - s^3 H_{n-1}G_n G_{n-1})$.

(f) $H_{3n-1} = \frac{1}{r}(H_{n+1}G_n G_{n+1} + rsH_n G_n G_{n-1} - s^3 H_{n-1}G_{n-1}G_{n-2})$.

(g) $H_{4n} = H_n(G_{n+1}^3 + s^2 G_n^2 G_{n-1} + 2sG_n^2 G_{n+1}) + sH_{n-1}G_n(\left(s + r^2\right) G_n^2 + 3sG_{n-1}G_{n+1})$.

Now, we give some identities by using identity (7.7).

Corollary 32. *For all integers n, we have the following identities:*

(a) $(-W_1^2 + sW_0^2 + rW_1W_0)W_{2n} = ((-W_1 + rW_0)W_{n+1} + sW_0W_n)W_n + s(W_0W_{n+1} - W_1W_n)W_{n-1}$.

(b) $(-W_1^2 + sW_0^2 + rW_1W_0)W_{2n+1} = ((-W_1 + rW_0)W_{n+1} + sW_0W_n)W_{n+1} + s(W_0W_{n+1} - W_1W_n)W_n$.

(c) $(-W_1^2 + sW_0^2 + rW_1W_0)W_{2n-1} = ((-W_1 + rW_0)W_n + sW_0W_{n-1})W_n + s(W_0W_n - W_1W_{n-1})W_{n-1}$.

(d) $(-W_1^2 + sW_0^2 + rW_1W_0)W_{3n} = ((-W_1 + rW_0)W_{n+1} + sW_0W_n)W_{2n} + s(W_0W_{n+1} - W_1W_n)W_{2n-1}$.

(e) $(-W_1^2 + sW_0^2 + rW_1W_0)W_{3n+1} = ((-W_1 + rW_0)W_{n+1} + sW_0W_n)W_{2n+1} + s(W_0W_{n+1} - W_1W_n)W_{2n}$.

(f) $(-W_1^2 + sW_0^2 + rW_1W_0)W_{3n-1} = ((-W_1 + rW_0)W_n + sW_0W_{n-1})W_{2n} + s(W_0W_n - W_1W_{n-1})W_{2n-1}$.

(g) $(-W_1^2 + sW_0^2 + rW_1W_0)W_{4n} = ((-W_1 + rW_0)W_{n+1} + sW_0W_n)W_{3n} + s(W_0W_{n+1} - W_1W_n)W_{3n-1}$.

Proof. **(a)** Take $m = n$ in (7.7).

(b) Take $n \to n + 1$ and $m = n$ in (7.7).

(c) Take $n \to n - 1$ in (7.7).

(d) Take $n \to 2n$ and $m = n$ in (7.7).

(e) Take $n \to 2n + 1$ and $m = n$ in (7.7).

(f) Take $n \to 2n$ and $m = n - 1$ in (7.7).

(g) Take $n \to 3n$ and $m = n$ in (7.7).

□

8 Identities

We now present some identities for the generalized Fibonacci (Horadam) polynomials related with the roots of its characteristic equation.

Lemma 33. *Let $n, m \in \mathbb{Z}$. Then*

(a) $(W_1 - (r - \alpha)W_0)\alpha^n = \alpha W_n + sW_{n-1}$ *and* $(W_1 - (r - \alpha)W_0)\alpha^{n-1} = W_n - (r - \alpha)W_{n-1}$.

(b) $(W_1-(r-\beta)W_0)\,\beta^n=\beta W_n+sW_{n-1}$ *and* $(W_1-(r-\beta)W_0)\,\beta^{n-1}=W_n-(r-\beta)W_{n-1}$.

(c) *If* $y^2-ry-s=0$, *then*

(i) $(W_1-(r-y)W_0)\,y^n=yW_n+sW_{n-1}$.

(ii) $(W_1-(r-y)W_0)\,y^{n-1}=W_n-(r-y)W_{n-1}$.

(iii) $(W_1-(r-y)W_0)^2\,y^{n+m}=W_mW_ny^2+s(W_nW_{m-1}+W_mW_{n-1})y+s^2W_{m-1}W_{n-1}$.

(d)

(i) $-sW_n^2+s^2W_{n-1}^2+srW_nW_{n-1}=(W_1^2-sW_0^2-rW_1W_0)(-s)^n$.

(ii) $W_{n+1}^2-sW_n^2-rW_{n+1}W_n=(W_1^2-sW_0^2-rW_1W_0)(-s)^n$.

(e)

(i)

1. $\alpha^n=\frac{1}{2}(H_n+\sqrt{r^2+4s}G_n)$.

2. $\alpha^n=\frac{1}{2}(2G_{n+1}+(\sqrt{r^2+4s}-r)G_n)$.

3. $\sqrt{r^2+4s}\alpha^n=\frac{1}{2}(2H_{n+1}+(\sqrt{r^2+4s}-r)H_n)$.

(ii)

1. $\beta^n=\frac{1}{2}(H_n-\sqrt{r^2+4s}G_n)$.

2. $\beta^n=\frac{1}{2}(2G_{n+1}-(\sqrt{r^2+4s}+r)G_n)$.

3. $\sqrt{r^2+4s}\beta^n=\frac{1}{2}(-2H_{n+1}+(\sqrt{r^2+4s}+r)H_n)$.

(iii) $H_n^2=(r^2+4s)G_n^2+4(-s)^n$,

i.e.,

$H_n^2-(r^2+4s)G_n^2=4(-s)^n$.

(iv) $H_{2n}=(r^2+4s)G_n^2+2(-s)^n$.

(v) $H_n^2-H_{2n}=2(-s)^n$,

i.e.,

$H_n^2=H_{2n}+2(-s)^n$.

(vi) $(H_m^2 - H_{2m})(H_n^2 - H_{2n}) = 4(-s)^{m+n}$.

(f)

(i) $(G_n^2 H_m^2 + G_m^2 H_n^2) - (r^2+4s)(G_m^2 G_n^2 + G_n^2 G_m^2) = 4((-s)^m G_n^2 + 4(-s)^n G_m^2)$.

(ii) $G_n^2 H_m^2 - G_m^2 H_n^2 = 4((-s)^m G_n^2 - (-s)^n G_m^2)$.

(iii) $2H_m^2 H_n^2 - (r^2+4s)(G_m^2 H_n^2 + G_n^2 H_m^2) = 4((-s)^m H_n^2 + (-s)^n H_m^2)$.

(iv) $(r^2+4s)(G_n^2 H_m^2 - G_m^2 H_n^2) = 4((-s)^m H_n^2 - (-s)^n H_m^2)$.

Proof. We use Binet's formula of W_n.

(a) and (b) If the roots α and β of characteristic equation (1.2) are distinct, i.e., $\alpha \neq \beta$, then $W_n = \dfrac{W_1 - \beta W_0}{\alpha - \beta}\alpha^n - \dfrac{W_1 - \alpha W_0}{\alpha - \beta}\beta^n$ and so we get

$$\begin{aligned}
\alpha W_n + sW_{n-1} &= \alpha W_n - \alpha\beta W_{n-1} \\
&= \alpha\left(\frac{W_1 - \beta W_0}{\alpha - \beta}\alpha^n - \frac{W_1 - \alpha W_0}{\alpha - \beta}\beta^n\right) \\
&\quad -\alpha\beta\left(\frac{W_1 - \beta W_0}{\alpha - \beta}\alpha^n\alpha^{-1} - \frac{W_1 - \alpha W_0}{\alpha - \beta}\beta^n\beta^{-1}\right) \\
&= (W_1 - \beta W_0)\,\alpha^n \\
&= (W_1 - (r-\alpha)W_0)\,\alpha^n
\end{aligned}$$

and

$$\begin{aligned}
\beta W_n + sW_{n-1} &= \beta W_n - \alpha\beta W_{n-1} \\
&= \beta\left(\frac{W_1 - \beta W_0}{\alpha - \beta}\alpha^n - \frac{W_1 - \alpha W_0}{\alpha - \beta}\beta^n\right) \\
&\quad -\alpha\beta\left(\frac{W_1 - \beta W_0}{\alpha - \beta}\alpha^n\alpha^{-1} - \frac{W_1 - \alpha W_0}{\alpha - \beta}\beta^n\beta^{-1}\right) \\
&= (W_1 - \alpha W_0)\,\beta^n \\
&= (W_1 - (r-\beta)W_0)\,\beta^n
\end{aligned}$$

since $\alpha + \beta = r$. If the roots α and β of characteristic equation (1.2) are equal, i.e., $\alpha = \beta$, then $W_n = (nW_1 - \alpha\,(n-1)\,W_0)\alpha^{n-1} = (nW_1 -$

$\beta\,(n-1)\,W_0)\beta^{n-1}$ and so we have, since $r=2\alpha$,

$$\begin{aligned}
\alpha W_n + sW_{n-1} &= \alpha W_n - \alpha\beta W_{n-1} \\
&= \alpha W_n - \alpha^2 W_{n-1} \\
&= \alpha(nW_1 - \alpha\,(n-1)\,W_0)\alpha^{n-1} - \alpha^2((n-1)W_1 \\
&\quad -\alpha\,((n-2)\,W_0)\alpha^{n-2} \\
&= (W_1 - \alpha W_0)\,\alpha^n \\
&= (W_1 - (r-\alpha)W_0)\,\alpha^n
\end{aligned}$$

i.e.,

$$\alpha W_n + sW_{n-1} = (W_1 - (r-\alpha)W_0)\,\alpha^n$$

and since $\alpha=\beta$ we obtain

$$\beta W_n + sW_{n-1} = (W_1 - (r-\beta)W_0)\,\beta^n.$$

Therefore, by dividing $(W_1-(r-\alpha)W_0)\,\alpha^n = \alpha W_n + sW_{n-1}$ and $(W_1-(r-\beta)W_0)\,\beta^n = \beta W_n + sW_{n-1}$, respectively, with α and β we obtain $(W_1-(r-\alpha)W_0)\,\alpha^{n-1} = W_n-(r-\alpha)W_{n-1}$ and $(W_1-(r-\beta)W_0)\,\beta^{n-1} = W_n-(r-\beta)W_{n-1}$.

