Earthline Journal of Mathematical Sciences
E-ISSN: 2581-8147
Volume 10, Number 2, 2022, Pages 289-304
https://doi.org/10.34198/ejms.10222.289304

Results of Semigroup of Linear Operators Generating a Regular Weak*-continuous Semigroup

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Abstract

This paper present results of ω-order preserving partial contraction mapping generating a regular weak*-continuous semigroup. We consider a semigroup on a Banach space X and $B : X^{\odot} \to X^*$ is bounded, then the intertwining formula was used to define a semigroup $T^B(t)$ on X^* which extends the perturbed semigroup $T_0^B(t)$ on $X^{\odot}$ using the variation of constants formula. We also investigated a certain class of weak*-continuous semigroups on dual space X^* which contains both adjoint semigroups and their perturbations by operators $B : X^{\odot} \to X^*$.

1 Introduction

A linear operator A is the infinitesimal generator of a uniformly continuous semigroup if and only if A is a bounded linear operator, and perturbation theory comprises methods for finding an approximate solution to a problem. In perturbation theory, the solution is expressed as a power series in a

Received: June 1, 2022; Accepted: July 6, 2022
2020 Mathematics Subject Classification: 06F15, 06F05, 20M05.
Keywords and phrases: ω-OCP_n, perturbation, C_0-semigroup, regular weak*-continuous semigroup.
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small parameter ε. The first term is the known solution to the solvable problem. Successive terms in the series at higher powers of ε usually become smaller. Hille-Yosida theorem characterizes the generators of strongly continuous one-parameter semigroups of linear operators on Banach spaces. Assume $Fav(T(t))$ is a Favard class of semigroup, $X_n \subseteq X$ is a finite set, $\omega - OCP_n$ the ω-order preserving partial contraction mapping, M_m be a matrix, $L(X)$ be a bounded linear operator on X, P_n a partial transformation semigroup, $\rho(A)$ a resolvent set, $\sigma(A)$ a spectrum of A and $A \in \omega - OCP_n$ is a generator of C_0-semigroup. This paper consist of results of ω-order preserving partial contraction mapping generating a regular weak*-continuous semigroup.

Akinyele *et al.* [1], established some perturbation results of the infinitesimal generator in the semigroup of the linear operator, also in [2] Akinyele *et al.*, obtained some results of semigroup of linear operator in spectra theory. Balakrishnan [3], introduced an operator calculus for infinitesimal generators of semigroup. Banach [4], established and introduced the concept of Banach spaces. Batty *et al.* [5], showed some asymptotic behavior of semigroup of operators. Chill and Tomilov [6], deduced some resolvent approach to stability operator semigroup. Davies [7], introduced linear operators and their spectra. Engel and Nagel [8], presented one-parameter semigroup for linear evolution equations. Nagel *et al.* [9], identified extrapolation spaces for unbounded operators. Neerven [10], deduced some results on adjoint of semigroup of linear operators. Omosowon *et al.* [11], proved some analytic results of semigroup of linear operator with dynamic boundary conditions, and also in [12], Omosowon *et al.*, established dual Properties of ω-order Reversing Partial Contraction Mapping in Semigroup of Linear Operator. Rauf and Akinyele [13], obtained ω-order preserving partial contraction mapping and established its properties, also in [14], Rauf *et al.* introduced some results of stability and spectra properties on semigroup of linear operator. Vrabie [15], proved some results of C_0-semigroup and its applications. Yosida [16], established some results on differentiability and representation of one-parameter semigroup of linear operators.

2 Preliminaries

Definition 2.1 (C_0-semigroup) [15]
A C_0-*semigroup* is a strongly continuous one parameter semigroup of bounded linear operator on Banach space.

Definition 2.2 (ω-OCP_n) [13]
A transformation $\alpha \in P_n$ is called ω-*order preserving partial contraction mapping* if $\forall x, y \in$ Dom$\alpha : x \leq y \implies \alpha x \leq \alpha y$ and at least one of its transformation must satisfy $\alpha y = y$ such that $T(t+s) = T(t)T(s)$ whenever $t, s > 0$ and otherwise for $T(0) = I$.

Definition 2.3 (Perturbation) [1]
Let $A : D(A) \subseteq X \to X$ be the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ and consider a second operator $B : D(B) \subseteq X \to X$ such that the sum $A + B$ generates a strongly continuous semigroup $(S(t))_{t\geq 0}$. We say that A *is perturbed by operator* B or that B *is a perturbation of* A.

