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New Class of Multivalent Functions with Negative Coefficients

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Abstract

In the present paper, we define a new class $\mathcal{NA}(n,p,\lambda,\alpha,\beta)$ of multivalent functions which are holomorphic in the unit disk $\Delta = \{s \in \mathbb{C} : |s| < 1\}$. A necessary and sufficient condition for functions to be in the class $\mathcal{NA}(n,p,\lambda,\alpha,\beta)$ is obtained. Also, we get some geometric properties like radii of starlikeness, convexity and close-to-convexity, closure theorems, extreme points, integral means inequalities and integral operators.

1. Introduction

Let $\mathcal{A}_p$ be symbolize the function class of the form:

$$k(s) = s^p + \sum_{n=p+1}^{\infty} d_n s^n \ \ (s \in \Delta;\ n \geq p+1; p \in \mathbb{N} = \{1,2,\dots\}), \qquad (1.1)$$

which are holomorphic and multivalent in the open unit disk $\Delta = \{s \in \mathbb{C} : |s| < 1\}$.

Let $\mathcal{N}_p$ be symbolize the function subclass of $\mathcal{A}_p$ containing of functions of the form:

$$k(s) = s^p - \sum_{n=p+1}^{\infty} d_n s^n \ \ (s \in \Delta; d_n \geq 0;\ n \geq p+1; p \in \mathbb{N} = \{1,2,\dots\}). \quad (1.2)$$

For function $k(s) \in \mathcal{N}_p$, given by (1.2), and $h(s) \in \mathcal{N}_p$ given by

$$h(s) = s^p - \sum_{n=p+1}^{\infty} c_n s^n \ \ (s \in \Delta; c_n \geq 0;\ n \geq p+1; p \in \mathbb{N} = \{1,2,\dots\}), \qquad (1.3)$$

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the convolution (or Hadamard product) of $k(s)$ and $h(s)$ is defined by

$$(k * h)(s) = s^p - \sum_{n=p+1}^{\infty} d_n c_n s^n = (h * k)(s). \tag{1.4}$$

A function $k(s) \in \mathcal{A}_p$ is called multivalent starlike of order γ $(0 \leq \gamma < p)$, if $k(s)$ satisfies the condition:

$$Re\left(\frac{sk'(s)}{k(s)}\right) > \gamma \qquad (s \in \Delta; 0 \leq \gamma < p;\ p \in \mathbb{N} = \{1,2,\ldots\}). \tag{1.5}$$

Also, a function $k(s) \in \mathcal{A}_p$ is called multivalent convex of order $\gamma(0 \leq \gamma < p)$, if $k(s)$ satisfies the condition:

$$Re\left(1 + \frac{sk''(s)}{k'(s)}\right) > \gamma \qquad (s \in \Delta; 0 \leq \gamma < p;\ p \in \mathbb{N} = \{1,2,\ldots\}). \tag{1.6}$$

Symbolize by $S_n^*(p,\gamma)$ the class of multivalent starlike functions of order γ. Symbolize by $C_n(p,\gamma)$ the class of multivalent convex functions of order γ, which were studied by Owa [11]. It is observed that

$$k(s) \in C_n(p,\gamma) \text{ if and only if } \frac{sk'(s)}{p} \in S_n^*(p,\gamma).$$

A function $k(s) \in \mathcal{A}_p$ is called multivalent close to convex of order $\gamma(0 \leq \gamma < p)$, if $k(s)$ satisfies the condition:

$$Re\left(\frac{k'(s)}{s^{p-1}}\right) > \gamma \qquad (s \in \Delta; 0 \leq \gamma < p;\ p \in \mathbb{N} = \{1,2,\ldots\}). \tag{1.7}$$

We recall the principle of subordination between two holomorphic functions $k(s)$ and $h(s)$ in Δ. It is known that $k(s)$ is subordinate to $h(s)$, written as $k(s) \prec h(s)$, $s \in \Delta$, if there is a $w(s)$ holomorphic in Δ, with $w(0) = 0$ and $|w(s)| < 1$, $s \in \Delta$, such that $k(s) = h(w(s))$. Moreover, $k(s) \prec h(s)$ is equivalent to $k(0) = h(0)$ and $k(\Delta) \subset h(\Delta)$, if $k(s)$ is univalent in Δ.

Definition 1.1. Let $k(s) \in \mathcal{N}_p$ given by (1.2), is said to be in the class $\mathcal{NA}(n,p,\lambda,\alpha,\beta)$ if and only if satisfies the following inequality:

$$Re\left\{\frac{sk'(s) + s^2k''(s)}{\lambda sk'(s) + (1-\lambda)k(s)}\right\} > \alpha\left|\frac{sk'(s) + s^2k''(s)}{\lambda sk'(s) + (1-\lambda)k(s)} - p\right| + \beta, \tag{1.8}$$

where $s \in \Delta, 0 \leq \beta < p, \alpha \geq 0, 0 \leq \lambda \leq 1, n \geq p+1$ and $p \in \mathbb{N} = \{1,2,\ldots\}$.

We note that by customizing the parameter α, λ, p, we get the following various subclasses as studied by different researchers:

1) If $\lambda = 1$, the class $\mathcal{NA}(n, p, \lambda, \alpha, \beta)$ reduces to the class $UCV(p, \alpha, \beta)$ which is introduced by Khairnar and More [8].

2) If $\lambda = 1$ and $\alpha = 0$, the class $\mathcal{NA}(n, p, \lambda, \alpha, \beta)$ reduces to the class $C_n(p, \alpha)$ which is studied by Owa [11] and Sălăgean et al. [12].

3) If $\lambda = 1$ and $p = 1$, the class $\mathcal{NA}(n, p, \lambda, \alpha, \beta)$ reduces to the class $UCT(\alpha, \beta)$ which is studied by Bharati et al. [6].

