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A Certain Subclass of Multivalent Functions Associated with Borel Distribution Series

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Abstract

In this paper, we determine the necessary and sufficient conditions for the power series $f(z)$ whose coefficients are probabilities of the Borel distribution to be in the family $J(p,\lambda,\alpha,\beta,\gamma)$ of analytic functions which defined in the open unit disk. We derive a number of important geometric properties, such as, coefficient estimates, integral representation, radii of starlikeness and convexity. Also we discuss the extreme points and neighborhood property for functions belongs to this family.

1. Introduction

Indicate by A_p the family of all functions f of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic and multivalent in the open unit disk $U = \{z \in \mathbb{C}: |z| < 1\}$.

Also, let W_p denote the subfamily of A_p consisting of functions of the form:

$$f(z) = z^p - \sum_{n=p+1}^{\infty} a_n z^n \ (a_n \geq 0, p \in N). \tag{1.2}$$

The function $f \in W_p$ is said to be starlike of order $\rho(0 \leq \rho < p)$ if it satisfies the

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condition:

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \rho \qquad (z \in U),$$

A function $f \in W_p$ is said to be convex of order $\rho(0 \leq \rho < p)$ if it satisfies the condition

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \rho \qquad (z \in U).$$

A function $f \in W_p$ is said to be close-to-convex of order $\rho(0 \leq \rho < p)$ if it satisfies the condition

$$Re\left\{\frac{f'(z)}{z^{p-1}}\right\} > \rho \quad (z \epsilon U).$$

Let the function f and g be analytic in U. We say that the function f is subordinate to g, if there exists a Schwarz function w analytic in U with $w(0)=0$ and $|w(z)|< 1$ $(z \in U)$ such that $f(z)=g(w(z))$. This subordinate is denoted by $f \prec g$ or $f(z) \prec g(z)(z \in U)$. It is well known that (see [6]), if the function g is univalent in U, then $f \prec g$ if and only if $f(0)=g(0)$ and $f(U) \subset g(U)$.

Denote by $S^*(\alpha)$ and $C(\alpha)$ the families of starlike and convex functions of orderρ, respectively. These families were introduced and studied by Silverman [9].

The elementary distributions such as the Poisson, the Pascal, the Logarithmic, the Binomial have been partially studied in the Geometric Function Theory from a theoretical point of view (see [1,3,4,7,8,10,11]).

A discrete random variable x is said to have a Borel distribution if it takes the values $1,2,3,\dots$ with the probabilities $\frac{e^{-\lambda}}{1!}, \frac{2\lambda e^{-2\lambda}}{2!}, \frac{9\lambda^2 e^{-3\lambda}}{3!}, \dots$ respectively, where λ is called the parameter.

Very recently, Wanas and Khuttar [12] introduced the Borel distribution (BD) whose probability mass function is

$$P(x=r)=\frac{(\lambda r)^{r-1}e^{-\lambda r}}{r!}, \quad r=1,2,3,\dots .$$

Wanas and Khuttar [12] introduced a series whose coefficients are probabilities of the Borel distribution (BD)

$$\mathcal{N}_p(\lambda; z) = z^p - \sum_{n=p+1}^{\infty} \frac{\left(\lambda(n-p)\right)^{n-p-1} e^{-\lambda(n-p)}}{(n-p)!} z^n = z^p - \sum_{n=p+1}^{\infty} \Phi_{n,p}(\lambda)\, z^n,$$

where $0 < \lambda \leq 1$ and

$$\Phi_{n,p}(\lambda) = \frac{\left(\lambda(n-p)\right)^{n-p-1} e^{-\lambda(n-p)}}{(n-p)!}.$$

We consider a linear operator $D(p,\lambda)f : W_p \longrightarrow W_p$ defined by the convolution or Hadamard product

$$D(p,\lambda)f(z) = \mathcal{N}_p(\lambda; z) * f(z) = z^p - \sum_{n=p+1}^{\infty} \frac{\left(\lambda(n-p)\right)^{n-p-1} e^{-\lambda(n-p)}}{(n-p)!} a_n z^n,$$

where $a_n \geq 0, 0 < \lambda \leq 1$ and $z \in U$.