Definition 2.4 (Regular weak*-continuous semigroup) [10]
A weak*-continuous semigroup $T^X(t)$ on a dual Banach space X^* is called *regular* if for all $t, s > 0$ and $x^* \in X^*$ we have

$$T^X(t)\left(weak * \int_0^s T^*(\sigma)x^* d\sigma\right) = weak^* \int_0^s T^X(t+\sigma)x^* d\sigma.$$

Example 1
2×2 matrix $[M_m(\mathbb{N} \cup \{0\})]$
Suppose

$$A = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$$

and let $T(t) = e^{tA}$, then

$$e^{tA} = \begin{pmatrix} e^{2t} & e^{I} \\ e^{t} & e^{2t} \end{pmatrix}.$$

Example 2
3×3 matrix $[M_m(\mathbb{N} \cup \{0\})]$
Suppose

$$A = \begin{pmatrix} 2 & 2 & 3 \\ 2 & 2 & 2 \\ 1 & 2 & 2 \end{pmatrix}$$

and let $T(t) = e^{tA}$, then

$$e^{tA} = \begin{pmatrix} e^{2t} & e^{2t} & e^{3t} \\ e^{2t} & e^{2t} & e^{2t} \\ e^{t} & e^{2t} & e^{2t} \end{pmatrix}.$$

Example 3
3×3 matrix $[M_m(\mathbb{C})]$, we have for each $\lambda > 0$ such that $\lambda \in \rho(A)$ where $\rho(A)$ is a resolvent set on X.
Suppose we have

$$A = \begin{pmatrix} 2 & 2 & 3 \\ 2 & 2 & 2 \\ 1 & 2 & 2 \end{pmatrix}$$

and let $T(t) = e^{tA_\lambda}$, then

$$e^{tA_\lambda} = \begin{pmatrix} e^{2t\lambda} & e^{2t\lambda} & e^{3t\lambda} \\ e^{2t\lambda} & e^{2t\lambda} & e^{2t\lambda} \\ e^{t\lambda} & e^{2t\lambda} & e^{2t\lambda} \end{pmatrix}.$$

Theorem 2.1 Hille-Yoshida [15]
A linear operator $A : D(A) \subseteq X \to X$ is the infinitesimal generator for a C_0-semigroup of contraction if and only if

(i) *A is densely defined and closed,*

(ii) *$(0, +\infty) \subseteq \rho(A)$ and for each $\lambda > 0$, we have*

$$\|R(\lambda, A)\|_{L(X)} \leq \frac{1}{\lambda}. \tag{2.1}$$

3 Main Results

This section present results of semigroup of linear operator by using ω-OCP_n to generate regular weak*-continuous semigroup:

Theorem 3.1
Assume $A : D(A) \subseteq X \to X$ is the infinitesimal generator of a C_0-semigroup $\{T(t) : t \geq 0\}$ on X and $B : X^{\odot} \to X^*$ is bounded such that $A, B \in \omega - OCP_n$, then we have:

(i) *The semigroup $T^B(t)$ on X^* is regular;*

(ii) *The perturbed integrated semigroup $S^B(t)$ satisfies the variation of constants formula*

$$S^B(t)x = S(t)x + \lim_{\lambda} \int_0^t S^B(t-s)BT_0(s)\lambda R(\lambda, A)xds.$$

Proof:
We need to check weak*-continuity and regularity. Weak*-continuity is a consequence of the variation of constants formula for $T^B(t)$, the uniform boundedness of the operators $\lambda R(\lambda, A^*)$ appearing therein and the weak*-continuity of $T^*(t)$.