(c) (i) and (ii) are obtained from (a) and (b). For (iii), use the property $y^{n+m}=y^n y^m$ and (i).

(d)

(i) By using (a), (b) and the identity $(W_1-\alpha W_0)(W_1-\beta W_0) = W_1^2 - sW_0^2 - rW_1W_0$, we get

$$\begin{aligned}
(W_1-\beta W_0)\,\alpha^n\,(W_1-\alpha W_0)\,\beta^n &= (\alpha W_n + sW_{n-1})(\beta W_n + sW_{n-1}) \\
&\Rightarrow \\
(W_1-\beta W_0)\,(W_1-\alpha W_0)\,(\alpha\beta)^n &= \alpha\beta W_n^2 + s^2W_{n-1}^2 + s\,(\alpha+\beta)\,W_{n-1}W_n \\
&\Rightarrow \\
(W_1^2 - sW_0^2 - rW_1W_0)(-s)^n &= -sW_n^2 + s^2W_{n-1}^2 + srW_{n-1}W_n.
\end{aligned}$$

(ii) Setting $W_{n-1}=\frac{1}{s}(W_{n+1}-rW_n)$ in (i) yields (ii).

(e)

(i)-(ii) Note that $(\alpha-\beta)^2=r^2+4s$. If the roots α and β of characteristic equation (1.2) are distinct, i.e., $\alpha\neq\beta$, i.e., $r^2+4s\neq 0$, then $G_n=\dfrac{\alpha^n-\beta^n}{\alpha-\beta}$ and $H_n=\alpha^n+\beta^n$ and so we get

$$\begin{aligned}\frac{1}{2}(H_n+\sqrt{r^2+4s}G_n) &= \frac{1}{2}((\alpha^n+\beta^n)+(\alpha-\beta)\frac{\alpha^n-\beta^n}{\alpha-\beta})=\alpha^n,\\ \frac{1}{2}(H_n-\sqrt{r^2+4s}G_n) &= \frac{1}{2}((\alpha^n+\beta^n)-(\alpha-\beta)\frac{\alpha^n-\beta^n}{\alpha-\beta})=\beta^n.\end{aligned}$$

If the roots α and β of characteristic equation (1.2) are equal, i.e., $\alpha=\beta$, i.e., $r^2+4s=0$, then $G_n=n\alpha^{n-1}$ and $H_n=2\alpha^n$ and so we have

$$\begin{aligned}\frac{1}{2}(H_n+\sqrt{r^2+4s}G_n) &= \frac{1}{2}(2\alpha^n+0.n\alpha^{n-1})=\alpha^n,\\ \frac{1}{2}(H_n-\sqrt{r^2+4s}G_n) &= \frac{1}{2}(2\alpha^n+0.n\alpha^{n-1})=\beta^n.\end{aligned}$$

Then, if we use the following identities

$$\begin{aligned}H_n &= 2G_{n+1}-rG_n,\\ (r^2+4s)G_n &= 2H_{n+1}-rH_n,\end{aligned}$$

in (i) (1) and (ii) (1) respectively, we obtain the remaining identities.

(iii) For (iii), multiply (i) and (ii) side by side and use $\alpha\beta=-s$.

(iv) Use the identity $2H_{2n}=(r^2+4s)G_n^2+H_n^2$ given in Corollary 27 and (iii).

(v) Subtract (iv) from (iii).

(vi) Use (v)

(f)

(i) Use (e) (iii).

(ii) Use (e) (iii).

(iii) Use (e) (iii).

(iv) Use (e) (iii).

□

Taking $W_n = G_n$ with $G_0 = 0, G_1 = 1$ in the last Lemma, we get the following Corollary.

Corollary 34. *Let* $n \in \mathbb{Z}$. *Then*

(a) $\alpha^n = \alpha G_n + sG_{n-1}$ *and* $\alpha^{n-1} = G_n - \beta G_{n-1}$.

(b) $\beta^n = \beta G_n + sG_{n-1}$ *and* $\beta^{n-1} = G_n - \alpha G_{n-1}$.

(c) *If* $y^2 - ry - s = 0$, *then*

(i) $y^n = yG_n + sG_{n-1}$.

(ii) $y^{n-1} = G_n - (r - y)G_{n-1}$.

(iii) $y^{n+m} = G_m G_n y^2 + s(G_n G_{m-1} + G_m G_{n-1})y + s^2 G_{m-1} G_{n-1}$.

(d)

(i) $-sG_n^2 + s^2 G_{n-1}^2 + srG_{n-1}G_n = (-s)^n$.

(ii) $G_{n+1}^2 - sG_n^2 - rG_{n+1}G_n = (-s)^n$.

If we take $W_n = H_n$ with $H_0 = 2, H_1 = r$ in the last Lemma, we get the following Corollary.

Corollary 35. *Let* $n \in \mathbb{Z}$. *Then*

(a) $(\alpha - \beta)\alpha^n = \alpha H_n + sH_{n-1}$ *and* $(\alpha - \beta)\alpha^{n-1} = H_n - \beta H_{n-1}$.

(b) $(\beta - \alpha)\beta^n = \beta H_n + sH_{n-1}$ *and* $(\beta - \alpha)\beta^{n-1} = H_n - \alpha H_{n-1}$.

(c) *If* $y^2 - ry - s = 0$, *then*

(i) $2y - ry^n = yH_n + sH_{n-1}$.

(ii) $(2y - r)\, y^{n-1} = H_n - (r - y)H_{n-1}$.

(iii) $(2y - r)^2\, y^{n+m} = H_m H_n y^2 + s(H_n H_{m-1} + H_m H_{n-1})y + s^2 H_{m-1} H_{n-1}$.

(d)

(i) $-sH_n^2 + s^2 H_{n-1}^2 + srH_{n-1}H_n = -(r^2 + 4s)(-s)^n$.

(ii) $H_{n+1}^2 - sH_n^2 - rH_{n+1}H_n = -(r^2 + 4s)(-s)^n$.

We know from Lemma 8 that

$$(-W_1^2 + sW_0^2 + rW_0W_1)H_n = (-2W_1 + rW_0)W_{n+1} + (rW_1 + 2sW_0)W_n. \quad (8.1)$$

From (8.1) we get

$$(-W_1^2 + sW_0^2 + rW_0W_1)(W_1H_n + sW_0H_{n-1}) = (-W_1^2 + sW_0^2 + rW_0W_1)(rW_n + 2sW_{n-1})$$

i.e.,

$$W_1H_n + sW_0H_{n-1} = rW_n + 2sW_{n-1}. \quad (8.2)$$

We also know from Lemma 7 that

$$(-W_1^2 + sW_0^2 + rW_1W_0)G_n = W_0W_{n+1} - W_1W_n. \quad (8.3)$$

We can give Catalan's identity in the following forms.

Theorem 36. *For all integers* m, n, *we have the following identities.*

(a)

(i) *(Catalan's identity)* $W_{n+m}W_{n-m} = W_n^2 - (-s)^{n-m}(W_1^2 - sW_0^2 - rW_0W_1)G_m^2$.

(ii) $(-W_1^2 + sW_0^2 + rW_1W_0)W_{n+m}W_{n-m} = (-W_1^2 + sW_0^2 + rW_1W_0)W_n^2 + (-s)^{n-m}(W_0W_{m+1} - W_1W_m)^2$.