We now recall the following Lemmas that will be used to prove our main results.

Lemma 1.1 [5]. *If f and g are analytic in U with $f \prec g$, then*

$$\int_0^{2\pi} \left|f(re^{i\theta})\right|^{\mu} d\theta \leq \int_0^{2\pi} \left|g(re^{i\theta})\right|^{\mu} d\theta,$$

where $\mu > 0, z = re^{i\theta}$ and $(0 < r < 1)$.

Lemma 1.2 [2]. *Let $\alpha \geq 0$. Then $Re(w) > \alpha$ if and only if $|w - (p+\alpha)| < |w + (p-\alpha)|$, where $w$be any complex number.*

2. Main Results

We begin this section by defining the family $J(p,\lambda,\alpha,\beta,\gamma)$ as follows:

Definition 2.1. A function f of the form (1.2) is said to be in the class $J(p,\lambda,\alpha,\beta,\gamma)$ if satisfies the following condition:

$$Re\left\{\frac{z(D(p,\lambda)f(z))' + \gamma z^2\left(D(p,\lambda)f(z)\right)''}{\begin{array}{c}\gamma z\left[\left(D(p,\lambda)f(z)\right)' + \beta z\left(D(p,\lambda,)f(z)\right)''\right] \\ +(1-\gamma)\left[\beta z\left(D(p,\lambda)f(z)\right)' + (1-\beta)D(p,\lambda)f(z)\right]\end{array}}\right\} > \alpha, \qquad (2.1)$$

where $0 \leq \alpha < 1,\ 0 \leq \beta < 1,\ 0 \leq \gamma < 1$ and $0 < \lambda \leq 1$.

Theorem 2.1. *Let* $f \in W_p$, *then* $f \in J(p, \lambda, \alpha, \beta, \gamma)$ *if and only if*

$$\sum_{n=p+1}^{\infty} (\gamma(n-1)+1)(n-\alpha-(n-1)\alpha\beta)\Phi_{n,p}(\lambda)a_n$$
$$\leq (\gamma(p-1)+1)(p-\alpha-(p-1)\alpha\beta), \tag{2.2}$$

where $0 < \lambda \leq 1,\ 0 \leq \alpha < 1,\ 0 < \beta \leq 1,\ 0 \leq \gamma \leq 1$ *and* $p \in N$.

The result is sharp for the function

$$f(z) = z^p - \frac{(\gamma(p-1)+1)(p-\alpha-(p-1)\alpha\beta)}{(\gamma(n-1)+1)(n-\alpha-(n-1)\alpha\beta)\Phi_{n,p}(\lambda)} z^n\ ,$$
$$(n \geq p+1,\ \ p \in N). \tag{2.3}$$

Proof. Assume that $f \in J(p, \lambda, \alpha, \beta, \gamma)$, so we have

$$Re\left\{\frac{z(D(p,\lambda)f(z))' + \gamma z^2\,(D(p,\lambda)f(z))''}{\begin{array}{c}\gamma z\left[\left(D(p,\lambda)f(z)\right)' + \beta z\left(D(p,\lambda,)f(z)\right)''\right]\\ +(1-\gamma)\left[\beta z\left(D(p,\lambda)f(z)\right)' + (1-\beta)D(p,\lambda)f(z)\right]\end{array}}\right\} > \alpha.$$

Then

$$Re\left\{\frac{pz^p - \sum_{n=p+1}^{\infty}\Phi_{n,p}(\lambda)na_nz^n + \gamma p(p-1)z^p - \sum_{n=p+1}^{\infty}\Phi_{n,p}(\lambda)\gamma n(n-1)a_nz^n}{\begin{array}{c}\gamma pz^p - \sum_{n=p+1}^{\infty}\Phi_{n,p}(\lambda)\,\gamma na_nz^n + \gamma\beta p(p-1)z^p - \sum_{n=p+1}^{\infty}\Phi_{n,p}\,(\lambda)\gamma\beta n(n-1)a_nz^n\\ +(1-\gamma)\beta pz^p - \sum_{n=p+1}^{\infty}\Phi_{n,p}(\lambda)(1-\gamma)\beta na_nz^n + (1-\gamma)(1-\beta)\left(z^p - \sum_{n=p+1}^{\infty}\Phi_{n,p}(\lambda)a_nz^n\right)\end{array}}\right\} > \alpha.$$