4) If $\lambda = 1, p = 1$ and $\alpha = 0$, the class $\mathcal{NA}(n, p, \lambda, \alpha, \beta)$ reduces to the class $C(\alpha)$ which is studied by Silverman [13].

5) If $\lambda = 0$, $p = 1$ and $\alpha = 0$, the class $\mathcal{NA}(n, p, \lambda, \alpha, \beta)$ reduces to the class $H(1, \beta)$ which is studied by Lashin [9].

Lemma 1.1 [3]. *Let $y = v + iu$ be complex number. Then $Re(y) \geq \beta$ if and only if $|y - (p + \beta)| \leq |y + (p - \beta)|$, where $\beta \geq 0$.*

Lemma 1.2 [3]. *Let $y = v + iu$ and β, α be real numbers. Then $Re(y) \geq \alpha|y - p| + \beta$ if and only if $Re\{y(1 + \alpha e^{i\theta}) - p\alpha e^{i\theta}\} \geq \beta$.*

Lemma 1.3 [10]. *If k and h are holomorphic in Δ with $k \prec h$, then*

$$\int_0^{2\pi} |k(re^{i\theta})|^{\mu} d\theta \leq \int_0^{2\pi} |h(re^{i\theta})|^{\mu} d\theta,$$

where $\mu > 0, s = re^{i\theta}$ and $(0 < r < 1)$.

Some of the following properties studied for other classes in [1, 2, 4, 5, 7, 14].

2. Coefficient Inequality

From the following theorem, we get the necessary and sufficient condition for the function $k(s)$ to be in the class $\mathcal{NA}(n, p, \lambda, \alpha, \beta)$.

Theorem 2.1. *Let $k(s)$ be in the form* (1.2). *Then $k(s)$ is in the class $\mathcal{NA}(n, p, \lambda, \alpha, \beta)$ if and only if*

$$\sum_{n=p+1}^{\infty} [n^2(\alpha + 1) - (\beta + \alpha p)(1 + \lambda n - \lambda)]d_n \leq p^2(\alpha + 1) - (\beta + \alpha p)(1 + \lambda p - \lambda), \quad (2.1)$$

where $s \in \Delta, 0 \leq \beta < p, \alpha \geq 0, 0 \leq \lambda \leq 1, n \geq p+1$ *and* $p \in \mathbb{N} = \{1,2,\dots\}$.

The result is sharp for the function $k(s)$ *given by*

$$k(s) = s^p - \frac{p^2(\alpha+1) - (\beta+\alpha p)(1+\lambda p - \lambda)}{n^2(\alpha+1) - (\beta+\alpha p)(1+\lambda n - \lambda)} s^n, \quad (n \geq p+1; p \in \mathbb{N}). \quad (2.2)$$

Proof. Let $k(s) \in \mathcal{NA}(n,p,\lambda,\alpha,\beta)$. Then $k(s)$ satisfies the inequality (1.8). By using Lemma 1.2, the inequality (1.8) is equivalent to

$$Re\left\{\frac{(sk'(s) + s^2k''(s))\ (1+\alpha e^{i\theta})}{\lambda sk'(s) + (1-\lambda)k(s)} - p\alpha e^{i\theta}\right\} \geq \beta$$

$$(0 \leq \beta < p, \alpha \geq 0, 0 \leq \lambda \leq 1, n \geq p+1,\ p \in \mathbb{N} \text{ and} -\pi < \theta \leq \pi).$$

Or equivalently,

$$Re\left\{\frac{(sk'(s) + s^2k''(s))(1+\alpha e^{i\theta})}{\lambda sk'(s) + (1-\lambda)k(s)} - \frac{p\alpha e^{i\theta}\big(\lambda sk'(s) + (1-\lambda)k(s)\big)}{\lambda sk'(s) + (1-\lambda)k(s)}\right\} \geq \beta. \quad (2.3)$$

Let

$$A(s) = (sk'(s) + s^2k''(s)\,)(1+\alpha e^{i\theta}) - p\alpha e^{i\theta}\big(\lambda sk'(s) + (1-\lambda)k(s)\big).$$

$$B(s) = \lambda sk'(s) + (1-\lambda)k(s).$$

Then by Lemma 1.1, (2.3) is equivalent to

$$|A(s) + (p-\beta)B(s)| \geq |A(s) - (p+\beta)B(s)| \quad for \quad 0 \leq \beta < p.$$

Now

$$|A(s) + (p-\beta)B(s)|$$

$$= \left| \begin{array}{l} \left(ps^p - \sum_{n=p+1}^{\infty} nd_ns^n + p(p-1)s^p - \sum_{n=p+1}^{\infty} n(n-1)d_ns^n\right)(1+\alpha e^{i\theta}) \\ -p\alpha e^{i\theta}\left(\lambda ps^p - \sum_{n=p+1}^{\infty} \lambda nd_ns^n + (1-\lambda)s^p - \sum_{n=p+1}^{\infty} (1-\lambda)d_ns^n\right) \\ +(p-\beta)\left(\lambda ps^p - \sum_{n=p+1}^{\infty} \lambda nd_ns^n + (1-\lambda)s^p - \sum_{n=p+1}^{\infty} (1-\lambda)d_ns^n\right) \end{array} \right|$$

$$= \left| \begin{array}{c} \left(ps^p - \sum_{n=p+1}^{\infty} nd_n s^n + p(p-1)s^p - \sum_{n=p+1}^{\infty} n(n-1)d_n s^n \right)\left(1+\alpha e^{i\theta}\right) \\ +\left(p-\beta-p\alpha e^{i\theta}\right)\left(\lambda p s^p - \sum_{n=p+1}^{\infty} \lambda n d_n s^n + (1-\lambda)s^p - \sum_{n=p+1}^{\infty} (1-\lambda)d_n s^n \right) \end{array} \right|$$

$$= \left| \begin{array}{c} p^2\left(1+\alpha e^{i\theta}\right) s^p + \left(p-\beta-p\alpha e^{i\theta}\right)(1+\lambda p-\lambda)s^p \\ - \sum_{n=p+1}^{\infty} n^2\left(1+\alpha e^{i\theta}\right)d_n s^n - \sum_{n=p+1}^{\infty} \left(p-\beta-p\alpha e^{i\theta}\right)(1+\lambda n-\lambda)d_n s^n \end{array} \right|$$

$$\geq \left(p^2(1+\alpha)+(p-\beta-p\alpha)(1+\lambda p-\lambda)\right)|s|^p$$

$$- \sum_{n=p+1}^{\infty} \left(n^2(1+\alpha)+(p-\beta-p\alpha)(1+\lambda n-\lambda)\right) d_n |s|^n.$$

