Earthline Journal of Mathematical Sciences
E-ISSN: 2581-8147
Volume 10, Number 2, 2022, Pages 227-239
https://doi.org/10.34198/ejms.10222.227239

Coefficient Bounds for New Subclasses of m-Fold Symmetric Holomorphic Bi-Univalent Functions

Ali Mohammed Ramadhan[1] and Najah Ali Jiben Al-Ziadi[2,*]

[1] Department of Mathematics, College of Education, University of Al-Qadisiyah, Diwaniya, Iraq
e-mail: edu-math.post15@qu.edu.iq

[2] Department of Mathematics, College of Education, University of Al-Qadisiyah, Diwaniya, Iraq
e-mail: najah.ali@qu.edu.iq

Abstract

In the present paper, we investigate two new subclasses $\mathcal{AR}_{\Sigma_m}(\delta,\lambda;\alpha)$ and $\mathcal{AR}_{\Sigma_m}(\delta,\lambda;\beta)$ of Σ_m consisting of m-fold symmetric holomorphic bi-univalent functions in the open unit disk Δ. For functions from the two classes described here, we obtain estimates on the initial bounds $|d_{m+1}|$ and $|d_{2m+1}|$. In addition, we get new special cases for our results.

1. Introduction

Let Δ be the unit disk $\{s : s \in \mathbb{C} \text{ and } |s| < 1\}$, symbolized by $\mathcal{A}$ the class of functions holomorphic in Δ, fulfilling the condition

$$k(0) = k'(0) - 1 = 0.$$

Then each function k in $\mathcal{A}$ has the Taylor-Maclaurin expansion

$$k(s) = s + \sum_{n=2}^{\infty} d_n s^n. \tag{1.1}$$

Further, by $\mathcal{S}$ we shall symbolize the class of all functions in $\mathcal{A}$ which are univalent in Δ.

The Koebe one-quarter theorem [4] shows that the image of Δ under every function k from $\mathcal{S}$ contains a disk of radius 1/4. As a result, any univalent function has an inverse

Received: June 17, 2022; Accepted: June 30, 2022

2020 Mathematics Subject Classification: 30C45, 30C50.

Keywords and phrases: holomorphic function, m-fold symmetric holomorphic function, bi-univalent function, m-fold symmetric holomorphic bi-univalent function, coefficient bounds.

*Corresponding author

k^{-1} which fulfilled

$$k^{-1}\big(k(s)\big) = s \qquad (s \in \Delta)$$

and

$$k\big(k^{-1}(r)\big) = r \quad \left(|r| < r_0\,(k);\; r_0(k) \geq \frac{1}{4}\right),$$

where

$$k^{-1}(r) = h(r) = r - d_2 r^2 + (2d_2^2 - d_3)r^3 - (5d_2^3 - 5d_2 d_3 + d_4)r^4 + \cdots. \quad (1.2)$$

A function $k \in \mathcal{A}$ is said to be bi-univalent in Δ if both $k(s)$ and $k^{-1}(s)$ are univalent in Δ. We symbolize by Σ the class of all bi-univalent functions in Δ given by the Taylor-Maclaurin series expansion (1.1). Lewin [7] discussed the class of bi-univalent functions Σ and gotten a bound $|d_2| \leq 1.51$. Motivated by the work of Lewin [7], Brannan and Clunie [3] hypothesised that $|d_2| \leq \sqrt{2}$. Some examples of bi-univalent functions are $\frac{s}{1-s}$, $-log(1-s)$ and $\frac{1}{2}\log\left(\frac{1+s}{1-s}\right)$ (see also Srivastava et al. [15]). The coefficient estimate problem involving the bound of $|d_n|$ $\big(n \in \mathbb{N}\backslash\{1,2\}\big)$ for each $f \in \Sigma$ is still an open problem [15].

For each function $f \in \mathcal{S}$, the function

$$g(s) = \sqrt[m]{k(s^m)} \quad \big(s \in \Delta, m \in \mathbb{N}\big) \quad (1.3)$$

is univalent and maps the unit disk Δ into a region with m-fold symmetry. A function is told to be m-fold symmetric (see [5, 10]) if it has the following normalized form:

$$k(s) = s + \sum_{n=1}^{\infty} d_{mn+1} s^{mn+1} \quad \big(s \in \Delta, m \in \mathbb{N}\big). \quad (1.4)$$

We symbolize by $\mathcal{S}_m$ the class of m-fold symmetric holomorphic univalent functions in Δ, which are normalized by the series expansion (1.4). In fact, the functions in the class $\mathcal{S}$ are one-fold symmetric (that is, $m = 1$).

