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Integration by Parts in Differential Summation Form

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Abstract

In this paper, we establish general differential summation formulas for integration by parts (IBP), more importantly a powerful tool that promotes exploration and creativity.

1. Introduction

Definition 1.1. A differential calculus is a branch of mathematics concerned with the determination, properties and application of derivatives and differentials.

Definition 1.2. An integral calculus or antiderivative or primitive assigns numbers to functions in a way that can be described as displacement, area, volume and other concepts that arise by combining infinitesimal data. Integration is one of the two main operations of calculus, with its inverse, differentiation. Integration without limits is called indefinite integrals, while integration with limit is called definite integrals. It is the fundamental theorem of calculus that connects differentiation with the definite integrals.

Definition 1.3 (Fundamental Theorem of Calculus 1)**.** If f is a continuous real-valued function defined on a closed interval $[a, b]$, then, once an antiderivative F of f is known,

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the definite integral of *f* over that interval is given by

$$\int_a^b f(x)dx = F(x)|_a^b = F(b) - F(a).$$

By product rule, if $f(x)$ and $g(x)$ are differentiable functions, then

$$\frac{d}{dx}(f(x)g(x)) = f(x)g'(x) + f'(x)g(x).$$

Integrating both sides, we have

$$\int f(x)g'(x)dx + \int f'(x)g(x)dx = f(x)g(x),$$

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx.$$

Let $u = f(x)$ and $v = g(x)$. Then we obtain the familiar integration by parts formula

$$\int udv = uv - \int vdu. \tag{1}$$

2. Literature Review

Integral calculus has been the most difficult aspect of mathematics for Secondary School students and preliminary Higher Institution students, most especially integration by parts. Students often find it very difficult in the choice of *u* and *dv*. Horowitz [3] gives the technique called tabular integration by parts, the method was used to solve some difficult integration by parts, not only that, the method was used to proof some mathematical formulas such as Laplace Transforms Formula, Taylor's Formula, and Residue Theorem for Meromorphic Functions Formula. Knill [4] used the integration by parts formula and tabular integration by parts to solve integration by parts problems, the two results shows that tabular integration by parts is so powerful, not time consuming and reliable. Murty [1] illustrated the procedure of integration by parts with five examples of the type

$$\int x^2 \sin xdx, \quad \int e^x \sin xdx, \quad \int x^2 \ln xdx, \quad \int \sin^2 xdx, \quad \int \sin 3x \cos 5xdx.$$

The method of tabular integration by parts was used to solve the problems.

The general formula for integral by parts of the form $\int e^{ax} \cos bxdx$ and $\int e^{ax} \sin bxdx$ is given by

$$I_n = \int e^{ax} \cos bxdx = \frac{e^{ax}}{a^2 + b^2}(b \sin bx + a \cos bx) + c, \tag{2}$$

$$I_n = \int e^{ax} \sin bxdx = \frac{e^{ax}}{a^2 + b^2}(a \sin bx - b \cos bx) + c. \tag{3}$$

Equations (2) and (3) can be applied directly to solve problems like

$$\int e^{2x} \sin 3xdx = \frac{e^{2x}}{13}(2 \sin 3x - 3 \cos 3x) + c.$$

Find the Fourier cosine transform of $f(x) = e^{-2x} + 4e^{-3x}$

$$F(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sxdx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty (e^{-2x} + 4e^{-3x}) \cos sxdx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-2x} \cos sxdx + \sqrt{\frac{2}{\pi}} \int_0^\infty 4e^{-3x} \cos sxdx$$

$$= 2\sqrt{\frac{2}{\pi}} \left[\frac{1}{s^2 + 4} + \frac{6}{s^2 + 9} \right].$$

This paper gives some formulas to evaluate integration by parts without thinking about the choose of u and dv. Students with no background in integration can also make use of the formulas in solving their integration by parts problems.

3. Main Results

Our main results were proved by the principle of mathematical induction.