The regularity checked as follows. By extending the semigroup $T_0^B(t)$ to a C_0-semigroup on X_{-1} with generator $A_{-1} + B$ for all $A, B \in \omega - OCP_n$. Then by denoting the extensions of $T_0(t)$ and $T_0^B(t)$ to the space X_{-1} by $T_{-1}(t)$ and $T_{-1}^B(t)$ respectively, then we have the following variation of constants formulas:

$$\begin{aligned} T_0^B(t)x_0 &= T_0(t)x_0 + \int_0^t T_{-1}^B(t-s)BT_0(s)x_0ds \\ &= T_0(t)x_0 + \int_0^t T_{-1}(t-s)BT_0^B(s)x_0ds \end{aligned} \tag{3.1}$$

for all $x_0 \in X_0$ and the Bochner integrals being X_{-1}. Suppose the part of $A^* + B$ in $X^{\odot}$ generates a C_0-semigroup $T_0^B(t)$ on $X^{\odot}$ which satisfies

$$T_0^B(t)x^{\odot} = T^{\odot}(t)x^{\odot} + \int_0^t T^*(t-s)BT_0^B(s)x^{\odot}ds \quad (x^{\odot} \in X^{\odot}). \tag{3.2}$$

Moreover, both $T^{\odot}(t)$ and $T_0^B(t)$ leave $D(A^*)$ invariant. Therefore the intertwining formula extends $T_0^B(t)$ to semigroup $T^B(t)$ on X^*, which satisfies for all $x^* \in X^*$ and $A, B \in \omega - OCP_n$, we have

$$\begin{aligned} T^B(t)x^* &= T^*(t)x^* + weak^* \lim_{\lambda} \left(weak^* \int_0^t T*(t-s)B\lambda R(\lambda, A^*)T^B(s)x^*ds \right) \\ &= T^*(t)x^* + weak^* \lim_{\lambda} \left(weak^* \int_0^t T^B(t-s)B\lambda R(\lambda, A^*)T^*(s)x^*ds \right) \end{aligned} \tag{3.3}$$

Then $(A^{\odot})_{-1} + B = (A^{\odot} + B)_{-1}$ generates the C_0-semigroup $T_{-1}^B(t) = (T_0^B(t))_{-1}$ on $(X^{\odot})_{-1} = (X^*)_{-1}$. Identifying X^* with a sequence of $(X^*)_{-1}$, we have

$$T_{-1}^B(t)\Big|_{X^*} = T^B(t)$$

and

$$T_{-1}^B(t) \int_0^s T_{-1}^B(\sigma)x^*d\sigma = \int_0^s T_{-1}^B(t+\sigma)x^*d\sigma \tag{3.4}$$

for all $x^* \in X^*$ and $B \in \omega - OCP_n$; this is because the integral is Bochner in $(X^*)_{-1}$. Now we need to show that for all $y \in X^*$, the $(X^*)_{-1}$-Bochner integral $\int_0^s T_{-1}^B(\sigma)y^*d\sigma$ equals $weak^* \int_0^s T_{-1}^B(\sigma)y^*d\sigma$. But identifying $D(A)$ with a linear subspace of $D(A^{\odot *}) \simeq ((X^{\odot})_{-1})^* \simeq ((X^*)_{-1})^*$, then we have for any $x \in D(A)$

$$\begin{aligned} \langle x, \int_0^s T_{-1}^B(\sigma)y^*d\sigma\rangle &= \int_0^s \langle x, T_{-1}^B(\sigma)y^*\rangle d\sigma \\ &= \langle x, weak^* \int_0^s T_{-1}^B(\sigma)y^*d\sigma\rangle \end{aligned} \tag{3.5}$$

and the result follows from the denseness of $D(A)$ which poves (i).

To prove (ii), this follows from integrating

$$T_{-1}^{B}x_{-1} = T_{-1}(t)x_{-1} + \lim_{\lambda} \int_0^t T_{-1}^{B}(t-s)B\lambda R(\lambda, A_{-1})T_{-1}(s)x_{-1}ds$$
$$= T_{-1}(t)x_{-1} + \lim_{\lambda} \int_0^t T_{-1}(t-s)B\lambda R(\lambda, A_{-1})T_{-1}^{B}(s)x_{-1}ds \quad (3.6)$$

in X_{-1}, then by the dominated convergence theorem and Fubini theorem, we have

$$S^{B}(t)x = S(t)x + \int_0^t \lim_{\lambda} \int_0^{\eta} T_{-1}^{B}(\eta - s)B\lambda R(\lambda, A_{-1})T_{-1}(s)xdsd\eta$$
$$= S(t)x + \lim_{\lambda} \int_0^t \int_s^{\eta} T_{-1}^{B}(\eta - s)B\lambda R(\lambda, A_{-1})T_{-1}(s)xd\eta ds$$
$$= S(t)x + \lim_{\lambda} \int_0^t s^{B}(t-s)BT_0(s)\lambda R(\lambda, A)xds. \quad (3.7)$$

The last integral is still in the sense of X_{-1}. But its integrand is continuous as a function $[0, t] \to X$, so the integral actually exists as a Bochner integral in X. Hence the prove is completed.