Or equivalently

$$Re\left\{\frac{(\gamma(p-1)+1)(p-\alpha-(p-1)\alpha\beta)z^p - \sum_{n=p+1}^{\infty}(\gamma(n-1)+1)(n-\alpha-(n-1)\alpha\beta)\Phi_{n,p}(\lambda)a_nz^n}{\alpha\left(1+\gamma(p-1)\right)\left(1+\beta(p-1)\right)z^p - \sum_{n=p+1}^{\infty}\Phi_{n,p}\,(\lambda)\alpha\left(1+\gamma(n-1)\right)\left(1+\beta(n-1)\right)a_nz^n}\right\} > 0.$$

This inequality is correct for a $z \in U$. Letting $z \to 1^-$ yields

$$Re\left\{(\gamma(p-1)+1)(p-\alpha-(p-1)\alpha\beta)z^p - \sum_{n=p+1}^{\infty}(\gamma(n-1)+1)(n-\alpha-(n-1)\alpha\beta)\Phi_{n,p}(\lambda)a_n\right\} > 0.$$

Therefore

$$\sum_{n=p+1}^{\infty}(\gamma(n-1)+1)(n-\alpha-(n-1)\alpha\beta)\Phi_{n,p}(\lambda)a_n$$
$$\leq(\gamma(p-1)+1)(p-\alpha-(p-1)\alpha\beta).$$

Conversely, let (2.2) hold. We will prove that (2.1) is correct and then $f \in J(p,\lambda,\alpha,\beta,\gamma)$. By Lemma 1.2, we put

$$w=\frac{z(D(p,\lambda)f(z))'+\gamma z^2\,(D(p,\lambda)f(z))''}{\gamma z\left[\big(D(p,\lambda)f(z)\big)'+\beta z\big(D(p,\lambda,)f(z)\big)''\right]+(1-\gamma)\left[\beta z\big(D(p,\lambda)f(z)\big)'+(1-\beta)D(p,\lambda)f(z)\right]}.$$

Or show that

$$T=\frac{1}{|N(z)|}\left|z\big(D(p,\lambda)f(z)\big)'+\gamma z^2\big(D(p,\lambda)f(z)\big)''-(p+\alpha)z\big(D(p,\lambda)f(z)\big)'-(p+\alpha)\gamma\,(D(p,\lambda)f(z))''\right|=Q,$$

where

$$N(z)=\gamma z\left[\big(D(p,\lambda)f(z)\big)'+\beta z\big(D(p,\lambda,)f(z)\big)''\right]+(1-\gamma)\left[\beta z\big(D(p,\lambda)f(z)\big)'+(1-\beta)\big(D(p,\lambda)f(z)\big)\right]$$

and it is easy to verify that $Q-T>0$.

And so the proof is complete.

Corollary 2.1. *If* $f\in J(p,\lambda,\alpha,\beta,\gamma)$, *then*

$$a_n\leq\frac{(\gamma(p-1)+1)(p-\alpha-(p-1)\alpha\beta)}{(\gamma(n-1)+1)(n-\alpha-(n-1)\alpha\beta)\Phi_{n,p}(\lambda)},\ (n\geq p+1, p\in N).$$

Theorem2.2. *Let a function* $f\in J(p,\lambda,\alpha,\beta,\gamma)$. *Then* f *is p-valently close-to-convex of order* ρ $(0\leq\rho<p)$ *in the disk* $|z|<R_1$, *where*

$$R_1=\inf_n\left\{\frac{(p-\rho)(\gamma(n-1)+1)(n-\alpha-(n-1)\alpha\beta)\Phi_{n,p}(\lambda)}{(\gamma(p-1)+1)(p-\alpha-(p-1)\alpha\beta)}\right\}^{\frac{1}{n-p}},$$
$$(n\geq p+1;\ p\in N).$$

The result is sharp, with the extermal function f *given by* (2.3).