Theorem 3.1. *If* $n \in \mathbb{N}$ *and* $a \neq 0,$ *then*

$$\int x^n \sin axdx = \sum_{r=0}^{n} {}^nP_r x^{n-r} \frac{1}{a^{2r+1}} \frac{d^r}{dx^r}(-\cos ax). \tag{4}$$

Proof. Let equation (4) be $P(n)$. Consider $P(1)$, we obtain

$$\int x \sin ax dx = \sum_{r=0}^{1} {}^{1}P_r x^{1-r} \frac{1}{a^{2r+1}} \frac{d^r}{dx^r}(-\cos ax)$$

$$= \frac{1}{a^2}(-ax\cos ax + \sin ax) + c.$$

Hence, $P(1)$ is true. Assuming, $P(k)$ is true for some $k \in \mathbb{N}$,

$$\int x^k \sin ax dx = \sum_{r=0}^{k} {}^{k}P_r x^{k-r} \frac{1}{a^{2r+1}} \frac{d^r}{dx^r}(-\cos ax). \tag{5}$$

Consider $P(k+1)$ and apply equation (5), we have the following

$$\int x^{k+1} \sin ax = uv - \int v du$$

$$u = x^{k+1},\ du = (k+1)x^k \text{ and } dv = \sin ax,\ v = -\frac{1}{a}\cos ax$$

$$= -\frac{1}{a}x^{k+1}\cos ax + \frac{k+1}{a}\int x^k \cos ax dx$$

$$= -\frac{1}{a}x^{k+1}\cos ax + \frac{k+1}{a}\left[\frac{1}{a}x^k \sin ax - \int \frac{1}{a}kx^{k-1}\sin ax dx\right]$$

$$= -\frac{1}{a}x^{k+1}\cos ax + \frac{k+1}{a^2}x^k \sin ax - \frac{k(k+1)}{a^2}\int x^{k-1}\sin ax dx$$

$$= -\frac{1}{a}x^{k+1}\cos ax + \frac{k+1}{a^2}x^k \sin ax$$

$$-\frac{k(k+1)}{a^2}\sum_{r=0}^{k-1} {}^{k-1}P_r x^{k-1-r} \frac{1}{a^{2r+1}} \frac{d^r}{dx^r}(-\cos ax)$$

$$= \sum_{r=0}^{k+1} {}^{k+1}P_r x^{k+1-r} \frac{1}{a^{2r+1}} \frac{d^r}{dx^r}(-\cos ax).$$

Since $P(k+1)$ is true, also true for all values of $n \in \mathbb{N}$. □

Theorem 3.2. *If* $n \in \mathbb{N}$ *and* $a \neq 0,$ *then*

$$\int x^n \cos ax dx = \sum_{r=0}^{n} {}^{n}P_r x^{n-r} \frac{1}{a^{2r+1}} \frac{d^r}{dx^r}(\sin ax). \tag{6}$$

Proof. See Theorem 3.1. □

Theorem 3.3. *If* $n \in \mathbb{N}$ *and* $a \neq 0,$ *then*

$$\int x^n \cosh ax dx = \sum_{r=0}^{n} {}^{n}P_r x^{n-r} \frac{1}{a^{2r+1}} (-1)^r \frac{d^r}{dx^r}(\sinh ax). \tag{7}$$

Proof. (By induction) Denote equation (7) by $Q(n)$. Consider $Q(1)$, we have

$$\int x \cosh ax dx = \sum_{r=0}^{1} {}^{1}P_r x^{1-r} \frac{1}{a^{2r+1}} (-1)^r \frac{d^r}{dx^r}(\sinh ax)$$

$$= \frac{1}{a^2}(ax \sinh ax - \cosh ax) + c.$$

Hence, $Q(1)$ is true. Assuming, $Q(k)$ is true for some $k \in \mathbb{N}$, then

$$\int x^k \cosh ax dx = \sum_{r=0}^{k} {}^{k}P_r x^{k-r} \frac{1}{a^{2r+1}} (-1)^r \frac{d^r}{dx^r}(\sinh ax). \tag{8}$$