Similarly,

$$|A(s)-(p+\beta)B(s)|$$

$$= \left| \begin{array}{c} \left(ps^p - \sum_{n=p+1}^{\infty} nd_n s^n + p(p-1)s^p - \sum_{n=p+1}^{\infty} n(n-1)\, d_n s^n \right)\left(1+\alpha e^{i\theta}\right) \\ -\left(p+\beta+p\alpha e^{i\theta}\right)\left(\lambda p s^p - \sum_{n=p+1}^{\infty} \lambda n d_n s^n + (1-\lambda)s^p - \sum_{n=p+1}^{\infty} (1-\lambda)d_n s^n \right) \end{array} \right|$$

$$\leq \left((p+\beta+p\alpha)(1+\lambda p-\lambda)-p^2(1+\alpha)\right)|s|^p$$

$$+ \sum_{n=p+1}^{\infty} \left(n^2(1+\alpha)-(p+\beta+p\alpha)(1+\lambda n-\lambda)\right) d_n |s|^n.$$

Therefore

$$|A(s)+(p-\beta)B(s)| - |A(s)-(p+\beta)B(s)|$$

$$\geq p^2(\alpha+1)-(\beta+\alpha p)(1+\lambda p-\lambda)$$

$$- \sum_{n=p+1}^{\infty} [n^2(\alpha+1)-(\beta+\alpha p)(1+\lambda n-\lambda)]d_n \geq 0.$$

Hence

$$\sum_{n=p+1}^{\infty}[n^2(\alpha+1)-(\beta+\alpha p)(1+\lambda n-\lambda)]d_n \leq p^2(\alpha+1)-(\beta+\alpha p)(1+\lambda p-\lambda).$$

Conversely, by considering (2.1), we must show that

$$Re\left\{\frac{(sk'(s)+s^2k''(s))\,(1+\alpha e^{i\theta})}{\lambda sk'(s)+(1-\lambda)k(s)}-\frac{(p\alpha e^{i\theta}+\beta)(\lambda sk'(s)+(1-\lambda)k(s))}{\lambda sk'(s)+(1-\lambda)k(s)}\right\}\geq 0. \quad (2.4)$$

Upon choosing the values of s on the positive real axis where $0\leq s=r<1$, $Re(-e^{i\theta})\geq -|e^{i\theta}|=-1$ and letting $r\to 1^-$, we conclude to (2.4) by using (2.1) in left hand of (2.4).

Corollary 2.1. *Let* $k(s)$ *be in the class* $\mathcal{NA}(n,p,\lambda,\alpha,\beta)$. *Then*

$$d_n \leq \frac{p^2(\alpha+1)-(\beta+\alpha p)(1+\lambda p-\lambda)}{n^2(\alpha+1)-(\beta+\alpha p)(1+\lambda n-\lambda)}, \quad (2.5)$$

where $0\leq\beta<p, \alpha\geq 0, 0\leq\lambda\leq 1, n\geq p+1$ *and* $p\in\mathbb{N}=\{1,2,\ldots\}$.

3. Radii of Starlikeness, Convexity and Close-to-Convexity

In the next theorems, we will find the radii of starlikeness, convexity and close-to-convexity for the functions in the class $\mathcal{NA}(n,p,\lambda,\alpha,\beta)$.

Theorem 3.1. *Let* $k(s)\in\mathcal{NA}(n,p,\lambda,\alpha,\beta)$. *Then the function* $k(s)$ *is multivalent starlike of order* $\gamma(0\leq\gamma<p)$ *in the disk* $|s|<R_1$, *where*

$$R_1=\inf_n\left[\frac{(p-\gamma)\left(n^2(\alpha+1)-(\beta+\alpha p)(1+\lambda n-\lambda)\right)}{(n-\gamma)\left(p^2(\alpha+1)-(\beta+\alpha p)(1+\lambda p-\lambda)\right)}\right]^{\frac{1}{n-p}}, \quad (n\geq p+1; p\in\mathbb{N}).$$

The result is sharp for the function $k(s)$ *given by* (2.2).

Proof. It is sufficient to prove

$$\left|\frac{sk'(s)}{k(s)}-p\right|\leq p-\gamma \quad (0\leq\gamma<p),$$

for $|s|<R_1$, we get

$$\left|\frac{sk'(s)}{k(s)}-p\right|\leq\frac{\sum_{n=p+1}^{\infty}(n-p)d_n|s|^{n-p}}{1-\sum_{n=p+1}^{\infty}d_n|s|^{n-p}}.$$

Thus

$$\left|\frac{sk'(s)}{k(s)} - p\right| \leq p - \gamma,$$

if

$$\sum_{n=p+1}^{\infty} \frac{n-\gamma}{p-\gamma} d_n |s|^{n-p} \leq 1. \tag{3.1}$$

Therefore, by using Theorem 2.1, (3.1) will be true if

$$\frac{n-\gamma}{p-\gamma}|s|^{n-p} \leq \frac{n^2(\alpha+1) - (\beta+\alpha p)(1+\lambda n-\lambda)}{p^2(\alpha+1) - (\beta+\alpha p)(1+\lambda p-\lambda)}.$$

Hence

$$|s| \leq \left[\frac{(p-\gamma)\big(n^2(\alpha+1) - (\beta+\alpha p)(1+\lambda n-\lambda)\big)}{(n-\gamma)\big(p^2(\alpha+1) - (\beta+\alpha p)(1+\lambda p-\lambda)\big)}\right]^{\frac{1}{n-p}}, \qquad (n \geq p+1; p \in \mathbb{N}).$$

Setting $|s| = R_1$, we obtain the desired result.