(iii) $(-s)^m W_{n+m} W_{n-m} = (-s)^m W_n^2 - (-s)^n (W_1^2 - sW_0^2 - rW_0W_1) G_m^2.$

(iv) $(-W_1^2 + sW_0^2 + rW_1W_0)(-s)^m W_{n+m} W_{n-m} = (-W_1^2 + sW_0^2 + rW_1W_0)(-s)^m W_n^2 + (-s)^n (W_0 W_{m+1} - W_1 W_m)^2.$

(b)

(i) $G_{n+m} G_{n-m} = G_n^2 - (-s)^{n-m} G_m^2.$

(ii) $(-s)^m G_{n+m} G_{n-m} = (-s)^m G_n^2 - (-s)^n G_m^2.$

(iii) $H_{n+m} H_{n-m} = H_n^2 + (r^2 + 4s)(-s)^{n-m} G_m^2.$

(iv) $(-s)^m H_{n+m} H_{n-m} = (-s)^m H_n^2 + (r^2 + 4s)(-s)^n G_m^2.$

(c)

(i) $G_{n+1} G_{n-1} = G_n^2 - (-s)^{n-1}.$

(ii) $sG_{n+1} G_{n-1} = sG_n^2 + (-s)^n.$

(iii) $H_{n+1} H_{n-1} = H_n^2 + (r^2 + 4s)(-s)^{n-1}.$

(iv) $sH_{n+1} H_{n-1} = sH_n^2 - (r^2 + 4s)(-s)^n.$

(d)

(i) $G_{n+m} G_{n-m} + H_{n+m} H_{n-m} = G_n^2 + H_n^2 + ((r^2 + 4s) - 1)(-s)^{n-m} G_m^2.$

(ii) $G_{n+m} G_{n-m} - H_{n+m} H_{n-m} = G_n^2 - H_n^2 - ((r^2 + 4s) + 1)(-s)^{n-m} G_m^2.$

(iii) $G_{n+m} G_{n-m} H_{n+m} H_{n-m} = (G_n^2 - (-s)^{n-m} G_m^2)(H_n^2 + (r^2 + 4s)(-s)^{n-m} G_m^2).$

(iv) $G_{n-m-1} G_{n-m} G_{n+m} G_{n+m+1} = (G_n^2 - (-s)^{n-m} G_m^2)(G_n^2 - (-s)^{n-m-1} G_{m+1}^2).$

(v) $H_{n-m-1} H_{n-m} H_{n+m} H_{n+m+1} = (H_n^2 + (r^2+4s)(-s)^{n-m} G_m^2)(H_n^2 + (r^2 + 4s)(-s)^{n-m-1} G_{m+1}^2).$

(e)

(i) $G_{n+1} G_{n-1} + H_{n+1} H_{n-1} = G_n^2 + H_n^2 + ((r^2 + 4s) - 1)(-s)^{n-1}.$

(ii) $G_{n+1}G_{n-1} - H_{n+1}H_{n-1} = G_n^2 - H_n^2 - ((r^2+4s)+1)(-s)^{n-1}$.

(iii) $G_{n+1}G_{n-1}H_{n+1}H_{n-1} = (G_n^2 - (-s)^{n-1})(H_n^2 + (r^2+4s)(-s)^{n-1})$.

(iv) $G_{n-2}G_{n-1}G_{n+1}G_{n+2} = (G_n^2 - (-s)^{n-1})(G_n^2 - r^2(-s)^{n-2})$.

(v) $H_{n-2}H_{n-1}H_{n+1}H_{n+2} = (H_n^2 + (r^2+4s)(-s)^{n-1})(H_n^2 + r^2(r^2+4s)(-s)^{n-2})$.

(f)

(i) $G_{n+m}G_{n-m} + (-s)^m H_{n+m}H_{n-m} = G_n^2 + (-s)^m H_n^2 + ((r^2+4s)(-s)^n - (-s)^{n-m})G_m^2$.

(ii) $G_{n+m}G_{n-m} - (-s)^m H_{n+m}H_{n-m} = G_n^2 - (-s)^m H_n^2 - ((r^2+4s)(-s)^n + (-s)^{n-m})G_m^2$.

(iii) $H_{n+m}H_{n-m} + (-s)^m G_{n+m}G_{n-m} = H_n^2 + (-s)^m G_n^2 + ((r^2+4s)(-s)^{n-m} - (-s)^n)G_m^2$.

(iv) $H_{n+m}H_{n-m} - (-s)^m G_{n+m}G_{n-m} = H_n^2 - (-s)^m G_n^2 + ((r^2+4s)(-s)^{n-m} + (-s)^n)G_m^2$.

(g)

(i) $G_{n+1}G_{n-1} - sH_{n+1}H_{n-1} = G_n^2 - sH_n^2 + ((r^2+4s)(-s)^n - (-s)^{n-1})$.

(ii) $G_{n+1}G_{n-1} + sH_{n+1}H_{n-1} = G_n^2 + sH_n^2 - ((r^2+4s)(-s)^n + (-s)^{n-1})$.

(iii) $H_{n+1}H_{n-1} - sG_{n+1}G_{n-1} = H_n^2 - sG_n^2 + ((r^2+4s)(-s)^{n-1} - (-s)^n)$.

(iv) $H_{n+1}H_{n-1} + sG_{n+1}G_{n-1} = H_n^2 + sG_n^2 + ((r^2+4s)(-s)^{n-1} + (-s)^n)$.

Proof.

(a)

(i) This is the Catalan's identity (see Theorem 11).

(ii) By using (i) and (8.3) we get required identity.

(iii) Factor both sides of (i) with $(-s)^m$.

(iv) Factor both sides of (ii) with $(-s)^m$.

(b)

(i) Set $W_n = G_n$ in (i) or (iii).

(ii) Take $W_n = G_n$ in (iii).

(iii) Set $W_n = H_n$ in (i).

(iv) Take $W_n = H_n$ in (iii).

(c) Set $m = 1$ in (b).

(d)

(i)-(ii)-(iii) Use (b) (i) and (b) (iii).

(iv) Use (b) (i).

(v) Use (b) (iii).

(e) Set $m = 1$ in (d).

(f)

(i) Use (b) (i) and (b) (iv).

(ii) Use (b) (i) and (b) (iv).

(iii) Use (b) (ii) and (b) (iii).

(iv) Use (b) (ii) and (b) (iii).

(g) Set $m = 1$ in (f).

$\square$

9 Several Expressions of the Generalized Fibonacci Polynomials

We adopt the conventions $0! = 1$ and

$$\binom{n}{m} = \begin{cases} \dfrac{n!}{(n-m)!m!} & , \quad \text{if } n \geq m \geq 0 \\ 1 & , \quad \text{if } n = m = -1 \\ 0 & , \quad \text{otherwise} \end{cases}.$$

Note that

$$\binom{n-1-m}{m-1} + \binom{n-1-m}{m-2} = \binom{n-m}{m-1}.$$

The explicit expression of the generalized Fibonacci polynomials can be given as follows.

Theorem 37. *For any $n \geq 1$, the generalized Fibonacci (Horadam) polynomials satisfy the explicit expression*

$$W_n = \sum_{m=0}^{\lfloor n/2 \rfloor} \binom{n-m}{m} W_0 r^{n-2m} s^m + \sum_{m=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-1-m}{m} (W_1 - rW_0) r^{n-1-2m} s^m. \tag{9.1}$$

Proof. Clearly, using ordinary generating function of the generalized Fibonacci polynomials W_n, see Lemma 3, we get

$$\begin{aligned} & \sum_{n=0}^{\infty} W_n y^n \\ = \; & (W_0 + (W_1 - rW_0)y) \frac{1}{1 - ry - sy^2} \\ = \; & (W_0 + (W_1 - rW_0)y) \sum_{m=0}^{\infty} \sum_{n=0}^{m} \binom{m}{n} r^{m-n} s^n y^{m+n} \end{aligned}$$

$$\begin{aligned}
&= W_0 \sum_{m=0}^{\infty}\sum_{n=0}^{m}\binom{m}{n} r^{m-n}s^n y^{m+n} + (W_1 - rW_0)\sum_{m=0}^{\infty}\sum_{n=0}^{m}\binom{m}{n} r^{m-n}s^n y^{m+n+1} \\
&= W_0 \sum_{n=0}^{\infty}\sum_{m=0}^{\lfloor n/2\rfloor}\binom{n-m}{m} r^{n-2m}s^m y^n + (W_1 - rW_0)\sum_{n=0}^{\infty}\sum_{m=0}^{\lfloor (n-1)/2\rfloor}\binom{n-1-m}{m} r^{n-1-2m}s^m y^n \\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{\lfloor n/2\rfloor}\binom{n-m}{m} W_0 r^{n-2m}s^m + \sum_{m=0}^{\lfloor (n-1)/2\rfloor}\binom{n-1-m}{m}(W_1 - rW_0) r^{n-1-2m}s^m\right) y^n,
\end{aligned}$$

i.e.,

$$\sum_{n=0}^{\infty} W_n y^n = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{\lfloor n/2\rfloor}\binom{n-m}{m} W_0 r^{n-2m}s^m + \sum_{m=0}^{\lfloor (n-1)/2\rfloor}\binom{n-1-m}{m}(W_1 - rW_0) r^{n-1-2m}s^m\right) y^n$$

and so, by comparing both sides and equating the coefficients of y^n we get the required result (9.1). □

Taking $W_n = G_n$ with $G_0 = 0, G_1 = 1$ and $W_n = H_n$ with $H_0 = 2, H_1 = r$, respectively, in the last Theorem, we get the following Corollary.