Theorem 3.2
Suppose $A : D(A) \subseteq X \to X$ is the infinitesimal generator of a regular, weak-continuous semi-group $\{T^{X}(t); t \geq 0\}$ on X^* such that $A \in \omega - OCP_n$. Then:*

(i) *Define operators $s^*(t)$ on X^* by*

$$S^{X}(t)x^* := weak^* \int_0^t T^{X}(s)x^*ds;$$

(ii) $R(\lambda, A^{X})T^{X}(t) = T^{X}(t)R(\lambda, A^{X})$.

In particular, the operators $S^{X}(t)$ define a non-degenerate locally Lipschitz integrated semigroup on X^* and $T^{X}(t)$ is an intertwined semigroup with intertwining operator A^{X}.

Proof:

By the uniform boundedness theorem, $\|T^X(t)\|$ is bounded in a neighbourhood of $t = 0$. The semigroup property then implies that

$$\|T^X(t)\| \leqq Me^{\omega t}$$

for some M and ω. This easily implies that $S^X(t)$ is locally Lipschitz with respect to t. Clearly, $S^X(0) = 0$ and $t \longmapsto S^X(t)x^*$ is continuous. The regularity assumption means that we have

$$T^X(t)S^X(s) = S^X(s)T^X(t). \tag{3.8}$$

Integrating (3.8), we have

$$S(t)S(s)x = \int_0^t (s(s+\eta) - S(\eta))xd\eta \tag{3.9}$$

for all $x \in X$. It remains to check that $S^X(t)$ in non-degenerate. But if $S^X(t)x^* = 0$ for all $t > 0$, then for all $x \in X$ we have that

$$\int_0^t \langle T^X(s)x^*, x\rangle = 0$$

for all $t > 0$. This implies that

$$\langle T^*(t)x^*, x\rangle = 0$$

for all t. Since x is arbitrary, the weak*-continuity of $T^X(t)$ implies that $x^* = 0$. If $T^X(t)$ is a regular, weak*-continuous semigroup on X^*, then we define the generator of $T^X(t)$ to the generator A^X of the associated integrated semigroup $S^X(t)$, and this proves (i).

To prove (ii), we have that for arbitrary $x^{\in} X^*$ and $A \in \omega - OCP_n$,

$$\begin{aligned}
R(\lambda, A^X)T^X(t)x^* &= \lambda \int_0^\infty e^{-\lambda s} S^X(s)T^X(t)x^* ds \\
&= \lambda \int_0^\infty e^{-\lambda s} T^X(t)S^X(s)x^* ds \\
&= T^X(t)\lambda \int_0^\infty e^{-\lambda s} S^X(s)x^* ds \\
&= T^X(t)R(\lambda, A^X)x^*.
\end{aligned}$$

We used the fact that $t \longmapsto S^X(t)x^*$ is Bochner integrable and this archived the prove.

Theorem 3.3

Let $A : D(A) \subseteq X \to X$ be the infinitesimal generator of a C_0-semigroup $\{T(t); t \geq 0\}$ where $A \in \omega - OCP_n$. Then the following assertions are equivalent:

(i) $X^{\odot}$ *defines an equivalent norm on* X*;*

(ii) *There is a constant $C > 0$ such that*

$$\lim_{t\to 0} \sup t^{-1}\|S(t)x\| \geqq C\|x\| \quad \textit{for all} \quad x \in X;$$

(iii) *There is a constant $C > 0$ such that*

$$\lim_{\lambda\to\infty} \sup \|\lambda R(\lambda, A)x\| \geq C\|x\| \quad \textit{for all} \quad x \in X \quad \textit{and} \quad A \in \omega - OCP_n;$$

(iv) $S^X(t)x^* \in X^{\odot}$ *for all $t > 0$ and*

$$\langle Kx^{\odot\odot}, x^*\rangle = \lim_{t\to 0} \frac{1}{t}\langle x^{\odot\odot}, S^X(t)x^*\rangle.$$

Proof:

To prove (i), let $C > 0$ be the norming constant of the norm $\|\cdot\|$ induced by $X^{\odot}$. Suppose $x \in X$, $A \in \omega - OCP_n$ and $\varepsilon > 0$ be arbitrary. There is an $x^{\odot} \in X^{\odot}$ of norm one such that

$$|\langle x^{\odot}, x\rangle| > C\|x\| - \varepsilon.$$

Hence for λ sufficiently large, also

$$\|\lambda R(\lambda, A)x\| \geqq |\langle x^{\odot}, \lambda R(\lambda, A)x\rangle| = |\langle \lambda R(\lambda, A)^* x^{\odot}, x\rangle| > C\|x\| - \varepsilon \tag{3.10}$$

and we have

$$\|\lambda R(\lambda, A)x\| \geqq C\|x\|$$

for all $x \in X$ and $A \in \omega - OCP_n$ which obtained (iii).

(ii) Follows similarly from (i) by observing that

$$S^*(t)\Big|_{X^\odot} = \int_0^t T^\odot(s)$$

where $T^\odot(t)$ is the semigroup of a restricted a map i_0^* which induces an isomorphism $X^\odot \simeq (X_0)^\odot$ under which we have

$$i_0^* T^\odot(t) = T^\odot(t) i_0^*. \tag{3.11}$$

Moreover, for all $x \in X$ and $x^\odot \in X^\odot$ we have

$$\langle x^\odot, x\rangle = \lim_{n\to\infty} \langle i_0^* x^\odot, x_n\rangle$$

where (x_n) is any bounded sequence in X_0 such that $R(\mu, A)x_n \to R(\mu, A)x$ in X for all $A \in \omega - OCP_n$ and $x_n \to x$ in X_{-1}. This identity is an easy consequence of the fact that integrated semigroup generated by a Hille-Yosida operator is unique. So for $t \to 0$ and $x^\odot \in X^\odot$ we have $t^{-1}S^*(t)x^\odot \to x^\odot$.

Now let (iii). For $x \in X$ and $\varepsilon > 0$, choose $x^* \in X^*$ of norm one and $\lambda > 0$ such that

$$|\langle x^*, \lambda R(\lambda, A)x\rangle| > C\|x\| - \varepsilon.$$

Then also,

$$|\langle \lambda R(\lambda, A)^* x^*, x\rangle| > C\|x\| - \varepsilon$$

and (i) follows from $\lambda R(\lambda, A)^* x^* \in X^\odot$. The implication (ii) $\Longrightarrow$ (i) is proved similarly.

To prove (iv). The first statement follows from $X^\odot$ is the space of strong continuity of $T^X(t)$ and we have

$$T^X(t)\Big|_{X^\odot} = T^\odot(t). \tag{3.12}$$

Indeed, by a trivial direct computation we shows that $t \longmapsto S^X(t)x^*$ is strongly continuous. Since $\|t^{-1}S^*(t)\|$ is bounded in a neighbourhood of $t = 0$ and since for every $x^* \in X^*$ we have

$$\lim_{t\to 0}\frac{1}{t}R(\lambda, A^X)S^X(t)x^* = \lim_{t\to 0}\frac{1}{t}\int_0^t T^{\odot}(s)R(\lambda, A^X)x^* ds \\ - R(\lambda, A^X)x^* \qquad (3.13)$$

and by applying (3.11) in (3.13), we have

$$\langle Kx^{\odot\odot}, x^*\rangle = \lim_{t\to 0}\frac{1}{t}\langle x^{\odot\odot}, S^*(t)x^*\rangle.$$

Hence, the prove is completed.

Theorem 3.4

Suppose $A \in \omega - OCP_n$ is the generator of a regular weak-continuous semigroup $\{T^X(t); t \geqq 0\}$ on a dual space X^*. Then:*

(i) $|x^*| := \sup_{\|x^{\odot\odot}\|>1} |\langle Kx^{\odot\odot}, x^*\rangle|$ *defines an equivalent norm on X^*;*

(ii) *Every regular weak*-continuous semigroup $T^X(t)$ is the restriction to a closed subspace of an adjoint semigroup where $T^X(t) = T^{\odot\odot *}(t)\Big|_{X^*}$.*

Proof:

By applying (i), (ii) and (iii) of Theorem 3.3, we have that for every $t > 0$, $x \in X$ of norm one and $x^* \in X^*$ we have

$$\frac{1}{t}\|S^X(t)x^*\| \geqq \frac{1}{t}|\int_0^t \langle T^X(s)x^*, x\rangle ds|.$$

Letting $t \to 0$, it follows from the weak*-continuity of $T^X(t)$ that

$$\limsup_{t\to 0}\frac{1}{t}\|S^X(t)x^*\| \geqq |\langle x^*, x\rangle|.$$

This holds for every $x \in X$ of norm one, and therefore

$$\limsup_{t\to 0}\frac{1}{t}\|S^X(t)x^*\| \geqq \|x^*\|$$

which proves (i).