Theorem 3.2. *Let $k(s) \in \mathcal{NA}(n,p,\lambda,\alpha,\beta)$. Then the function $k(s)$ is multivalent convex of order $\gamma (0 \leq \gamma < p)$ in the disk $|s| < R_2$, where*

$$R_2 = \inf_n \left[\frac{p(p-\gamma)\big(n^2(\alpha+1) - (\beta+\alpha p)(1+\lambda n-\lambda)\big)}{n(n-\gamma)\big(p^2(\alpha+1) - (\beta+\alpha p)(1+\lambda p-\lambda)\big)}\right]^{\frac{1}{n-p}}, \quad (n \geq p+1; p \in \mathbb{N}).$$

The result is sharp for the function $k(s)$ given by (2.2).

Proof. It is sufficient to show that

$$\left|1 + \frac{sk''(s)}{k'(s)} - p\right| \leq p - \gamma \quad (0 \leq \gamma < p),$$

for $|s| < R_2$, we get

$$\left|1 + \frac{sk''(s)}{k'(s)} - p\right| \leq \frac{\sum_{n=p+1}^{\infty} n(n-p) d_n |s|^{n-p}}{p - \sum_{n=p+1}^{\infty} n d_n |s|^{n-p}}.$$

Thus

$$\left|1 + \frac{sk''(s)}{k'(s)} - p\right| \leq p - \gamma,$$

if

$$\sum_{n=p+1}^{\infty} \frac{n(n-\gamma)}{p(p-\gamma)} d_n |s|^{n-p} \leq 1. \tag{3.2}$$

Therefore, by using Theorem 2.1, (3.2) will be true if

$$\frac{n(n-\gamma)}{p(p-\gamma)} |s|^{n-p} \leq \frac{n^2(\alpha+1)-(\beta+\alpha p)(1+\lambda n-\lambda)}{p^2(\alpha+1)-(\beta+\alpha p)(1+\lambda p-\lambda)},$$

and hence

$$|s| \leq \left[\frac{p(p-\gamma)\big(n^2(\alpha+1)-(\beta+\alpha p)(1+\lambda n-\lambda)\big)}{n(n-\gamma)\big(p^2(\alpha+1)-(\beta+\alpha p)(1+\lambda p-\lambda)\big)}\right]^{\frac{1}{n-p}}, (n \geq p+1; p \in \mathbb{N}).$$

Setting $|s| = R_2$, we get the desired result.

Theorem 3.3. *Let $k(s) \in \mathcal{NA}(n,p,\lambda,\alpha,\beta)$. Then the function $k(s)$ is multivalent close to convex of order $\gamma (0 \leq \gamma < p)$ in the disk $|s| < R_3$, where*

$$R_3 = \inf_n \left[\frac{(p-\gamma)\big(n^2(\alpha+1)-(\beta+\alpha p)(1+\lambda n-\lambda)\big)}{n\big(p^2(\alpha+1)-(\beta+\alpha p)(1+\lambda p-\lambda)\big)}\right]^{\frac{1}{n-p}}, \quad (n \geq p+1; p \in \mathbb{N}).$$

The result is sharp for the function $k(s)$ given by (2.2).

Proof. It is sufficient to show that

$$\left|\frac{k'(s)}{s^{p-1}} - p\right| \leq p-\gamma \qquad (0 \leq \gamma < p),$$

for $|s| < R_3$, we get

$$\left|\frac{k'(s)}{s^{p-1}} - p\right| \leq \sum_{n=p+1}^{\infty} n d_n |s|^{n-p}.$$

Thus

$$\left|\frac{k'(s)}{s^{p-1}} - p\right| \leq p-\gamma,$$

if

$$\sum_{n=p+1}^{\infty} \frac{n d_n |s|^{n-p}}{p-\gamma} \leq 1. \tag{3.3}$$

Therefore, by using Theorem 2.1, (3.3) will be true if

$$\frac{n}{p-\gamma}|s|^{n-p} \leq \frac{n^2(\alpha+1)-(\beta+\alpha p)(1+\lambda n-\lambda)}{p^2(\alpha+1)-(\beta+\alpha p)(1+\lambda p-\lambda)},$$

and hence

$$|s| \leq \left[\frac{(p-\gamma)\big(n^2(\alpha+1)-(\beta+\alpha p)(1+\lambda n-\lambda)\big)}{n\big(p^2(\alpha+1)-(\beta+\alpha p)(1+\lambda p-\lambda)\big)}\right]^{\frac{1}{n-p}}, \quad (n \geq p+1; p \in \mathbb{N}).$$

The result is sharp for the function $k(s)$ given by (2.2).

4. Closure Theorems

Theorem 4.1. *Let the functions k_v defined as*

$$k_v(s) = s^p - \sum_{n=p+1}^{\infty} d_{n,v}\, s^n, \quad \big(d_{n,v} \geq 0, n \geq p+1, p \in \mathbb{N}, v = 1,2,\dots,l\big), \quad (4.1)$$

be in the class $\mathcal{NA}(n,p,\lambda,\alpha,\beta)$ for every $v = 1,2,\dots,l$.