In [16] Srivastava et al. defined m-fold symmetric holomorphic bi-univalent function analogues to the concept of m-fold symmetric holomorphic univalent functions. They gave some important results, such as each function $k \in \Sigma$ generates an m-fold symmetric holomorphic bi-univalent function for each $m \in \mathbb{N}$, in their study. Furthermore, for the normalized form of k given by (1.4), they obtained the series expansion for k^{-1} as follows:

$$k^{-1}(r) = h(r) = r - d_{m+1}r^{m+1} + [(m+1)d_{m+1}^2 - d_{2m+1}]r^{2m+1} - \left[\frac{1}{2}(m+1)(3m+2)d_{m+1}^3 - (3m+2)d_{m+1}d_{2m+1} + d_{3m+1}\right]r^{3m+1} + \cdots, \quad (1.5)$$

where $k^{-1} = h$. We denote by Σ_m the class of m-fold symmetric holomorphic bi-univalent functions in Δ. For $m = 1$, formula (1.5) coincides with formula (1.2) of the class Σ.

Recently, many penmen investigated bounds for various subclasses of m-fold symmetric holomorphic bi-univalent functions (see [1, 2, 6, 11, 12, 13, 14, 17]).

The aim of this paper is to submit two new subclasses of the function class bi-univalent functions in which both k and k^{-1} are m-fold symmetric holomorphic bi-univalent functions and derive bounds on initial coefficients $|d_{m+1}|$ and $|d_{2m+1}|$ for functions in each of these new subclasses. Several related classes are also investigated and connections to earlier known outcomes are made.

We employ the following lemma [4] of Caratheodary class to deduce our main results.

Lemma 1.1. *If* $p \in \mathcal{P}$, *then* $|c_n| \le 2$ *for each* $n \in \mathbb{N}$, *where* $\mathcal{P}$ *is the family of all functions* p, *holomorphic in* Δ, *for which*

$$R(p(s)) > 0 \quad where \quad p(s) = 1 + c_1 s + c_2 s^2 + \cdots \quad (s \in \Delta).$$

2. Coefficient Bounds for the Function Class $\mathcal{AR}_{\Sigma_m}(\delta, \lambda; \alpha)$

Definition 2.1. A function $k(s) \in \Sigma_m$ given by (1.4) is told to be in the class $\mathcal{AR}_{\Sigma_m}(\delta, \lambda; \alpha)$ if the following conditions are fulfilled:

$$\left|arg\left[(1-\delta)\left(\frac{sk'(s)}{\lambda sk'(s) + (1-\lambda)k(s)}\right) + \delta\left(\frac{sk''(s) + k'(s)}{\lambda sk''(s) + k'(s)}\right)\right]\right| < \frac{\alpha\pi}{2} (s \in \Delta) \quad (2.1)$$

and

$$\left|arg\left[(1-\delta)\left(\frac{rh'(r)}{\lambda rh'(r) + (1-\lambda)h(r)}\right) + \delta\left(\frac{rh''(r) + h'(r)}{\lambda rh''(r) + h'(r)}\right)\right]\right| < \frac{\alpha\pi}{2} (r \in \Delta), \quad (2.2)$$

where the function $h = k^{-1}$ is given by (1.5) and $(0 \le \delta \le 1; 0 \le \lambda < 1;\ 0 < \alpha \le 1)$.