Corollary 38. *(r,s)-Fibonacci polynomials satisfy the explicit expression*

$$G_n = \sum_{m=0}^{\lfloor (n-1)/2\rfloor}\binom{n-1-m}{m} r^{n-1-2m}s^m, \qquad n \geq 1,$$

and (r,s)-Lucas polynomials satisfy the explicit expression

$$\begin{aligned}
H_n &= \sum_{m=0}^{\lfloor n/2\rfloor}\left(\binom{n-m}{m}\times 2 - \binom{n-1-m}{m}\right)\times r^{n-2m}s^m, \qquad n \geq 1, \\
&= \sum_{m=0}^{\lfloor n/2\rfloor}\frac{n}{n-m}\binom{n-m}{m} r^{n-2m}s^m, \qquad n \geq 1.
\end{aligned}$$

We deduce with the help of Binet's formula of W_n the following.

Theorem 39. *For any $n \geq 0$, if $\alpha \neq \beta$, then*

$$\begin{aligned}
2^{n-1}W_n &= \frac{1}{2}\left(\sum_{m=0}^{n}\frac{1}{\alpha-\beta}\left((W_1-\beta W_0) - (W_1-\alpha W_0)(-1)^m\right)\binom{n}{m} r^{n-m}\left(\sqrt{r^2+4s}\right)^m\right) \\
&= \frac{1}{2}\left(\sum_{m=0}^{n}\frac{1}{\alpha-\beta}\left((W_1-\beta W_0) - (W_1-\alpha W_0)(-1)^m\right)\binom{n}{m} r^{n-m}(\alpha-\beta)^m\right).
\end{aligned}$$

Proof. If the roots α and β of characteristic equation (1.2) are distinct, i.e., $\alpha \neq \beta$, i.e., $r^2+4s \neq 0$, then $W_n = \frac{W_1-\beta W_0}{\alpha-\beta}\alpha^n - \frac{W_1-\alpha W_0}{\alpha-\beta}\beta^n$ and so we get

$$\begin{aligned}
2^{n-1}W_n &= 2^{n-1}\left(\frac{W_1-\beta W_0}{\alpha-\beta}\alpha^n - \frac{W_1-\alpha W_0}{\alpha-\beta}\beta^n\right) \\
&= \frac{1}{2}\left(\frac{W_1-\beta W_0}{\alpha-\beta}(2\alpha)^n - \frac{W_1-\alpha W_0}{\alpha-\beta}(2\beta)^n\right) \\
&= \frac{1}{2}\left(\frac{W_1-\beta W_0}{\alpha-\beta}\sum_{m=0}^{n}\binom{n}{m}r^{n-m}\left(\sqrt{r^2+4s}\right)^m \right. \\
&\quad \left. -\frac{W_1-\alpha W_0}{\alpha-\beta}\sum_{m=0}^{n}(-1)^m\binom{n}{m}r^{n-m}\left(\sqrt{r^2+4s}\right)^m\right) \\
&= \frac{1}{2}\left(\sum_{m=0}^{n}\frac{1}{\alpha-\beta}\left((W_1-\beta W_0)-(W_1-\alpha W_0)(-1)^m\right)\binom{n}{m}r^{n-m}\left(\sqrt{r^2+4s}\right)^m\right) \\
&= \frac{1}{2}\left(\sum_{m=0}^{n}\left((W_1-\beta W_0)-(W_1-\alpha W_0)(-1)^m\right)\binom{n}{m}r^{n-m}\left(\sqrt{r^2+4s}\right)^{m-1}\right).
\end{aligned}$$

□

Note that if $\alpha \neq \beta$, then

$$\frac{W_1-\beta W_0}{\alpha-\beta} - \frac{W_1-\alpha W_0}{\alpha-\beta}(-1)^m = \begin{cases} W_0 & , \quad m=2k \text{ even} \\ \dfrac{2W_1-rW_0}{\alpha-\beta} & , \quad m=2k+1 \text{ odd} \end{cases}.$$

So

$$\frac{G_1-\beta G_0}{\alpha-\beta} - \frac{G_1-\alpha G_0}{\alpha-\beta}(-1)^m = \frac{1}{\alpha-\beta} - \frac{1}{\alpha-\beta}(-1)^m = \begin{cases} 0 & , \quad m=2k \text{ even} \\ \frac{2}{\alpha-\beta} & , \quad m=2k+1 \text{ odd} \end{cases}$$

and

$$\frac{H_1-\beta H_0}{\alpha-\beta} - \frac{H_1-\alpha H_0}{\alpha-\beta}(-1)^m = \frac{r-2\beta}{\alpha-\beta} - \frac{r-2\alpha}{\alpha-\beta}(-1)^m = \begin{cases} 2 & , \quad m=2k \text{ even} \\ 0 & , \quad m=2k+1 \text{ odd} \end{cases}.$$

Therefore, from the last Theorem we get the next corollary.

Corollary 40. *For any $n \geq 0$, if $\alpha \neq \beta$, then*

(a)

$$2^{n-1}G_n = \sum_{m=0}^{n}\binom{n}{2m+1}r^{n-2m-1}(r^2+4s)^m.$$

(b)

$$2^{n-1}H_n = \sum_{m=0}^{n} \binom{n}{2m} r^{n-2m} \left(r^2+4s\right)^m .$$

10 Some Combinatorial Results on Generalized Fibonacci Polynomials

Now, we present some combinatorial results on generalized Fibonacci (Horadam) polynomials.

Theorem 41. *Let $n \geq 0$ and $m \in \mathbb{Z}$. Then*

(a)

$$\sum_{k=0}^{n} \binom{n}{k} (-s)^{km} W_{(n-2k)m} = \begin{cases} W_0 H_m^n & , \text{ if } \alpha \neq \beta \\ 2^n \alpha^{mn} W_0 & , \text{ if } \alpha = \beta \end{cases} .$$

(b)

(i) *If $\alpha \neq \beta$, i.e., $r^2+4s \neq 0$, then*

$$\begin{aligned} \sum_{k=0}^{n} \binom{n}{k} (-s)^{(n-k)m} W_{km}^2 &= \frac{((W_1-\beta W_0)^2\alpha^{nm} + (W_1-\alpha W_0)^2\beta^{nm})H_m^n}{r^2+4s} \\ &\quad - \frac{(W_1-\beta W_0)(W_1-\alpha W_0)(-s)^{nm}2^{n+1}}{r^2+4s}, \end{aligned}$$

and if $\alpha = \beta$, i.e., $r^2+4s = 0$, then

$$\begin{aligned} \sum_{k=0}^{n} \binom{n}{k} (-s)^{(n-k)m} W_{km}^2 &= 2^{n-2}\alpha^{2mn-2}(m^2 n\,(n+1)\,W_1^2 \\ &\quad +\alpha^2(m^2n+m^2n^2-4mn+4)W_0^2 \\ &\quad -2mn\alpha(m+mn-2)W_0W_1). \end{aligned}$$

(ii) *If $\alpha \neq \beta$, i.e., $r^2+4s \neq 0$, then*

$$\begin{aligned} &\sum_{k=0}^{n}\binom{n}{k}(-s)^{(n-k)m}W_{km}G_{km} \\ = \; &\frac{((W_1-\beta W_0)\alpha^{nm}+(W_1-\alpha W_0)\beta^{nm})H_m^n}{r^2+4s} \\ &-\frac{((W_1-\beta W_0)+(W_1-\alpha W_0))(-s)^{nm}2^n}{r^2+4s}, \end{aligned}$$

and if $\alpha = \beta$, then

$$\begin{aligned} &\sum_{k=0}^{n}\binom{n}{k}(-s)^{(n-k)m}W_{km}G_{km} \\ = \; &mn2^{n-2}\alpha^{2mn-2}(m\,(n+1)\,W_1-\alpha\,(m+mn-2)\,W_0). \end{aligned}$$

(iii) *If $\alpha \neq \beta$, i.e., $r^2+4s \neq 0$, then*

$$\begin{aligned} &\sum_{k=0}^{n}\binom{n}{k}(-s)^{(n-k)m}W_{km}H_{km} \\ = \; &\frac{((W_1-\beta W_0)\alpha^{nm}-(W_1-\alpha W_0)\beta^{nm})H_m^n}{(r^2+4s)^{\frac{1}{2}}} \\ &+\frac{((W_1-\beta W_0)-(W_1-\alpha W_0))(-s)^{nm}2^n}{(r^2+4s)^{\frac{1}{2}}}, \end{aligned}$$

and if $\alpha = \beta$, then

$$\sum_{k=0}^{n}\binom{n}{k}(-s)^{(n-k)m}W_{km}H_{km} = 2^n\alpha^{2mn-1}(mnW_1-\alpha\,(mn-2)\,W_0).$$

Proof. We use Binet's formulas of W_n, G_n and H_n.