Proof. It is sufficient to show that

$$\left|\frac{f'(z)}{z^{p-1}}-p\right| \leq p-\rho \quad (0 \leq \rho < p),$$

for $|z| < R_1$, we have that

$$\left|\frac{f'(z)}{z^{p-1}}-p\right| \leq \sum_{n=p+1}^{\infty} \Phi_{n,p}(\lambda) n a_n |z|^{n-p} .$$

Thus

$$\left|\frac{f'(z)}{z^{p-1}}-p\right| \leq p-\rho,$$

if

$$\sum_{n=p+1}^{\infty} \frac{\Phi_{n,p}(\lambda) a_n |z|^{n-p}}{p-\rho} \leq 1. \tag{2.4}$$

Hence, by Theorem 2.1, (2.4) will be true if

$$\frac{1}{p-\rho}|z|^{n-p} \leq \frac{(\gamma(n-1)+1)(n-\alpha-(n-1)\alpha\beta)\Phi_{n,p}(\lambda)}{(\gamma(p-1)+1)(p-\alpha-(p-1)\alpha\beta)},$$

and hence

$$|z| \leq \left\{\frac{(p-\rho)(\gamma(n-1)+1)(n-\alpha-(n-1)\alpha\beta)\Phi_{n,p}(\lambda)}{(\gamma(p-1)+1)(p-\alpha-(p-1)\alpha\beta)}\right\}^{\frac{1}{n-p}},$$

$$(n \geq p+1;\ p \in N).$$

The result is sharp for the function f given by (2.3).

Theorem 2.3. *Let $f \in J(p,\lambda,\alpha,\beta,\gamma)$. Then f is p-valently starlike of order ρ $(0 \leq \rho < p)$ in the disk $|z| < R_2$, where*

$$R_2 = \inf_n \left\{\frac{(p-\rho)(\gamma(n-1)+1)(n-\alpha-(n-1)\alpha\beta)\Phi_{n,p}(\lambda)}{(n-\rho)(\gamma(p-1)+1)(p-\alpha-(p-1)\alpha\beta)}\right\}^{\frac{1}{n-p}},$$

$$(n \geq p+1;\ p \in N).$$

The result is sharp for the function f given by (2.3).

Proof. It is sufficient to show that

$$\left|\frac{zf'(z)}{f(z)} - p\right| \le p - \rho \quad (0 \le \rho < p),$$

for $|z| < R_2$, we have

$$\left|\frac{zf'(z)}{f(z)} - p\right| \le \frac{\sum_{n=p+1}^{\infty} \Phi_{n,p}(\lambda)(n-p)a_n|z|^{n-p}}{1 - \sum_{n=p+1}^{\infty} a_k|z|^{n-p}}.$$

Thus

$$\left|\frac{zf'(z)}{f(z)} - p\right| \le p - \rho,$$

if

$$\sum_{n=p+1}^{\infty} \frac{(n-\rho)}{(p-\rho)} \Phi_{n,p}(\lambda)a_n|z|^{n-p} \le 1. \tag{2.5}$$

Hence, by Theorem 2.1, (2.5) will be true if

$$\frac{(n-\rho)}{(p-\rho)}|z|^{n-p} \le \frac{(\gamma(n-1)+1)(n-\alpha-(n-1)\alpha\beta)}{(\gamma(p-1)+1)(p-\alpha-(p-1)\alpha\beta)},$$

and hence

$$|z| \le \left\{\frac{(p-\rho)(\gamma(n-1)+1)(n-\alpha-(n-1)\alpha\beta)}{(n-\rho)(\gamma(p-1)+1)(p-\alpha-(p-1)\alpha\beta)}\right\}^{\frac{1}{n-p}}, \quad (n \ge p+1;\ p \in N).$$

Setting $|z| = R_2$, we get the desired result.