Consider $Q(k+1)$ using equation (8), we obtain the following

$$\int x^{k+1} \cosh ax = uv - \int v du$$

$$u = x^{k+1},\ du = (k+1)x^k \text{ and } dv = \cosh ax,\ v = \frac{1}{a}\sinh ax$$

$$= \frac{1}{a} x^{k+1} \sinh ax - \frac{k+1}{a} \int x^k \sinh ax dx$$

$$= \frac{1}{a} x^{k+1} \sinh ax - \frac{k+1}{a}\left[\frac{1}{a} x^k \cosh ax - \int \frac{1}{a} k x^{k-1} \cosh ax dx\right]$$

$$= \frac{1}{a} x^{k+1} \sinh ax - \frac{k+1}{a^2} x^k \cosh ax + \frac{k(k+1)}{a^2} \int x^{k-1} \cosh ax dx$$

$$= \frac{1}{a} x^{k+1} \sinh ax - \frac{k+1}{a^2} x^k \cosh ax$$

$$+ \frac{k(k+1)}{a^2} \sum_{r=0}^{k-1} {}^{k-1}P_r x^{k-1-r} \frac{1}{a^{2r+1}} (-1)^r \frac{d^r}{dx^r} (\sinh ax)$$

$$= \sum_{r=0}^{k+1} {}^{k+1}P_r x^{k+1-r} \frac{1}{a^{2r+1}} (-1)^r \frac{d^r}{dx^r} (\sinh ax).$$

Since $Q(k+1)$ is true, also true for all values of $n \in \mathbb{N}$. □

Theorem 3.4. *If* $n \in \mathbb{N}$ *and* $a \neq 0$, *then*

$$\int x^n \sinh ax dx = \sum_{r=0}^{n} {}^{n}P_r x^{n-r} \frac{1}{a^{2r+1}} (-1)^r \frac{d^r}{dx^r} (\cosh ax). \tag{9}$$

Proof. See Theorem 3.3. □

Theorem 3.5. *If* $n \in \mathbb{N}$ *and* $a \neq 0$, *then*

$$\int x^n e^{ax} dx = e^{ax} \sum_{r=0}^{n} (-1)^r \frac{1}{a^{r+1}} \frac{d^r}{dx^r} (x^n). \tag{10}$$

Proof. Denote equation (10) by $C(n)$. Consider $C(1)$, we obtain

$$\int xe^{ax} dx = e^{ax} \sum_{r=0}^{1} (-1)^r \frac{1}{a^{r+1}} \frac{d^r}{dx^r} (x) = \frac{1}{a^2} (ax - 1) + c.$$

Hence, $C(1)$ is true. Suppose, $C(k)$ is true for some $k \in \mathbb{N}$, then

$$\int x^k e^{ax} dx = e^{ax} \sum_{r=0}^{k} (-1)^r \frac{1}{a^{r+1}} \frac{d^r}{dx^r} (x^k). \tag{11}$$

Consider $C(k+1)$ using equation (11), we obtain

$$\int x^{k+1} e^{ax} = uv - \int v du$$

$$u = x^{k+1},\ du = (k+1)x^k \text{ and } dv = e^{ax},\ v = \frac{1}{a}e^{ax}$$

$$= \frac{1}{a}x^{k+1}e^{ax} - \frac{(k+1)}{a}\int e^{ax}x^k dx$$

$$= \frac{1}{a}x^{k+1}e^{ax} - \frac{(k+1)}{a}e^{ax}\sum_{r=0}^{k}(-1)^r \frac{1}{a^{r+1}}\frac{d^r}{dx^r}(x^k)$$

$$= e^{ax}\left[\frac{1}{a}x^{k+1} - \frac{(k+1)}{a}\sum_{r=0}^{k}(-1)^r \frac{1}{a^{r+1}}\frac{d^r}{dx^r}(x^k)\right]$$