(a) If the roots α and β of characteristic equation (1.2) are distinct, i.e., $\alpha \neq \beta$, then we get

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}(-s)^{km}W_{(n-2k)m}\\
=\ &\sum_{k=0}^{n}\binom{n}{k}(\alpha\beta)^{km}\left(\frac{W_1-\beta W_0}{\alpha-\beta}\alpha^{(n-2k)m}-\frac{W_1-\alpha W_0}{\alpha-\beta}\beta^{(n-2k)m}\right)\\
=\ &\frac{1}{\alpha-\beta}((W_1-\beta W_0)-(W_1-\alpha W_0))(\alpha^m+\beta^m)^n\\
=\ &\frac{1}{\alpha-\beta}\left(\alpha-\beta\right)W_0(\alpha^m+\beta^m)^n\\
=\ &W_0(\alpha^m+\beta^m)^n\\
=\ &W_0H_m^n
\end{aligned}$$

and if the roots α and β of characteristic equation (1.2) are equal, i.e., $\alpha=\beta$, then we obtain

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}(-s)^{km}W_{(n-2k)m}\\
=\ &\sum_{k=0}^{n}\binom{n}{k}(\alpha\alpha)^{km}\left(((n-2k)mW_1-\alpha\left((n-2k)m-1\right)W_0)\alpha^{(n-2k)m-1}\right)\\
=\ &\sum_{k=0}^{n}\binom{n}{k}\alpha^{2km}\left(((n-2k)mW_1-\alpha\left((n-2k)m-1\right)W_0)\alpha^{(n-2k)m-1}\right)\\
=\ &\sum_{k=0}^{n}\binom{n}{k}((n-2k)mW_1-\alpha\left((n-2k)m-1\right)W_0)\alpha^{mn-1}\\
=\ &2^n\alpha^{mn}W_0.
\end{aligned}$$

(b) **(i)** If $\alpha\neq\beta$, then we get

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}(-s)^{(n-k)m}W_{km}^2\\
=\ &\sum_{k=0}^{n}\binom{n}{k}(\alpha\beta)^{(n-k)m}\left(\frac{W_1-\beta W_0}{\alpha-\beta}\alpha^{km}-\frac{W_1-\alpha W_0}{\alpha-\beta}\beta^{km}\right)^2
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(W_1-\beta W_0)^2}{(\alpha-\beta)^2}\alpha^{nm}\sum_{k=0}^{n}\binom{n}{k}\alpha^{km}\beta^{(n-k)m} \\
&\quad +\frac{(W_1-\alpha W_0)^2}{(\alpha-\beta)^2}\beta^{nm}\sum_{k=0}^{n}\binom{n}{k}\alpha^{(n-k)m})\beta^{km} \\
&\quad -2\frac{W_1-\beta W_0}{\alpha-\beta}\frac{W_1-\alpha W_0}{\alpha-\beta}\alpha^{nm}\beta^{nm}\sum_{k=0}^{n}\binom{n}{k} \\
&= \frac{(W_1-\beta W_0)^2}{(\alpha-\beta)^2}\alpha^{nm}(\alpha^m+\beta^m)^n+\frac{(W_1-\alpha W_0)^2}{(\alpha-\beta)^2}\beta^{nm}(\alpha^m+\beta^m)^n \\
&\quad -2\frac{W_1-\beta W_0}{\alpha-\beta}\frac{W_1-\alpha W_0}{\alpha-\beta}\alpha^{nm}\beta^{nm}2^n \\
&= \frac{(W_1-\beta W_0)^2\alpha^{nm}+(W_1-\alpha W_0)^2\beta^{nm}}{r^2+4s}H_m^n \\
&\quad -\frac{(W_1-\beta W_0)(W_1-\alpha W_0)}{r^2+4s}(-s)^{nm}2^{n+1}
\end{aligned}
$$

and if $\alpha=\beta$, then we obtain

$$
\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}(-s)^{(n-k)m}W_{km}^2 \\
&= \sum_{k=0}^{n}\binom{n}{k}(\alpha^2)^{(n-k)m}((kmW_1-\alpha\,(km-1)\,W_0)\alpha^{km-1})^2 \\
&= \sum_{k=0}^{n}\binom{n}{k}\alpha^{2(n-k)m}\alpha^{2(km-1)}(kmW_1-\alpha\,(km-1)\,W_0)^2 \\
&= 2^{n-2}\alpha^{2mn-2}(m^2n\,(n+1)\,W_1^2+\alpha^2(m^2n+m^2n^2-4mn+4)W_0^2 \\
&\quad -2mn\alpha(m+mn-2)W_0W_1).
\end{aligned}
$$

(ii) If $\alpha\neq\beta$, then we get

$$
\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}(-s)^{(n-k)m}W_{km}G_{km} \\
&= \sum_{k=0}^{n}\binom{n}{k}(\alpha\beta)^{(n-k)m}(\frac{W_1-\beta W_0}{\alpha-\beta}\alpha^{km}-\frac{W_1-\alpha W_0}{\alpha-\beta}\beta^{km})(\frac{\alpha^{km}-\beta^{km}}{\alpha-\beta})
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{(\alpha-\beta)^2}((W_1-\beta W_0)\sum_{k=0}^{n}\binom{n}{k}\alpha^{(n-k)m}\beta^{(n-k)m}\alpha^{2km} \\
&\quad +(W_1-\alpha W_0)\sum_{k=0}^{n}\binom{n}{k}\alpha^{(n-k)m})\beta^{(n-k)m}\beta^{2km} \\
&\quad -((W_1-\beta W_0)+(W_1-\alpha W_0))\sum_{k=0}^{n}\binom{n}{k}\alpha^{(n-k)m})\beta^{(n-k)m}\alpha^{km}\beta^{km}) \\
&= \frac{1}{(\alpha-\beta)^2}((W_1-\beta W_0)\alpha^{nm}(\alpha^m+\beta^m)^n+(W_1-\alpha W_0)\beta^{nm}(\alpha^m+\beta^m)^n \\
&\quad -((W_1-\beta W_0)+(W_1-\alpha W_0))\alpha^{nm}\beta^{nm}2^n) \\
&= \frac{((W_1-\beta W_0)\alpha^{nm}+(W_1-\alpha W_0)\beta^{nm})H_m^n}{r^2+4s} \\
&\quad -\frac{((W_1-\beta W_0)+(W_1-\alpha W_0))(-s)^{nm}2^n}{r^2+4s}
\end{aligned}
$$

and if $\alpha=\beta$, then we obtain

$$
\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}(-s)^{(n-k)m}W_{km}G_{km} \\
&= \sum_{k=0}^{n}\binom{n}{k}(\alpha^2)^{(n-k)m}(kmW_1-\alpha\,(km-1)\,W_0)\alpha^{km-1}km\alpha^{km-1} \\
&= \sum_{k=0}^{n}\binom{n}{k}\alpha^{2mn-2}(kmW_1-\alpha\,(km-1)\,W_0)km \\
&= mn2^{n-2}\alpha^{2mn-2}(m\,(n+1)\,W_1-\alpha\,(m+mn-2)\,W_0).
\end{aligned}
$$

(iii) If $\alpha\neq\beta$, then we get

$$
\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}(-s)^{(n-k)m}W_{km}H_{km} \\
&= \sum_{k=0}^{n}\binom{n}{k}(\alpha\beta)^{(n-k)m}(\frac{W_1-\beta W_0}{\alpha-\beta}\alpha^{km}-\frac{W_1-\alpha W_0}{\alpha-\beta}\beta^{km})(\alpha^{km}+\beta^{km})
\end{aligned}
$$

$$\begin{aligned}
&= \frac{1}{(\alpha-\beta)}((W_1-\beta W_0)\alpha^{nm}\sum_{k=0}^{n}\binom{n}{k}\alpha^{km}\beta^{(n-k)m} \\
&\quad -(W_1-\alpha W_0)\beta^{nm}\sum_{k=0}^{n}\binom{n}{k}\alpha^{(n-k)m})\beta^{km}+((W_1-\beta W_0) \\
&\quad -(W_1-\alpha W_0))\alpha^{nm}\beta^{nm}\sum_{k=0}^{n}\binom{n}{k}) \\
&= \frac{((W_1-\beta W_0)\alpha^{nm}-(W_1-\alpha W_0)\beta^{nm})H_m^n}{(r^2+4s)^{\frac{1}{2}}} \\
&\quad +\frac{((W_1-\beta W_0)-(W_1-\alpha W_0))(-s)^{nm}2^n}{(r^2+4s)^{\frac{1}{2}}}
\end{aligned}$$

and if $\alpha=\beta$, then we obtain

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}(-s)^{(n-k)m}W_{km}H_{km} \\
&= \sum_{k=0}^{n}\binom{n}{k}(\alpha^2)^{(n-k)m}(kmW_1-\alpha\,(km-1)\,W_0)\alpha^{km-1}\times 2\alpha^{km} \\
&= 2\sum_{k=0}^{n}\binom{n}{k}\alpha^{2mn-1}(kmW_1-\alpha\,(km-1)\,W_0) \\
&= 2^n\alpha^{2mn-1}(mnW_1-\alpha\,(mn-2)\,W_0).
\end{aligned}$$

□

Taking $W_n = G_n$ with $G_0 = 0, G_1 = 1$ in the last Theorem, we get the following Corollary.