To prove (ii). By (iv) of Theorem 3.3, we have

$$\begin{aligned}\langle T^{\odot\odot}(t)x^{\odot\odot}, x^*\rangle &= \lim_{s\to 0}\frac{1}{s}\langle T^{\odot\odot}(t)x^{\odot\odot}, S^X(s)x^*\rangle \\ &= \lim_{s\to 0}\frac{1}{s}\langle x^{\odot\odot}, T^{\odot}(t)S^X(s)x^8\rangle \\ &= \lim_{s\to 0}\frac{1}{s}\langle x^{\odot\odot}, S^X(s)T^X(t)x^*\rangle \\ &= \langle x^{\odot\odot}, T^X(t)x^*\rangle \qquad (3.14)\end{aligned}$$

and this achieved the proof.

Theorem 3.5

Let $A : D(A) \subseteq X \to X$ be the generator of a regular, weak-continuous semigroup $\{T^X(t); t \geqq 0\}$ such that $A \in \omega - OCP_n$. Then we have*

(i) *For all $x^{\odot\odot} \in X^{\odot\odot}$ and $x^* \in X^*$, we have*

$$\langle x^{\odot\odot}, R(\lambda, A^X)x^* = \int_0^\infty e^{-\lambda t}\langle x^{\odot\odot}, T^X(t)x^*\rangle dt$$

and

$$\langle x^{\odot\odot}, weak * \int_0^t T^X(s)x^* ds\rangle = \int_0^t \langle x^{\odot\odot}, T^X(s)x^*\rangle ds;$$

(ii) *$x^* \in D(A^*)$, $A \in \omega - OCP_n$ with $A^X x^* = y^*$ if and only if*

$$\sigma(X^*, X^{\odot\odot}) - \lim_{t\to 0}\frac{1}{t}(T^X(t)x^* - x^*) = y^*.$$

Proof:

In $(X^*)_{-1}$ we have $R(\lambda, A^X)x^* = \int_0^\infty e^{-\lambda t}T^X(t)x^* dt$. Identifying $((X^*)_{-1})^*$ with $D(A^{\odot *})$ by letting A be a densely defined generalized Hille-Yosida operator on X. Then θ defined an isomorphism of $D(A^*)$ onto $(X_{-1})^*$, which is independent of μ. Moreover, if A is a generator, then θ maps $D(A^{\odot})$ onto $(X_{-1})^{\odot}$. Then by identity let A be Hille-Yosida on X. Suppose A is a closed operator with $\lambda \in \rho(A)$, then

$D(A_{-1}) = X_0$ and $\lambda - A_{-1} : X_0 \to X_{-1}$ is a isomorphism such that A is the part of A_{-1} in X. If $\lambda \in \rho(A)$, then $\lambda \in \rho(A_{-1})$ and

$$R(\lambda, A) = R(\lambda, A_{-1})\Big|_X. \tag{3.15}$$

Then there is a natural isomorphism: $\varphi : X_{-1} \simeq (X_0)_{-1}$, combining this with isomorphism $\phi : (X^*)^{-1} \simeq (X_1)^*$ induces by an isomorphism $(X^{\odot})^{-1} \simeq (X_1)^{\odot}$, then we obtain a natural isomorphism $\xi : D(A^*_{\odot}) \simeq (X_{-1})^*$ by putting

$$\langle \xi x_0^{\odot}, x_{-1}\rangle := \langle \theta x_0^{\odot}, \varphi x_{-1}\rangle.$$