Then the function $m_1(s)$ defined as

$$m_1(s) = s^p - \sum_{n=p+1}^{\infty} e_n\, s^n, \quad (e_n \geq 0, n \geq p+1, p \in \mathbb{N}),$$

also belongs to the class $\mathcal{NA}(n,p,\lambda,\alpha,\beta)$, where

$$e_n = \frac{1}{l}\sum_{v=1}^{l} d_{n,v}, \qquad (n \geq p+1, p \in \mathbb{N}).$$

Proof. Since $k_v \in \mathcal{NA}(n,p,\lambda,\alpha,\beta)$ it follows from Theorem 2.1 that

$$\sum_{n=p+1}^{\infty} [n^2(\alpha+1)-(\beta+\alpha p)(1+\lambda n-\lambda)]d_{n,v} \leq p^2(\alpha+1)-(\beta+\alpha p)(1+\lambda p-\lambda),$$

for every $v = 1,2,\dots,l$. Therefore

$$\sum_{n=p+1}^{\infty} [n^2(\alpha+1)-(\beta+\alpha p)(1+\lambda n-\lambda)]e_n$$

$$= \sum_{n=p+1}^{\infty} [n^2(\alpha+1) - (\beta+\alpha p)(1+\lambda n - \lambda)] \left(\frac{1}{l} \sum_{v=1}^{l} d_{n,v} \right)$$

$$= \frac{1}{l} \sum_{v=1}^{l} \left(\sum_{n=p+1}^{\infty} [n^2(\alpha+1) - (\beta+\alpha p)(1+\lambda n - \lambda)] d_{n,v} \right)$$

$$\leq p^2(\alpha+1) - (\beta+\alpha p)(1+\lambda p - \lambda).$$

Using Theorem 2.1, it follows that $m_1(s) \in \mathcal{NA}(n,p,\lambda,\alpha,\beta)$.

Theorem 4.2. *Let the function k_v defined by (4.1) be in the class $\mathcal{NA}(n,p,\lambda,\alpha,\beta)$ for every $v = 1,2,\dots,l$. Then the function $m_2(s)$ defined as*

$$m_2(s) = \sum_{v=1}^{l} t_v \, k_v(s)$$

is also in the class $\mathcal{NA}(n,p,\lambda,\alpha,\beta)$, where

$$\sum_{v=1}^{l} t_v = 1 \ , \qquad (t_v \geq 0).$$

Proof. Using Theorem 2.1, for every $v = 1,2,\dots,l$, we get

$$\sum_{n=p+1}^{\infty} [n^2(\alpha+1) - (\beta+\alpha p)(1+\lambda n - \lambda)] d_{n,v} \leq p^2(\alpha+1) - (\beta+\alpha p)(1+\lambda p - \lambda).$$

But

$$m_2(s) = \sum_{v=1}^{l} t_v \, k_v(s) = \sum_{v=1}^{l} t_v \left(s^p - \sum_{n=p+1}^{\infty} d_{n,v} \, s^n \right) = s^p - \sum_{n=p+1}^{\infty} \left(\sum_{v=1}^{l} t_v d_{n,v} \right) s^n.$$

Therefore

$$\sum_{n=p+1}^{\infty} [n^2(\alpha+1) - (\beta+\alpha p)(1+\lambda n - \lambda)] \left(\sum_{v=1}^{l} t_v d_{n,v} \right)$$

$$= \sum_{v=1}^{l} t_v \left(\sum_{n=p+1}^{\infty} [n^2(\alpha+1) - (\beta+\alpha p)(1+\lambda n - \lambda)] d_{n,v} \right)$$

$$\leq \sum_{v=1}^{l} t_v \left(p^2(\alpha+1) - (\beta+\alpha p)(1+\lambda p-\lambda)\right)$$

$$= p^2(\alpha+1) - (\beta+\alpha p)(1+\lambda p-\lambda)$$

and the proof is complete.

Corollary 4.1. *The class $\mathcal{NA}(n,p,\lambda,\alpha,\beta)$ is close under convex linear combination.*

5. Extreme Points

We get here an extreme point of the class $\mathcal{NA}(n,p,\lambda,\alpha,\beta)$.

Theorem 5.1. *Let $k_p(s) = s^p$ and*

$$k_n(s) = s^p - \frac{p^2(\alpha+1) - (\beta+\alpha p)(1+\lambda p-\lambda)}{n^2(\alpha+1) - (\beta+\alpha p)(1+\lambda n-\lambda)} s^n, \tag{5.1}$$

where $s \in \Delta, 0 \leq \beta < p, \alpha \geq 0, 0 \leq \lambda \leq 1, n \geq p+1$ and $p \in \mathbb{N} = \{1,2,\ldots\}$.

Then $k(s) \in \mathcal{NA}(n,p,\lambda,\alpha,\beta)$ if and only if it can be expressed as:

$$k(s) = \gamma_p s^p + \sum_{n=p+1}^{\infty} \gamma_n k_n(s), \tag{5.2}$$

where $\left(\gamma_p \geq 0,\ \gamma_n \geq 0,\ n \geq p+1\right)$ and $\gamma_p + \sum_{n=p+1}^{\infty} \gamma_n = 1$.

Proof. Suppose that $k(s)$ is represented in the form (5.2). Then

$$k(s) = \gamma_p s^p + \sum_{n=p+1}^{\infty} \gamma_n \left[s^p - \frac{p^2(\alpha+1) - (\beta+\alpha p)(1+\lambda p-\lambda)}{n^2(\alpha+1) - (\beta+\alpha p)(1+\lambda n-\lambda)} s^n\right]$$

$$= s^p - \sum_{n=p+1}^{\infty} \frac{p^2(\alpha+1) - (\beta+\alpha p)(1+\lambda p-\lambda)}{n^2(\alpha+1) - (\beta+\alpha p)(1+\lambda n-\lambda)} \gamma_n s^n.$$

Hence

$$\sum_{n=p+1}^{\infty} \frac{n^2(\alpha+1) - (\beta+\alpha p)(1+\lambda n-\lambda)}{p^2(\alpha+1) - (\beta+\alpha p)(1+\lambda p-\lambda)}$$

$$\times \frac{p^2(\alpha+1) - (\beta+\alpha p)(1+\lambda p-\lambda)\gamma_n}{n^2(\alpha+1) - (\beta+\alpha p)(1+\lambda n-\lambda)}$$

$$= \sum_{n=p+1}^{\infty} \gamma_n = 1 - \gamma_p \leq 1.$$

Then $k(s) \in \mathcal{NA}(n, p, \lambda, \alpha, \beta)$.