$$= e^{ax}\sum_{r=0}^{k+1}(-1)^r \frac{1}{a^{r+1}}\frac{d^r}{dx^r}(x^{k+1}).$$

Hence, $C(k+1)$, also true for all values of $n \in \mathbb{N}$. □

Theorem 3.6. *If* $n \in \mathbb{N}$, *then*

$$\int x^n \ln x dx = \frac{x^{n+1}}{(n+1)^2}[(n+1)\ln x - 1]. \tag{12}$$

Proof. Let equation (12) be $D(n)$. Consider $D(1)$, we have

$$\int x \ln x dx = \frac{x^2}{4}(2\ln x - 1) + c.$$

Hence, $D(1)$ is true. Suppose that $D(k)$ is true for some $k \in \mathbb{N}$,

$$\int x^k \ln x = \frac{x^{k+1}}{(n+1)^2}[(n+1)\ln x - 1]. \tag{13}$$

Consider $D(k+1)$ using equation (13), we obtain the following

$$\int x^{k+1}\ln x dx = uv - \int v du$$

$$u = \ln x,\ v = \frac{1}{x} \text{ and } dv = x^{k+1},\ v = \frac{1}{k+2}x^{k+2}dx$$

$$= \frac{1}{k+2} x^{k+2} \ln x - \frac{1}{k+2} \int x^{k+1} dx$$

$$= \frac{1}{k+2} x^{k+2} \ln x - \frac{1}{(k+2)^2} x^{k+2}$$

$$= \frac{1}{(k+2)^2} x^{k+2} [(k+2) \ln x - 1].$$

Since $D(k+1)$ is true, also true for all values of $n \in \mathbb{N}$. □

4. Applications

Example 4.1. Determine the total charge entering a terminal between $t = 1s$ and $t = 2s$ if the current passing the terminal is $i = (3t^2 - t)e^t$ Ampere.

Solution. The total charge entering terminal is given by

$$Q = \int_{t_0}^{t_1} i(t) dt$$

$$= \int_1^2 (3t^2 - t) e^t \, dt$$

$$= \int_1^2 3t^2 e^t dt - \int_1^2 t e^t dt.$$

By Theorem 3.5, we obtain

$$Q = 3(t^2 - 2t + 2)e^t - (t-1)e^t |_1^2$$

$$= 5e^2 - 3e$$

$$= 28.7904350092761 \text{ Couloumbs.}$$

Example 4.2. Evaluate $\int x^2 \sin 3x dx.$

Solution. Instead of using equation (1), we can apply Theorem 3.1 directly by taking $n = 2$ as follows:

$$\int x^n \sin ax dx = \sum_{r=0}^{n} {}^n P_r x^{n-r} \frac{1}{a^{2r+1}} \frac{d^r}{dx^r}(-\cos ax)$$

$$\int x^2 \sin 3x dx = \sum_{r=0}^{2} {}^2 P_r x^{2-r} \frac{1}{a^{2r+1}} \frac{d^r}{dx^r}(-\cos 3x)$$

$$= \frac{1}{3} x^2(-\cos 3x) + 2x \frac{1}{3^3} \frac{d}{dx}(-\cos 3x) + 2\frac{1}{3^5} \frac{d^2}{dx^2}(-\cos 3x)$$

$$= -\frac{1}{3} x^2 \cos 3x + \frac{2}{9} x \sin 3x + \frac{2}{27} \cos 3x + c.$$

You can solve the above problem using equation (1) and compare the results.

Example 4.3. Find the energy delivered to an element at $t = 3s$ if the current entering its positive terminal is $i = 5\cos 60\pi t$ Ampere and the voltage across its terminals is $3t^3$.

Solution. Using energy formula and Theorem 3.2, we obtain

$$E = \int_{t_0}^{t_1} IV dt$$

$$= \int_0^3 15t^3 \cos(60\pi t)\, dt$$

$$= \left. \frac{(1800\pi^2 t^2 - 1)\cos(60\pi t) + 60(600\pi^3 t^3 - \pi t)\sin(60\pi t)}{144000\pi^4} \right|_0^3$$

$$= 0.0113986 \text{ Joules}.$$

Example 4.4. Integrate $\int x^2 \cos 3x dx.$

Solution. Theorem 3.2 can be applied directly by taking $n = 2$ and $a = 3,$ we obtain