Corollary 42. *Let $n \geq 0$ and $m \in \mathbb{Z}$. Then*

(a)

$$\sum_{k=0}^{n}\binom{n}{k}(-s)^{km}G_{(n-2k)m}=0.$$

(b) *If $\alpha \neq \beta$, then*

$$\sum_{k=0}^{n}\binom{n}{k}(-s)^{(n-k)m}G_{km}^2=\frac{1}{r^2+4s}(H_{nm}H_m^n-(-s)^{nm}2^{n+1}),$$

and if $\alpha = \beta$, *then*

$$\sum_{k=0}^{n} \binom{n}{k} (-s)^{(n-k)m} G_{km}^{2} = n\,(n+1)\,2^{n-2} m^{2} \alpha^{2mn-2}.$$

Taking $W_n = H_n$ with $H_0 = 2, H_1 = r$, in the last Theorem, we get the following Corollary.

Corollary 43. *Let* $n \geq 0$ *and* $m \in \mathbb{Z}$. *Then*

(a)

$$\sum_{k=0}^{n} \binom{n}{k} (-s)^{km} H_{(n-2k)m} = \begin{cases} 2H_m^n & , \text{ if } \alpha \neq \beta \\ 2^{n+1}\alpha^{mn} & , \text{ if } \alpha = \beta \end{cases}.$$

(b)

(i) *If* $\alpha \neq \beta$, *then*

$$\sum_{k=0}^{n} \binom{n}{k} (-s)^{(n-k)m} H_{km}^{2} = H_{nm} H_m^n + (-s)^{nm} 2^{n+1},$$

and if $\alpha = \beta$, *then*

$$\begin{aligned} &\sum_{k=0}^{n} \binom{n}{k} (-s)^{(n-k)m} H_{km}^{2} \\ = \; & 2^{n-2}\alpha^{2mn-2}\left(m^2 n\,(n+1)\,r^2 + 4\alpha^2(m^2 n + m^2 n^2 - 4mn + 4)\right. \\ & \left. -4mn\alpha(m + mn - 2)r\right). \end{aligned}$$

(ii) *If* $\alpha \neq \beta$, *then*

$$\sum_{k=0}^{n} \binom{n}{k} (-s)^{(n-k)m} H_{km} G_{km} = G_{nm} H_m^n,$$

and if $\alpha = \beta$, *then*

$$\begin{aligned} &\sum_{k=0}^{n} \binom{n}{k} (-s)^{(n-k)m} H_{km} G_{km} \\ = \; & mn2^{n-2}\alpha^{2mn-2}(m\,(n+1)\,r - 2\alpha\,(m + mn - 2)). \end{aligned}$$

Next, we give some combinatorial results on generalized Fibonacci (Horadam) polynomials.

Theorem 44. *Let $n \geq 0$ and $m \in \mathbb{Z}$. Then*

(a) *If $\alpha \neq \beta$, i.e., $r^2+4s \neq 0$, then*

$$\begin{aligned} &\sum_{k=0}^{n} \binom{n}{k} (-1)^k (-s)^{km} W_{(n-2k)m} \\ &= (r^2+4s)^{\frac{n-1}{2}} ((W_1-\beta W_0)-(W_1-\alpha W_0)(-1)^n) G_m^n \\ &= \begin{cases} (r^2+4s)^{\frac{n}{2}} W_0 G_m^n & \text{if } n \text{ is even} \\ (r^2+4s)^{\frac{n-1}{2}} (2W_1 - rW_0) G_m^n & \text{if } n \text{ is odd} \end{cases}, \end{aligned}$$

and if $\alpha = \beta$, i.e., $r^2+4s=0$, then

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k (-s)^{km} W_{(n-2k)m} = \begin{cases} 2m\alpha^{m-1}(W_1-\alpha W_0) & \text{if } n=1 \\ W_0 & \text{if } n=0 \\ 0 & \text{if } n \neq 0 \wedge n \neq 1 \end{cases}.$$

(b)

(i) *If $\alpha \neq \beta$, i.e., $r^2+4s \neq 0$, then*

$$\begin{aligned} &\sum_{k=0}^{n} \binom{n}{k} (-1)^k (-s)^{(n-k)m} W_{km}^2 \\ &= \begin{cases} \left(r^2+4s\right)^{\frac{n-2}{2}} ((W_1-\beta W_0)^2 (-1)^n \alpha^{nm} \\ \quad +(W_1-\alpha W_0)^2 \beta^{nm}) G_m^n & \text{if } n \neq 0 \\ W_0 & \text{if } n=0 \end{cases}, \end{aligned}$$

and if $\alpha = \beta$, i.e., $r^2+4s=0$, then

$$\begin{aligned} &\sum_{k=0}^{n} \binom{n}{k} (-1)^k (-s)^{(n-k)m} W_{km}^2 \\ &= \begin{cases} 2m^2\alpha^{4m-2}\left(W_1-\alpha W_0\right)^2 & \text{if } n=2 \\ m\alpha^{2m-2}\left(W_1-\alpha W_0\right)\left(-mW_1+\alpha\left(m-2\right)W_0\right) & \text{if } n=1 \\ W_0^2 & \text{if } n=0 \\ 0 & \text{if } n \neq 0 \wedge n \neq 1 \wedge n \neq 2 \end{cases}. \end{aligned}$$

(ii) *If* $\alpha \neq \beta$, *i.e.,* $r^2+4s \neq 0$, *then*

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k(-s)^{(n-k)m}W_{km}G_{km}$$
$$= \begin{cases} (r^2+4s)^{\frac{n-2}{2}}(((W_1-\beta W_0)\alpha^{nm}(-1)^n \\ \quad +(W_1-\alpha W_0)\beta^{nm})G_m^n & \text{if} \quad n \neq 0 \\ 0 & \text{if} \quad n=0 \end{cases},$$

and if $\alpha = \beta$, *then*

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k(-s)^{(n-k)m}W_{km}G_{km}$$
$$= \begin{cases} 2m^2\alpha^{4m-2}(W_1-\alpha W_0) & \text{if} \quad n=2 \\ -m\alpha^{2m-2}(mW_1-\alpha(m-1)W_0) & \text{if} \quad n=1 \\ 0 & \text{if} \quad n\neq 1 \wedge n \neq 2 \end{cases}.$$

(iii) *If* $\alpha \neq \beta$, *i.e.,* $r^2+4s \neq 0$, *then*

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k(-s)^{(n-k)m}W_{km}H_{km}$$
$$= \begin{cases} (r^2+4s)^{\frac{n-1}{2}}((W_1-\beta W_0)\alpha^{nm}(-1)^n \\ \quad -(W_1-\alpha W_0)\beta^{nm})G_m^n & \text{if} \quad n \neq 0 \\ 2W_0 & \text{if} \quad n=0 \end{cases},$$

and if $\alpha = \beta$, *then*

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k(-s)^{(n-k)m}W_{km}H_{km}$$
$$= \begin{cases} -2m\alpha^{2m-1}(W_1-\alpha W_0) & \text{if} \quad n=1 \\ 2W_0 & \text{if} \quad n=0 \\ 0 & \text{if} \quad n\neq 0 \wedge n \neq 1 \end{cases}.$$

Proof. We use Binet's formulas of W_n, G_n and H_n.