Conversely, suppose that $k(s) \in \mathcal{NA}(n, p, \lambda, \alpha, \beta)$. We may set

$$\gamma_n = \frac{n^2(\alpha+1) - (\beta + \alpha p)(1 + \lambda n - \lambda)}{p^2(\alpha+1) - (\beta + \alpha p)(1 + \lambda p - \lambda)} d_n,$$

where d_n is given by (2.5). Then

$$k(s) = s^p - \sum_{n=p+1}^{\infty} d_n s^n$$

$$= s^p - \sum_{n=p+1}^{\infty} \frac{p^2(\alpha+1) - (\beta + \alpha p)(1 + \lambda p - \lambda)}{n^2(\alpha+1) - (\beta + \alpha p)(1 + \lambda n - \lambda)} \gamma_n s^n$$

$$= s^p - \sum_{n=p+1}^{\infty} [s^p - k_n(s)]\gamma_n = \left(1 - \sum_{n=p+1}^{\infty} \gamma_n\right) s^p + \sum_{n=p+1}^{\infty} \gamma_n k_n(s)$$

$$= \gamma_p s^p + \sum_{n=p+1}^{\infty} \gamma_n k_n(s).$$

This completes the proof of Theorem 5.1.

6. Integral Means Inequalities

By using Theorem 2.1 and Lemma 1.3, we prove the following theorems.

Theorem 6.1. *Let* $\mu > 0$. *If* $k(s) \in \mathcal{NA}(n, p, \lambda, \alpha, \beta)$ *and suppose that* $k_t(s)$ *is defined by*

$$k_t(s) = s^p - \frac{p^2(\alpha+1) - (\beta + \alpha p)(1 + \lambda p - \lambda)}{t^2(\alpha+1) - (\beta + \alpha p)(1 + \lambda t - \lambda)} s^t, \quad (t \geq p + 1; p \in \mathbb{N}).$$

If there is a holomorphic function $w(s)$ *defined by*

$$\left(w(s)\right)^{t-p} = \frac{t^2(\alpha+1) - (\beta + \alpha p)(1 + \lambda t - \lambda)}{p^2(\alpha+1) - (\beta + \alpha p)(1 + \lambda p - \lambda)} \sum_{n=p+1}^{\infty} d_n s^{n-p}.$$

Then, for $s = re^{i\theta}$ *and* $(0 < r < 1)$,

$$\int_0^{2\pi} |k(s)|^\mu d\theta \le \int_0^{2\pi} |k_t(s)|^\mu d\theta, \quad (\mu > 0). \tag{6.1}$$

Proof. We must show that

$$\int_0^{2\pi} \left|1 - \sum_{n=p+1}^{\infty} d_n s^{n-p}\right|^\mu d\theta \le \int_0^{2\pi} \left|1 - \frac{p^2(\alpha+1) - (\beta+\alpha p)(1+\lambda p-\lambda)}{t^2(\alpha+1) - (\beta+\alpha p)(1+\lambda t-\lambda)} s^{t-p}\right|^\mu d\theta.$$

By using Lemma 1.3, it suffices to show that

$$1 - \sum_{n=p+1}^{\infty} d_n s^{n-p} \prec 1 - \frac{p^2(\alpha+1) - (\beta+\alpha p)(1+\lambda p-\lambda)}{t^2(\alpha+1) - (\beta+\alpha p)(1+\lambda t-\lambda)} s^{t-p}.$$

Put

$$1 - \sum_{n=p+1}^{\infty} d_n s^{n-p} = 1 - \frac{p^2(\alpha+1) - (\beta+\alpha p)(1+\lambda p-\lambda)}{t^2(\alpha+1) - (\beta+\alpha p)(1+\lambda t-\lambda)} \big(w(s)\big)^{t-p}.$$

We find that

$$\big(w(s)\big)^{t-p} = \frac{t^2(\alpha+1) - (\beta+\alpha p)(1+\lambda t-\lambda)}{p^2(\alpha+1) - (\beta+\alpha p)(1+\lambda p-\lambda)} \sum_{n=p+1}^{\infty} d_n s^{n-p},$$

that yield easily $w(0) = 0$.

In addition by using (2.1), we get

$$|w(s)|^{t-p} = \left|\frac{t^2(\alpha+1) - (\beta+\alpha p)(1+\lambda t-\lambda)}{p^2(\alpha+1) - (\beta+\alpha p)(1+\lambda p-\lambda)} \sum_{n=p+1}^{\infty} d_n s^{n-p}\right|$$

$$\le |s| \left|\sum_{n=p+1}^{\infty} \frac{n^2(\alpha+1) - (\beta+\alpha p)(1+\lambda n-\lambda)}{p^2(\alpha+1) - (\beta+\alpha p)(1+\lambda p-\lambda)} d_n\right|$$

$$\le |s| < 1.$$

Next, the proof for the first derivative.