Theorem 2.4. *Let $f \in J(p, \lambda, \alpha, \beta, \gamma)$. Then f is p-valently convex of order ρ $(0 \le \rho < p)$ in the disk $|z| < R_3$, where*

$$R_3 = \inf_n \left\{\frac{(p-\rho)(\gamma(n-1)+1)(n-\alpha-(n-1)\alpha\beta)}{(n-\rho)(\gamma(p-1)+1)(p-\alpha-(p-1)\alpha\beta)}\right\}^{\frac{1}{n-p}}, \quad (n \ge p+1;\ p \in N).$$

The result is sharp with the extermal function f given by (2.3).

Proof. It is sufficient to show that

$$\left|1 + \frac{zf''(z)}{f'(z)} - p\right| \le p - \rho \quad (0 \le \rho < p),$$

for $|z| < R_3$, we have

$$\left|1 + \frac{zf''(z)}{f'(z)} - p\right| \leq \frac{\sum_{n=p+1}^{\infty} \Phi_{n,p}(\lambda)\, n\,(n-p) a_n |z|^{n-p}}{p - \sum_{n=p+1}^{\infty} \Phi_{n,p}(\lambda) a_n |z|^{n-p}}.$$

Thus

$$\left|1 + \frac{zf''(z)}{f'(z)} - p\right| \leq p - \rho,$$

if

$$\sum_{n=p+1}^{\infty} \frac{n(n-\rho)}{p(p-\rho)} \Phi_{n,p}(\lambda) a_n |z|^{n-p} \leq 1. \tag{2.6}$$

Hence, by Theorem 2.1, (2.6) will be true if

$$\frac{n(n-\rho)}{p(p-\rho)} |z|^{n-p} \leq \frac{(\gamma(n-1)+1)(n-\alpha-(n-1)\alpha\beta)}{(\gamma(p-1)+1)(p-\alpha-(p-1)\alpha\beta)},$$

and hence

$$|z| \leq \left\{\frac{p(p-\rho)(\gamma(n-1)+1)(n-\alpha-(n-1)\alpha\beta)}{n(n-\rho)(\gamma(p-1)+1)(p-\alpha-(p-1)\alpha\beta)}\right\}^{\frac{1}{n-p}}, \quad (n \geq p+1;\ p \in N).$$

Setting $|z| = R_3$, we get the desired result.

Theorem 2.5. *Let the functions f_r defined by*

$$f_r(z) = z^p - \sum_{n=p+1}^{\infty} a_{n,r} z^n, \quad \left(a_{n,r} \geq 0,\ \ n \geq p+1,\ \ p \in N,\ \ r = 1,2,\dots,l\right), \tag{2.7}$$

be in the class $J(p,\lambda,\alpha,\beta,\gamma)$ for every $r = 1,2,\dots,l$.

Then the function h_1 defined by

$$h_1(z) = z^p - \sum_{n=p+1}^{\infty} e_n z^n, \quad (e_n \geq 0,\ \ n \geq p+1,\ \ p \in N),$$

also belongs to the class $J(p,\lambda,\alpha,\beta,\gamma)$ where

$$e_n = \frac{1}{l} \sum_{r=1}^{l} a_{n,r}, \quad (n \geq p+1,\ \ p \in N).$$

Proof. Since $f_r \in J(p,\lambda,\alpha,\beta,\gamma)$ it follows from Theorem 2.1 that

$$\sum_{n=p+1}^{\infty}(\gamma(n-1)+1)(n-\alpha-(n-1)\alpha\beta)\Phi_{n,p}(\lambda)a_{n,r}$$
$$\leq(\gamma(p-1)+1)(p-\alpha-(p-1)\alpha\beta),$$

for every $r=1,2,\dots,l$. Hence

$$\sum_{n=p+1}^{\infty}(\gamma(n-1)+1)(n-\alpha-(n-1)\alpha\beta)\Phi_{n,p}e_n$$

$$=\sum_{n=p+1}^{\infty}(\gamma(n-1)+1)(n-\alpha-(n-1)\alpha\beta)\Phi_{n,p}\left(\frac{1}{l}\sum_{r=1}^{l}a_{n,r}\right)$$

$$=\frac{1}{l}\sum_{r=1}^{l}\left(\sum_{n=p+1}^{\infty}(\gamma(n-1)+1)(n-\alpha-(n-1)\alpha\beta)\Phi_{n,p}(\lambda)a_{n,r}\right)$$

$$\leq(\gamma(p-1)+1)(p-\alpha-(p-1)\alpha\beta).$$

By Theorem 2.1, it follows that $h_1\in J(p,\lambda,\alpha,\beta,\gamma)$.