(a) If $\alpha \neq \beta$, then we get

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}(-1)^{k}(-s)^{km}W_{(n-2k)m} \\
&= \sum_{k=0}^{n}\binom{n}{k}(-1)^{k}(\alpha\beta)^{km}\left(\frac{W_1-\beta W_0}{\alpha-\beta}\alpha^{(n-2k)m}-\frac{W_1-\alpha W_0}{\alpha-\beta}\beta^{(n-2k)m}\right) \\
&= \frac{W_1-\beta W_0}{\alpha-\beta}\sum_{k=0}^{n}\binom{n}{k}(-1)^{k}\alpha^{(n-k)m}\beta^{km} \\
&\quad -\frac{W_1-\alpha W_0}{\alpha-\beta}\sum_{k=0}^{n}\binom{n}{k}(-1)^{k}\alpha^{km}\beta^{(n-k)m} \\
&= \frac{W_1-\beta W_0}{\alpha-\beta}(\alpha^m-\beta^m)^n-\frac{W_1-\alpha W_0}{\alpha-\beta}(\beta^m-\alpha^m)^n \\
&= \frac{1}{\alpha-\beta}((W_1-\beta W_0)-(W_1-\alpha W_0)(-1)^n)(\alpha^m-\beta^m)^n \\
&= (r^2+4s)^{\frac{n-1}{2}}((W_1-\beta W_0)-(W_1-\alpha W_0)(-1)^n)G_m^n
\end{aligned}$$

and if the roots α and β of characteristic equation (1.2) are equal, i.e., $\alpha=\beta$, then we obtain

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}(-1)^{k}(-s)^{km}W_{(n-2k)m} \\
&= \sum_{k=0}^{n}\binom{n}{k}(-1)^{k}(\alpha\alpha)^{km}\left(((n-2k)mW_1-\alpha\left((n-2k)m-1\right)W_0)\alpha^{(n-2k)m-1}\right) \\
&= \sum_{k=0}^{n}\binom{n}{k}(-1)^{k}\alpha^{2km}\left(((n-2k)mW_1-\alpha\left((n-2k)m-1\right)W_0)\alpha^{(n-2k)m-1}\right) \\
&= \sum_{k=0}^{n}\binom{n}{k}(-1)^{k}((n-2k)mW_1-\alpha\left((n-2k)m-1\right)W_0)\alpha^{mn-1} \\
&= \left\{\begin{array}{ccc} 2m\alpha^{m-1}(W_1-\alpha W_0) & \text{if} & n=1 \\ W_0 & \text{if} & n=0 \\ 0 & \text{if} & n\neq 0\wedge n\neq 1 \end{array}\right. .
\end{aligned}$$

(b) Note that

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{k}=\left\{\begin{array}{ccc} 1 & \text{if} & n=0 \\ 0 & \text{if} & n\neq 0 \end{array}\right. .$$

(i) If $\alpha \neq \beta$, then, if we get

$$
\begin{aligned}
& \sum_{k=0}^{n} \binom{n}{k} (-1)^k (-s)^{(n-k)m} W_{km}^2 \\
= & \sum_{k=0}^{n} \binom{n}{k} (-1)^k (\alpha\beta)^{(n-k)m} \left(\frac{W_1 - \beta W_0}{\alpha - \beta} \alpha^{km} - \frac{W_1 - \alpha W_0}{\alpha - \beta} \beta^{km} \right)^2 \\
= & \frac{(W_1 - \beta W_0)^2}{(\alpha - \beta)^2} \alpha^{nm} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \alpha^{km} \beta^{(n-k)m} \\
& + \frac{(W_1 - \alpha W_0)^2}{(\alpha - \beta)^2} \beta^{nm} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \alpha^{(n-k)m)} \beta^{km} \\
& -2 \frac{W_1 - \beta W_0}{\alpha - \beta} \frac{W_1 - \alpha W_0}{\alpha - \beta} \alpha^{nm} \beta^{nm} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \\
= & \frac{(W_1 - \beta W_0)^2}{(\alpha - \beta)^2} \alpha^{nm} (\beta^m - \alpha^m)^n + \frac{(W_1 - \alpha W_0)^2}{(\alpha - \beta)^2} \beta^{nm} (\alpha^m - \beta^m)^n \\
& -2 \frac{W_1 - \beta W_0}{\alpha - \beta} \frac{W_1 - \alpha W_0}{\alpha - \beta} \alpha^{nm} \beta^{nm} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \\
= & \left(r^2 + 4s\right)^{\frac{n-2}{2}} ((W_1 - \beta W_0)^2 (-1)^n \alpha^{nm} + (W_1 - \alpha W_0)^2 \beta^{nm}) G_m^n \\
& - \frac{(W_1 - \beta W_0)(W_1 - \alpha W_0)}{r^2 + 4s} (-s)^{nm} \sum_{k=0}^{n} \binom{n}{k} (-1)^k
\end{aligned}
$$

and so if $n = 0$, then

$$
\sum_{k=0}^{n} \binom{n}{k} (-1)^k (-s)^{(n-k)m} W_{km}^2 = \frac{W_1^2 + \left(r^2 + 3s\right) W_0^2 - r W_0 W_1}{4s + r^2}
$$

(in fact we can find directly $\sum_{k=0}^{n} \binom{n}{k} (-1)^k (-s)^{(n-k)m} W_{km}^2 = W_0^2$) and if $n \neq 0$, then

$$
\begin{aligned}
& \sum_{k=0}^{n} \binom{n}{k} (-1)^k (-s)^{(n-k)m} W_{km}^2 \\
= & \left(r^2 + 4s\right)^{\frac{n-2}{2}} ((W_1 - \beta W_0)^2 (-1)^n \alpha^{nm} + (W_1 - \alpha W_0)^2 \beta^{nm}) G_m^n.
\end{aligned}
$$

If $\alpha = \beta$, then we obtain

$$\begin{aligned}
& \sum_{k=0}^{n} \binom{n}{k} (-1)^k (-s)^{(n-k)m} W_{km}^2 \\
= & \sum_{k=0}^{n} \binom{n}{k} (-1)^k \alpha^{2(n-k)m} \alpha^{2(km-1)} (kmW_1 - \alpha\,(km-1)\,W_0)^2
\end{aligned}$$

and so the required result follows.

(ii) If $\alpha \neq \beta$, then we get

$$\begin{aligned}
& \sum_{k=0}^{n} \binom{n}{k} (-1)^k (-s)^{(n-k)m} W_{km} G_{km} \\
= & \sum_{k=0}^{n} \binom{n}{k} (-1)^k (\alpha\beta)^{(n-k)m} (\frac{W_1 - \beta W_0}{\alpha - \beta} \alpha^{km} - \frac{W_1 - \alpha W_0}{\alpha - \beta} \beta^{km}) \\
& \times (\frac{\alpha^{km} - \beta^{km}}{\alpha - \beta}) \\
= & \frac{1}{(\alpha - \beta)^2} ((W_1 - \beta W_0)\alpha^{nm} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \alpha^{km} \beta^{(n-k)m} \\
& + (W_1 - \alpha W_0)\beta^{nm} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \alpha^{(n-k)m}) \beta^{km} \\
& - ((W_1 - \beta W_0) + (W_1 - \alpha W_0))\alpha^{nm} \beta^{nm} \sum_{k=0}^{n} \binom{n}{k} (-1)^k) \\
= & \frac{1}{(\alpha - \beta)^2} ((W_1 - \beta W_0)\alpha^{nm} (\beta^m - \alpha^m)^n \\
& + (W_1 - \alpha W_0)\beta^{nm} (\alpha^m - \beta^m)^n \\
& - ((W_1 - \beta W_0) + (W_1 - \alpha W_0))\alpha^{nm} \beta^{nm} \sum_{k=0}^{n} \binom{n}{k} (-1)^k) \\
= & (r^2 + 4s)^{\frac{n-2}{2}} (((W_1 - \beta W_0)\alpha^{nm} (-1)^n + (W_1 - \alpha W_0)\beta^{nm}) G_m^n \\
& - ((W_1 - \beta W_0) + (W_1 - \alpha W_0))\alpha^{nm} \beta^{nm} \sum_{k=0}^{n} \binom{n}{k} (-1)^k)
\end{aligned}$$

and so if $n=0$, then since $\sum_{k=0}^{n}\binom{n}{k}(-1)^k = 1$, we have

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k(-s)^{(n-k)m}W_{km}G_{km} = 0$$

(in fact we can find directly $\sum_{k=0}^{n}\binom{n}{k}(-1)^k(-s)^{(n-k)m}W_{km}G_{km} = W_0G_0 = 0$) and if $n \neq 0$, then since $\sum_{k=0}^{n}\binom{n}{k}(-1)^k = 0$

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}(-1)^k(-s)^{(n-k)m}W_{km}G_{km}\\ = \ & (r^2+4s)^{\frac{n-2}{2}}(((W_1-\beta W_0)\alpha^{nm}(-1)^n + (W_1-\alpha W_0)\beta^{nm})G_m^n\end{aligned}$$

and if $\alpha = \beta$, then we obtain

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}(-1)^k(-s)^{(n-k)m}W_{km}G_{km}\\ = \ & \sum_{k=0}^{n}\binom{n}{k}(-1)^k(\alpha^2)^{(n-k)m}(kmW_1-\alpha\,(km-1)\,W_0)\alpha^{km-1}km\alpha^{km-1}\\ = \ & \sum_{k=0}^{n}\binom{n}{k}(-1)^k\alpha^{2mn-2}(kmW_1-\alpha\,(km-1)\,W_0)km\end{aligned}$$

and so the required result follows.