In particular, by regarding X as a subspace of X_{-1}, there is a natural action of an $x_0^{\odot} \in D(A_0^*)$ on an $x \in X$. Letting $i : X \to X_{-1}$ be the inclusion map, we claim that

$$\langle \xi x_0^{\odot}, ix\rangle = \langle K x_0^{\odot}, x\rangle.$$

Indeed,

$$\begin{aligned}
\langle \xi x_0^{\odot}, ix\rangle &= \lim_{\lambda} \langle \xi x_0^{\odot}, \lambda R(\lambda, A_{-1}) ix\rangle \\
&= \lim_{\lambda} \langle \theta x_0^{\odot}, \varphi \lambda R(\lambda, A_{-1}) ix\rangle \\
&= \lim_{\lambda} \langle (\mu - A_0^*) x_0^{\odot}, R(\mu, (A_0)_{-1}) \lambda R(\lambda, A) x\rangle \\
&= \lim_{\lambda} \langle x_0^{\odot}, \lambda R(\lambda, A) x\rangle \\
&= \langle K x_0^{\odot}, x\rangle,
\end{aligned} \tag{3.16}$$

so that for each $x^{\odot *} \in D(A^{\odot *})$ and $A \in \omega - OCP_n$ we have

$$\langle x^{\odot}, R(\lambda, A^X) x^*\rangle = \langle x^{\odot *}, \int_0^{\infty} e^{-\lambda t} T^X(t) x^* dt\rangle = \int_0^{\infty} \langle x^{\odot *}, e^{-\lambda t} T^X(t) x^*\rangle dt. \tag{3.17}$$

Using that the integral is Bochner in $(X^*)_{-1}$. Since $D(A^{\odot *})$ is dense in $X^{\odot\odot}$, the dominated convergence theorem implies that these identities hold for every $x^{\odot\odot} \in X^{\odot\odot}$, and this completes the prove of (i).

To prove (ii), first let $x^* \in D(A^X)$ and $A^X x^* = y^*$. Put

$$z^* := (\lambda - A^X) x^* = \lambda x^* - y^*.$$

A little computation involving the first identity in (i) of Theorem 3.5 shows that for $x^{\odot\odot} \in X^{\odot\odot}$,

$$\frac{1}{t}\langle x^{\odot\odot}, T^X(t)x^* - x^*\rangle = \langle x^{\odot\odot}, \frac{e\lambda t - 1}{t}x^*\rangle - \frac{e^t}{t}\int_0^t e^{-\lambda t}\langle x^{\odot\odot}, T^X(s)z^*\rangle ds. \quad (3.18)$$

Letting $t \to 0$ we obtain, using the $\sigma(X^*, X^{\odot\odot})$-continuity of $T^X(t)$,

$$\lim_{t\to 0}\frac{1}{t}\langle x^{\odot\odot}, T^X(t)x^* - x^*\rangle = \lambda\langle^{\odot\odot}, x^*\rangle - \langle x^{\odot\odot}, z^*\rangle = \langle x^{\odot\odot}, y^*\rangle. \quad (3.19)$$

Conversely, suppose that for some $x^* \in X^*$ the $\sigma(X^*, X^{\odot\odot})$-limits and equals y^*. Put

$$z^* := R(\lambda, A^X)(\lambda x^* - y^*).$$

Fix $x^{\odot\odot} \in X^{\odot\odot}$. Then

$$\begin{aligned}\langle x^{\odot\odot}, z^*\rangle &= \langle R(\lambda, A^{\odot\odot})x^{\odot\odot}, \lambda x^* - y^*\rangle \\ &= \lambda\langle R(\lambda, A^{\odot\odot})x^{\odot\odot}, x^*\rangle - \lim_{t\to 0} t^{-1}(T^{\odot *}(t) - I)R(\lambda, A^{\odot\odot})x^{\odot\odot}, x^*\rangle \\ &= \langle(\lambda - A^{\odot\odot})R(\lambda, A^{\odot\odot})x^{\odot\odot}, x^*\rangle \\ &= \langle x^{\odot\odot}, x^*\rangle. \end{aligned} \quad (3.20)$$

Therefore

$$x^* = z^* = R(\lambda, A^X)(\lambda x^* - y^*) \in D(A^X)$$

and

$$A^X x^* = y^*.$$

Hence the proof is completed.

Conclusion

In this paper, it has been established that ω-order preserving partial contraction mapping generates some results of regular weak*-continuous semigroup.

Acknowledgment

We acknowledge the management of the University of Ilorin for providing us with a suitable research laboratory and library to enable us carried out this research.

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