Theorem 6.2. *Let* $\mu > 0$. *If* $k(s) \in \mathcal{NA}(n,p,\lambda,\alpha,\beta)$ *and*

$$k_t(s) = s^p - \frac{p^2(\alpha+1)-(\beta+\alpha p)(1+\lambda p-\lambda)}{t^2(\alpha+1)-(\beta+\alpha p)(1+\lambda t-\lambda)} s^t, \quad (t \geq p+1; p \in \mathbb{N}).$$

Then, for $s = re^{i\theta}$ *and* $(0 < r < 1)$,

$$\int_0^{2\pi} |k'(s)|^\mu d\theta \leq \int_0^{2\pi} \left|k_t'(s)\right|^\mu d\theta, \quad (\mu > 0). \tag{6.2}$$

Proof. It is sufficient to show that

$$1 - \sum_{n=p+1}^{\infty} \frac{n}{p} d_n s^{n-p} \prec 1 - \frac{t\big(p^2(\alpha+1)-(\beta+\alpha p)(1+\lambda p-\lambda)\big)}{p\big(t^2(\alpha+1)-(\beta+\alpha p)(1+\lambda t-\lambda)\big)} s^{t-p}.$$

This follows because

$$|w(s)|^{t-p} = \left|\frac{p\big(t^2(\alpha+1)-(\beta+\alpha p)(1+\lambda t-\lambda)\big)}{t\big(p^2(\alpha+1)-(\beta+\alpha p)(1+\lambda p-\lambda)\big)} \sum_{n=p+1}^{\infty} \frac{n}{p} d_n s^{n-p}\right|$$

$$\leq |s| \left| \sum_{n=p+1}^{\infty} \frac{n^2(\alpha+1)-(\beta+\alpha p)(1+\lambda n-\lambda)}{p^2(\alpha+1)-(\beta+\alpha p)(1+\lambda p-\lambda)} d_n \right|$$

$$\leq |s| < 1.$$

Theorem 6.3. *Let* $h(s) = s^p - \sum_{n=p+1}^{\infty} c_n s^n (s \in \Delta; c_n \geq 0;\ n \geq p+1; p \in \mathbb{N} = \{1,2,\dots\})$ *and* $k(s) \in \mathcal{NA}(n,p,\lambda,\alpha,\beta)$ *be of the form* (1.2) *and let for some* $t \in \mathbb{N}$,

$$\frac{Q_t}{c_t} = \min_{n \geq p+1} \frac{Q_n}{c_n},$$

where

$$Q_n = \frac{n^2(\alpha+1)-(\beta+\alpha p)(1+\lambda n-\lambda)}{p^2(\alpha+1)-(\beta+\alpha p)(1+\lambda p-\lambda)}.$$

Also, let for such $t \in \mathbb{N}$, *the functions* k_t *and* h_t *be defined by*

$$k_t(s) = s^p - \frac{p^2(\alpha+1)-(\beta+\alpha p)(1+\lambda p-\lambda)}{t^2(\alpha+1)-(\beta+\alpha p)(1+\lambda t-\lambda)} s^t,$$

$$h_t(s) = s^p - c_t s^t. \tag{6.3}$$

If there is a holomorphic function $w(s)$ defined by

$$\left(w(s)\right)^{t-p} = \frac{t^2(\alpha+1)-(\beta+\alpha p)(1+\lambda t-\lambda)}{\left(p^2(\alpha+1)-(\beta+\alpha p)(1+\lambda p-\lambda)\right)c_t}\sum_{n=p+1}^{\infty} d_n c_n s^{n-p},$$

then, for $\mu > 0, s = re^{i\theta}$ and $(0 < r < 1)$,

$$\int_0^{2\pi} |(k*h)(s)|^{\mu} d\theta \leq \int_0^{2\pi} |(k_t*h_t)(s)|^{\mu} d\theta, \qquad (\mu > 0).$$

Proof. Convolution of k(s) and h(s) is defined by

$$(k*h)(s) = s^p - \sum_{n=p+1}^{\infty} d_n c_n\, s^n.$$

Similarly, from (6.3), we get

$$(k_t*h_t)(s) = s^p - \frac{\left(p^2(\alpha+1)-(\beta+\alpha p)(1+\lambda p-\lambda)\right)c_t}{t^2(\alpha+1)-(\beta+\alpha p)(1+\lambda t-\lambda)} s^t.$$

To prove the theorem, we must show that for $\mu > 0, s = re^{i\theta}$ and $(0 < r < 1)$,

$$\int_0^{2\pi} \left|1 - \sum_{n=p+1}^{\infty} d_n c_n s^{n-p}\right|^{\mu} d\theta$$

$$\leq \int_0^{2\pi} \left|1 - \frac{\left(p^2(\alpha+1)-(\beta+\alpha p)(1+\lambda p-\lambda)\right)c_t}{t^2(\alpha+1)-(\beta+\alpha p)(1+\lambda t-\lambda)} s^{t-p}\right|^{\mu} d\theta.$$

Therefore, using Lemma 1.3, it is sufficient to show that

$$1 - \sum_{n=p+1}^{\infty} d_n c_n s^{n-p} \prec 1 - \frac{\left(p^2(\alpha+1)-(\beta+\alpha p)(1+\lambda p-\lambda)\right)c_t}{t^2(\alpha+1)-(\beta+\alpha p)(1+\lambda t-\lambda)} s^{t-p}. \qquad (6.4)$$

If the subordination (6.4) is correct, then there is a holomorphic function $w(s)$ with $|w(s)| < 1$ and $w(0) = 0$ such that

$$1 - \sum_{n=p+1}^{\infty} d_n c_n s^{n-p} = 1 - \frac{\left(p^2(\alpha+1)-(\beta+\alpha p)(1+\lambda p-\lambda)\right)c_t}{t^2(\alpha+1)-(\beta+\alpha p)(1+\lambda t-\lambda)} \left(w(s)\right)^{t-p}.$$

According to the assumption of the theorem, there is a holomorphic function $w(s)$ given by

$$\left(w(s)\right)^{t-p} = \frac{t^2(\alpha+1)-(\beta+\alpha p)(1+\lambda t-\lambda)}{\left(p^2(\alpha+1)-(\beta+\alpha p)(1+\lambda p-\lambda)\right)c_t}\sum_{n=p+1}^{\infty} d_n c_n s^{n-p},$$

which readily yield $w(0)=0$. So for such function $w(s)$, using the assumption in the coefficient inequality for the class $\mathcal{NA}(n,p,\lambda,\alpha,\beta)$, we have

$$|w(s)|^{t-p} = \left|\frac{t^2(\alpha+1)-(\beta+\alpha p)(1+\lambda t-\lambda)}{\left(p^2(\alpha+1)-(\beta+\alpha p)(1+\lambda p-\lambda)\right)c_t}\sum_{n=p+1}^{\infty} d_n c_n s^{n-p}\right|$$

$$\leq |s|\left|\frac{t^2(\alpha+1)-(\beta+\alpha p)(1+\lambda t-\lambda)}{\left(p^2(\alpha+1)-(\beta+\alpha p)(1+\lambda p-\lambda)\right)c_t}\sum_{n=p+1}^{\infty} d_n c_n\right|$$

$$\leq |s| < 1.$$

Therefore, the subordination (6.4) holds true.