Theorem 2.6. *Let the functions f_r defined by* (2.7) *be in the class $J\in(p,\lambda,\alpha,\beta,\gamma)$ for every $r=1,2,\dots,l$. Then the function h_2 defined by*

$$h_2(z)=\sum_{r=1}^{l}c_rf_r(z)$$

is also in the class $J\in(p,\lambda,\alpha,\beta,\gamma)$ where

$$\sum_{r=1}^{l}c_r=1,\ \ (c_r\geq 0).$$

Proof. By Theorem 2.1 for every $r=1,2,\dots,l$, we have

$$\sum_{n=p+1}^{\infty}(\gamma(n-1)+1)(n-\alpha-(n-1)\alpha\beta)\Phi_{n,p}(\lambda)a_{n,r}$$

$$\leq(\gamma(p-1)+1)(p-\alpha-(p-1)\alpha\beta).$$

But

$$h_2(z) = \sum_{r=1}^{l} c_r f_r(z) = \sum_{r=1}^{l} c_r \left(z^p - \sum_{n=p+1}^{\infty} a_{n,r} z^n \right) = z^p - \sum_{n=p+1}^{\infty} \left(\sum_{r=1}^{l} c_r a_{n,r} \right) z^n.$$

Therefore

$$\sum_{n=p+1}^{\infty} (\gamma(n-1)+1)(n-\alpha-(n-1)\alpha\beta)\Phi_{n,p}(\lambda) \left(\sum_{r=1}^{l} c_r a_{n,r} \right)$$

$$= \sum_{r=1}^{l} c_r \left(\sum_{n=p+1}^{\infty} (\gamma(n-1)+1)(n-\alpha-(n-1)\alpha\beta)\Phi_{n,p}(\lambda) a_{n,r} \right)$$

$$\leq \sum_{r=1}^{l} c_r \, (\gamma(p-1)+1)(p-\alpha-(p-1)\alpha\beta)$$

$$= (\gamma(p-1)+1)(p-\alpha-(p-1)\alpha\beta)$$

and the proof is complete.

Theorem 2.7. *Let* $\tau > 0$. *If* $f \in J(p, \lambda, \alpha, \beta, \gamma)$ *and suppose that* f_s *is defined by*

$$f_s(z) = z^p - \frac{(\gamma(p-1)+1)(p-\alpha-(p-1)\alpha\beta)}{(\gamma(s-1)+1)(s-\alpha-(s-1)\alpha\beta)\Phi_{s,p}(\lambda)} z^s,$$

$$(s \geq p+1;\ p \in N).$$

If there exists an analytic function w *defined by*

$$\big(w(z)\big)^{s-p} = \frac{(\gamma(s-1)+1)(s-\alpha-(s-1)\alpha\beta)\Phi_{s,p}(\lambda)}{(\gamma(p-1)+1)(p-\alpha-(p-1)\alpha\beta)} \sum_{n=p+1}^{\infty} a_n z^{n-p}.$$

Then, for $z = re^{i\theta}$ *and* $(0 < r < 1)$,

$$\int_0^{2\pi} |f(z)|^\tau d\theta \leq \int_0^{2\pi} |f_s(z)|^\tau d\theta, \ \ (\tau > 0). \tag{2.8}$$

Proof. We show that

$$\int_0^{2\pi} \left| 1 - \sum_{n=p+1}^{\infty} a_n z^{n-p} \right|^\tau d\theta$$

$$\leq \int_0^{2\pi} \left|1 - \frac{(\gamma(p-1)+1)(p-\alpha-(p-1)\alpha\beta)}{(\gamma(s-1)+1)(s-\alpha-(s-1)\alpha\beta)\Phi_{s,p}(\lambda)} z^{s-p}\right|^{\tau} d\theta.$$