$$\int x^n \cos ax dx = \sum_{r=0}^{n} {}^n P_r x^{n-r} \frac{1}{a^{2r+1}} \frac{d^r}{dx^r}(\sin ax)$$

$$\int x^2 \cos 3x dx = \sum_{r=0}^{2} {}^2P_r x^{2-r} \frac{1}{a^{2r+1}} \frac{d^r}{dx^r}(\sin 3x)$$

$$= \frac{1}{3}x^2 \sin 3x + 2x\frac{1}{3^3}\frac{d}{dx}(\sin 3x) + 2\frac{1}{3^5}\frac{d^2}{dx^2}(\sin 3x)$$

$$= x^2\frac{1}{3}(\sin 3x) + 2x\frac{1}{3^3}(3\cos 3x) + 2\frac{1}{3^5}(-9\sin 3x)$$

$$= \frac{1}{3}x^2 \sin 3x + \frac{2}{9}x\cos 3x - \frac{2}{27}\sin 3x + c.$$

Example 4.5. Evaluate $\int x^3 e^{2x} dx.$

Solution. Using Theorem 3.5 with $n = 3$ and $a = 3,$ we obtain

$$\int x^n e^{ax} dx = e^{ax}\sum_{r=0}^{n}(-1)^r \frac{1}{a^{r+1}}\frac{d^r}{dx^r}(x^n)$$

$$\int x^3 e^{2x} dx = e^{2x}\sum_{r=0}^{3}(-1)^r \frac{1}{a^{r+1}}\frac{d^r}{dx^r}(x^3)$$

$$= e^{2x}\left(\frac{1}{2}x^3 - \frac{3}{4}x^2 + \frac{3}{4}x - \frac{3}{8}\right)$$

$$= \frac{e^{2x}}{8}(4x^3 - 6x^2 + 6x - 3) + c.$$

Example 4.6. Integrate $\int_0^1 x^2(1-x)\cos m\pi x dx.$

Solution. Let

$$a_m = \int_0^1 x^2(1-x)\cos m\pi x dx$$

$$= \int_0^1 (x^2 \cos m\pi x - x^3 \cos m\pi x) dx$$

$$= 2\left[\sum_{r=0}^{2} {}^{2}P_r x^{2-r} \frac{1}{(m\pi)^{2r+1}} (-1)^r \frac{d^r}{dx^r} (\sin m\pi x)\right.$$

$$\left.- \sum_{r=0}^{3} {}^{3}P_r x^{3-r} \frac{1}{(m\pi)^{2r+1}} (-1)^r \frac{d^r}{dx^r} (\sin m\pi x)\right]$$

$$= 2\left[\frac{x^2}{m\pi} \sin m\pi x + \frac{2x}{(m\pi)^2} \cos m\pi x - \frac{2}{(m\pi)^3} \sin m\pi x\right]\Bigg|_0^1$$

$$- 2\left[\frac{x^3}{m\pi} \sin m\pi x + \frac{3x^2}{(m\pi)^2} \cos m\pi x - \frac{6x}{(m\pi)^3} \sin m\pi x - \frac{6}{(m\pi)^4} \cos m\pi x\right]\Bigg|_0^1$$

$$= 2\left(\frac{2(-1)^m}{(m\pi)^2}\right) - 2\left(-\frac{3(-1)^m}{(m\pi)^2} + \frac{6(-1)^m}{(m\pi)^4} - \frac{6}{(m\pi)^4}\right)$$

$$= \frac{2(-1)^{m+1}}{(m\pi)^2} + \frac{12}{(m\pi)^4} ((-1)^m - 1).$$

We can see clearly from the above examples that the method of summation is faster and reliable than using equation (1).

5. Conclusion

The research shows the direct application of the differential summation formula in solving integration by parts (IBP). Students can solve problems on integration by parts without any basic knowledge in integration but in differentiation. The formula $P(x = r) = {}^{n}C_r p^r q^{n-r}$ can be used to evaluate expansion, find coefficient of any powers and even the constant term of an expansion. These theorems can also be used to find the coefficient of any power and the constant term in any integration by parts without necessarily integrate completely.

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