(iii) If $\alpha \neq \beta$, then we get

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}(-1)^k(-s)^{(n-k)m}W_{km}H_{km}\\ = \ & \sum_{k=0}^{n}\binom{n}{k}(-1)^k(\alpha\beta)^{(n-k)m}(\frac{W_1-\beta W_0}{\alpha-\beta}\alpha^{km} - \frac{W_1-\alpha W_0}{\alpha-\beta}\beta^{km})\\ & \times(\alpha^{km}+\beta^{km})\\ = \ & \frac{1}{(\alpha-\beta)}((W_1-\beta W_0)\alpha^{nm}\sum_{k=0}^{n}\binom{n}{k}(-1)^k\alpha^{km}\beta^{(n-k)m}\\ & -(W_1-\alpha W_0)\beta^{nm}\sum_{k=0}^{n}\binom{n}{k}(-1)^k\alpha^{(n-k)m})\beta^{km}\end{aligned}$$

$$
\begin{aligned}
&\quad +((W_1-\beta W_0)-(W_1-\alpha W_0))\alpha^{nm}\beta^{nm}\sum_{k=0}^{n}\binom{n}{k}(-1)^k) \\
&= \frac{1}{(\alpha-\beta)}((W_1-\beta W_0)\alpha^{nm}(\beta^m-\alpha^m)^n-(W_1-\alpha W_0)\beta^{nm}(\alpha^m-\beta^m)^n \\
&\quad +((W_1-\beta W_0)-(W_1-\alpha W_0))\alpha^{nm}\beta^{nm}\sum_{k=0}^{n}\binom{n}{k}(-1)^k) \\
&= (r^2+4s)^{\frac{n-1}{2}}((W_1-\beta W_0)\alpha^{nm}(-1)^n-(W_1-\alpha W_0)\beta^{nm})G_m^n \\
&\quad +\frac{1}{(r^2+4s)^{\frac{1}{2}}}((W_1-\beta W_0)-(W_1-\alpha W_0))(-s)^{nm}\sum_{k=0}^{n}\binom{n}{k}(-1)^k
\end{aligned}
$$

and so if $n=0$, then since $\sum_{k=0}^{n}\binom{n}{k}(-1)^k=1$, we have

$$
\sum_{k=0}^{n}\binom{n}{k}(-1)^k(-s)^{(n-k)m}W_{km}H_{km}=2\frac{(\alpha-\beta)W_0}{\sqrt{r^2+4s}}=2\frac{\sqrt{r^2+4s}W_0}{\sqrt{r^2+4s}}=2W_0
$$

(in fact we can find directly $\sum_{k=0}^{n}\binom{n}{k}(-1)^k(-s)^{(n-k)m}W_{km}H_{km} = W_0H_0=2W_0$) and if $n\neq 0$, then, since $\sum_{k=0}^{n}\binom{n}{k}(-1)^k=0$,

$$
\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}(-1)^k(-s)^{(n-k)m}W_{km}H_{km} \\
&= (r^2+4s)^{\frac{n-1}{2}}((W_1-\beta W_0)\alpha^{nm}(-1)^n-(W_1-\alpha W_0)\beta^{nm})G_m^n
\end{aligned}
$$

and if $\alpha=\beta$, then we obtain

$$
\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}(-1)^k(-s)^{(n-k)m}W_{km}H_{km} \\
&= \sum_{k=0}^{n}\binom{n}{k}(-1)^k(\alpha^2)^{(n-k)m}(kmW_1-\alpha\,(km-1)\,W_0)\alpha^{km-1}\times 2\alpha^{km} \\
&= 2\sum_{k=0}^{n}\binom{n}{k}(-1)^k\alpha^{2mn-1}(kmW_1-\alpha\,(km-1)\,W_0)
\end{aligned}
$$

and so the required result follows. □

Taking $W_n = G_n$ with $G_0 = 0, G_1 = 1$ in the last Theorem, we get the following Corollary.

Corollary 45. *Let $n \geq 0$ and $m \in \mathbb{Z}$. Then*

(a) *if $\alpha \neq \beta$, then*

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}(-1)^k(-s)^{km}G_{(n-2k)m} &= (r^2+4s)^{\frac{n-1}{2}}(1-(-1)^n)G_m^n \\ &= \begin{cases} 0 & \text{if } n \text{ is even} \\ 2(r^2+4s)^{\frac{n-1}{2}}G_m^n & \text{if } n \text{ is odd}\end{cases},\end{aligned}$$

and if $\alpha = \beta$, then

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k(-s)^{km}G_{(n-2k)m} = \begin{matrix} 2m\alpha^{m-1} & \text{if } n=1 \\ 0 & \text{if } n\neq 1\end{matrix}.$$

(b) *If $\alpha \neq \beta$, then*

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}(-1)^k(-s)^{(n-k)m}G_{km}^2 \\ &= \begin{cases}\left(r^2+4s\right)^{\frac{n-2}{2}}(\alpha^{nm}(-1)^n+\beta^{nm})G_m^n & \text{if } n\neq 0 \\ 0 & \text{if } n=0\end{cases} \\ &= \begin{cases}\left(r^2+4s\right)^{\frac{n-2}{2}}H_{nm}G_m^n & \text{if } n\neq 0,\ n \text{ is even} \\ -\left(r^2+4s\right)^{\frac{n-1}{2}}G_{nm}G_m^n & \text{if } n\neq 0,\ n \text{ is odd} \\ 0 & \text{if } n=0\end{cases},\end{aligned}$$

and if $\alpha = \beta$, then

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k(-s)^{(n-k)m}G_{km}^2 = \begin{cases} 2m^2\alpha^{4m-2} & \text{if } n=2 \\ -m^2\alpha^{2m-2} & \text{if } n=1 \\ 0 & \text{if } n\neq 1 \wedge n\neq 2\end{cases}.$$

Taking $W_n = H_n$ with $H_0 = 2, H_1 = r$, in the last Theorem, we get the following Corollary.

Corollary 46. *Let* $n \geq 0$ *and* $m \in \mathbb{Z}$. *Then*

(a) *if* $\alpha \neq \beta$, *then*

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}(-1)^{k}(-s)^{km}H_{(n-2k)m} &= (r^{2}+4s)^{\frac{n}{2}}(1+(-1)^{n})G_{m}^{n} \\ &= \left\{\begin{array}{ccc} 2(r^{2}+4s)^{\frac{n}{2}}G_{m}^{n} & \text{if} & n \text{ is even} \\ 0 & \text{if} & n \text{ is odd}\end{array}\right. ,\end{aligned}$$

and if $\alpha = \beta$, *then*

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{k}(-s)^{km}H_{(n-2k)m} = \left\{\begin{array}{ccc} 2 & \text{if} & n=0 \\ 0 & \text{if} & n\neq 0\end{array}\right. .$$

(b)

(i) *If* $\alpha \neq \beta$, *then*

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}(-1)^{k}(-s)^{(n-k)m}H_{km}^{2} \\ &= \left\{\begin{array}{ccc} \left(r^{2}+4s\right)^{\frac{1}{2}n}((-1)^{n}\alpha^{nm}+\beta^{nm})G_{m}^{n} & \text{if} & n\neq 0 \\ 2 & \text{if} & n=0\end{array}\right. \\ &= \left\{\begin{array}{ccc} \left(r^{2}+4s\right)^{\frac{1}{2}n}H_{nm}G_{m}^{n} & \text{if} & n\neq 0,\ n \text{ is even} \\ -\left(r^{2}+4s\right)^{\frac{n+1}{2}}G_{nm}G_{m}^{n} & \text{if} & n\neq 0,\ n \text{ is odd} \\ 2 & \text{if} & n=0\end{array}\right. ,\end{aligned}$$

and if $\alpha = \beta$, *then*

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{k}(-s)^{(n-k)m}H_{km}^{2} = \left\{\begin{array}{ccc} 4 & \text{if} & n=0 \\ 0 & \text{if} & n\neq 0\end{array}\right. .$$

(ii) *If $\alpha \neq \beta$, then*

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k (-s)^{(n-k)m} H_{km} G_{km}$$
$$= \begin{cases} (r^2+4s)^{\frac{n-1}{2}} (\alpha^{nm}(-1)^n - \beta^{nm}) G_m^n & \text{if} \quad n \neq 0 \\ 0 & \text{if} \quad n = 0 \end{cases}$$
$$= \begin{cases} (r^2+4s)^{\frac{n}{2}} G_{nm} G_m^n & \text{if} \quad n \neq 0,\ n \text{ is even} \\ -(r^2+4s)^{\frac{n-1}{2}} H_{mn} G_m^n & \text{if} \quad n \neq 0,\ n \text{ is odd} \\ 0 & \text{if} \quad n = 0 \end{cases},$$

and if $\alpha = \beta$, then

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k (-s)^{(n-k)m} H_{km} G_{km} = \begin{cases} -2m\alpha^{2m-1} & \text{if} \quad n = 1 \\ 0 & \text{if} \quad n \neq 1 \end{cases}.$$

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