By applying Lemma 1.1, it suffices to show that

$$1 - \sum_{n=p+1}^{\infty} a_n z^{n-p} \prec 1 - \frac{(\gamma(p-1)+1)(p-\alpha-(p-1)\alpha\beta)}{(\gamma(s-1)+1)(s-\alpha-(s-1)\alpha\beta)\Phi_{s,p}(\lambda)} z^{s-p}.$$

Set

$$1 - \sum_{n=p+1}^{\infty} a_n z^{n-p} = 1 - \frac{(\gamma(p-1)+1)(p-\alpha-(p-1)\alpha\beta)}{(\gamma(s-1)+1)(s-\alpha-(s-1)\alpha\beta)\Phi_{s,p}(\lambda)} \big(w(z)\big)^{s-p}.$$

We find that

$$\big(w(z)\big)^{s-p} = \frac{(\gamma(s-1)+1)(s-\alpha-(s-1)\alpha\beta)\Phi_{s,p}(\lambda)}{(\gamma(p-1)+1)(p-\alpha-(p-1)\alpha\beta)} \sum_{n=p+1}^{\infty} a_n z^{n-p},$$

which readily yield $w(0) = 0$.

Furthermore using (2.2), we obtain

$$|w(z)|^{s-p} = \left|\frac{(\gamma(s-1)+1)(s-\alpha-(s-1)\alpha\beta)\Phi_{s,p}(\lambda)}{(\gamma(p-1)+1)(p-\alpha-(p-1)\alpha\beta)} \sum_{n=p+1}^{\infty} a_n z^{n-p}\right|$$

$$\leq |z| \left|\sum_{n=p+1}^{\infty} \frac{(\gamma(n-1)+1)(n-\alpha-(n-1)\alpha\beta)\Phi_{n,p}(\lambda)}{(\gamma(p-1)+1)(p-\alpha-(p-1)\alpha\beta)} a_n\right|$$

$$\leq |z| < 1.$$

Next, the proof for the first derivative

Theorem 2.8. *Let* $\tau > 0$. *If* $f \in J(p,\lambda,\alpha,\beta,\gamma)$ *and*

$$f_s(z) = z^p - \frac{(\gamma(p-1)+1)(p-\alpha-(p-1)\alpha\beta)}{(\gamma(s-1)+1)(s-\alpha-(s-1)\alpha\beta)\Phi_{s,p}(\lambda)} z^s,$$

$$(s \geq p+1;\ p \in N).$$

Then for $z = re^{i\theta}$ *and* $(0 < r < 1)$,

$$\int_0^{2\pi} |f'(z)|^\tau d\theta \leq \int_0^{2\pi} |f_s'(z)|^\tau d\theta, \ \ (\tau > 0). \tag{2.9}$$

Proof. It is sufficient to show that

$$1 - \sum_{n=p+1}^{\infty} \frac{n}{p} a_n z^{n-p} \prec 1 - \frac{s(\gamma(p-1)+1)(p-\alpha-(p-1)\alpha\beta)}{p(\gamma(s-1)+1)(s-\alpha-(s-1)\alpha\beta)\Phi_{s,p}(\lambda)} z^{s-p}.$$

This follows because

$$|w(z)|^{s-p} = \left| \frac{p(\gamma(s-1)+1)(s-\alpha-(s-1)\alpha\beta)\Phi_{s,p}(\lambda)}{s(\gamma(p-1)+1)(p-\alpha-(p-1)\alpha\beta)} \sum_{n=p+1}^{\infty} \frac{n}{p} a_n z^{n-p} \right|$$

$$\leq |z| \left| \sum_{n=p+1}^{\infty} \frac{(\gamma(n-1)+1)(n-\alpha-(n-1)\alpha\beta)\Phi_{n,p}(\lambda)}{(\gamma(p-1)+1)(p-\alpha-(p-1)\alpha\beta)} a_n \right|$$

$$\leq |z| < 1.$$

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