7. Integral Operators

In this segment, we consider integral transforms of functions in the class $\mathcal{NA}(n,p,\lambda,\alpha,\beta)$.

Theorem 7.1. *Let* $k(s)\in\mathcal{NA}(n,p,\lambda,\alpha,\beta)$ *be defined by* (1.2) *and* c *be any real number such that* $c>-p$. *Then the integral operator*

$$G(s) = \frac{c+p}{s^c}\int_0^s t^{c-1}\, k(t)dt(\, c>-p\,), \tag{7.1}$$

also in the class $\mathcal{NA}(n,p,\lambda,\alpha,\beta)$.

Proof. By virtue of (7.1) it follows from (1.2) that

$$G(s) = \frac{c+p}{s^c}\int_0^s t^{c-1}\left(t^p - \sum_{n=p+1}^{\infty} d_n\, t^n\right)dt$$

$$= \frac{c+p}{s^c}\int_0^s \left(t^{p+c-1} - \sum_{n=p+1}^{\infty} d_n\, t^{n+c-1}\right)dt$$

$$= s^p - \sum_{n=p+1}^{\infty} \left(\frac{c+p}{c+n}\right) d_n\, s^n \; = s^p - \sum_{n=p+1}^{\infty} h_n\, s^n,$$

where $h_n = \left(\frac{c+p}{c+n}\right) d_n$.

But

$$\sum_{n=p+1}^{\infty} [n^2(\alpha+1) - (\beta+\alpha p)(1+\lambda n-\lambda)]h_n$$

$$= \sum_{n=p+1}^{\infty} [n^2(\alpha+1) - (\beta+\alpha p)(1+\lambda n-\lambda)] \left(\frac{c+p}{c+n}\right) d_n.$$

Since $\left(\frac{c+p}{c+n}\right) \leq 1$ and by (2.1), the last expression is less than or equal to $p^2(\alpha+1) - (\beta+\alpha p)(1+\lambda p-\lambda)$. This ends the proof.

References

[1] R. Agarwal, J. Gupta and G. S. Paliwal, Geometric properties and neighborhood results for a subclass of analytic functions defined by convolution, *International Bulletin of Mathematical Research* 2(4) (2015), 5-15.

[2] M. K. Aouf, A. O. Mostafa and A. A. Hussain, Certain subclass of p-valent starlike and convex uniformaly functions defined by convolution, *Int. J. Open Problems Compt. Math.* 9(1) (2016), 36-60. https://doi.org/10.12816/0026353

[3] E. S. Aqlan, Some problems connected with geometric function theory, Ph.D. Thesis, Pune University, Pune, 2004.

[4] W. G. Atshan and N. A. J. Al-Ziadi, A new subclass of harmonic univalent functions, *J. Al-Qadisiyah Comput. Sci. Math.* 9(2) (2017), 26-32. https://doi.org/10.29304/jqcm.2017.9.2.140

[5] W. G. Atshan, On a certain class of multivalent functions defined by Hadamard product, *Journal of Asian Scientific Research* 3(9) (2013), 891-902.

[6] R. Bharati, R. Parvatham and A. Swaminathan, On subclasses of uniformly convex functions and corresponding class of starlike functions, *Tamkang J. Math.* 28(1) (1997), 17-32. https://doi.org/10.5556/j.tkjm.28.1997.4330

[7] S. H. Hadi, M. Darus, C. Part and J. R. Lee, Some geometric properties of multivalent functions associated with a new generalized q-Mittag-Leffler function, *AIMS Mathematics* 7(7) (2022), 11772–11783. https://doi.org/10.3934/math.2022656

[8] S. M. Khaimar and M. More, On a subclass of multivalent β-uniformly starlike and convex functions defined by a linear operator, *LAENG Internat. J. Appl. Math.* 39(3) (2009), Article ID IJAM_39_06.

[9] A. Y. Lashin, On a certain subclass of starlike functions with negative coefficients, *J. Ineq. Pure Appl. Math.* 10(2) (2009), 1-8.

[10] J. E. Littlewood, On inequalities in the theory of functions, *Proc. London Math. Soc.* 23(2) (1925), 481-519. https://doi.org/10.1112/plms/s2-23.1.481

[11] S. Owa, The quasi-Hadamard products of certain analytic functions, in *Current Topics in Analytic Function Theory*, H. M. Srivastava and S. Owa, (Editors), World Scientific Publishing Company, Singapore, New Jersey, London, and Hong Kong, 1992. https://doi.org/10.1142/9789814355896_0019

[12] G. S. Sălăgean, H. M. Hossen and M. K. Aouf, On certain class of p-valent functions with negative coefficients. II, *Studia Univ. Babes-Bolyai* 69(1) (2004), 77-85.

[13] H. Silverman, Univalent functions with negative coefficients, *Proc. Amer. Math. Soc.* 51 (1975), 109-116. https://doi.org/10.1090/S0002-9939-1975-0369678-0

[14] A. K. Wanas and H. M. Ahsoni, Some geometric properties for a class of analytic functions defined by beta negative binomial distribution series, *Earthline Journal of Mathematical Sciences* 9(1) (2022), 105-116. https://doi.org/10.34198/ejms.9122.105116