Theorem 2.1. *Let the function* $k(s)$*, given by (1.4), be in the class* $\mathcal{AR}_{\Sigma_m}(\delta, \lambda; \alpha)$*. Then*

$$|d_{m+1}| \leq \frac{2\alpha}{m\sqrt{|2\alpha(1-\lambda)(1-\lambda+\delta m-2\delta\lambda m-\delta\lambda m^2)+(1-\alpha)(1-\lambda)^2(1+\delta m)^2|}} \quad (2.3)$$

and

$$|d_{2m+1}| \leq \frac{2\alpha^2(m+1)}{m^2(1-\lambda)^2(1+\delta m)^2} + \frac{\alpha}{m(1-\lambda)(1+2\delta m)}. \quad (2.4)$$

Proof. Let $k \in \mathcal{AR}_{\Sigma_m}(\delta, \lambda; \alpha)$. Then

$$(1-\delta)\left(\frac{sk'(s)}{\lambda sk'(s)+(1-\lambda)k(s)}\right) + \delta\left(\frac{sk''(s)+k'(s)}{\lambda sk''(s)+k'(s)}\right) = [p(s)]^\alpha \quad (2.5)$$

and

$$(1-\delta)\left(\frac{rh'(r)}{\lambda rh'(r)+(1-\lambda)h(r)}\right) + \delta\left(\frac{rh''(r)+h'(r)}{\lambda rh''(r)+h'(r)}\right) = [q(r)]^\alpha, \quad (2.6)$$

where $h = k^{-1}$ and $p(s)$, $q(r)$ in $\mathcal{P}$ and have the forms

$$p(s) = 1 + p_m s^m + p_{2m} s^{2m} + p_{3m} s^{3m} + \cdots \quad (2.7)$$

and

$$q(r) = 1 + q_m r^m + q_{2m} r^{2m} + q_{3m} r^{3m} + \cdots. \quad (2.8)$$

It follows from (2.5) and (2.6) that

$$m(1-\lambda)(1+\delta m)d_{m+1} = \alpha p_m, \quad (2.9)$$

$$2m(1-\lambda)(1+2\delta m)d_{2m+1} - m(1-\lambda)(1+\lambda m)(1+2\delta m+\delta m^2)d_{m+1}^2$$

$$= \alpha p_{2m} + \frac{1}{2}\alpha(\alpha-1)p_m^2, \quad (2.10)$$

$$-m(1-\lambda)(1+\delta m)d_{m+1} = \alpha q_m, \quad (2.11)$$

and

$$m(1-\lambda)(1+2m-\lambda m+2\delta m+3\delta m^2-2\delta\lambda m^2-\delta\lambda m^3)d_{m+1}^2$$

$$-2m(1-\lambda)(1+2\delta m)d_{2m+1}$$

$$= \alpha q_{2m} + \frac{1}{2}\alpha(\alpha-1)q_m^2. \quad (2.12)$$

From (2.9) and (2.11), we get

$$p_m = -q_m \tag{2.13}$$

and

$$2m^2(1-\lambda)^2(1+\delta m)^2 d_{m+1}^2 = \alpha^2(p_m^2 + q_m^2). \tag{2.14}$$

From (2.10), (2.12) and (2.14), we find

$$\begin{aligned} &2m^2(1-\lambda)(1-\lambda+\delta m-2\delta\lambda m-\delta\lambda m^2)d_{m+1}^2 \\ &= \alpha(p_{2m}+q_{2m}) + \frac{\alpha(\alpha-1)}{2}(p_m^2+q_m^2) \\ &= \alpha(p_{2m}+q_{2m}) + \frac{\alpha(\alpha-1)}{2}(m(1-\lambda)(1+\delta m))^2 d_{m+1}^2. \end{aligned} \tag{2.15}$$

Therefore, we have

$$d_{m+1}^2 = \frac{\alpha^2(p_{2m}+q_{2m})}{2\alpha m^2(1-\lambda)(1-\lambda+\delta m-2\delta\lambda m-\delta\lambda m^2)+(1-\alpha)m^2(1-\lambda)^2(1+\delta m)^2}. \tag{2.16}$$

Stratifying Lemma 1.1 for the coefficients p_{2m} and q_{2m}, we get

$$|d_{m+1}| \le \frac{2\alpha}{m\sqrt{|2\alpha(1-\lambda)(1-\lambda+\delta m-2\delta\lambda m-\delta\lambda m^2)+(1-\alpha)(1-\lambda)^2(1+\delta m)^2|}}. \tag{2.17}$$

The last inequality gives the desired estimate on $|d_{m+1}|$ given in (2.3).

Next, the bound on $|d_{2m+1}|$ is then found by subtracting (2.12) from (2.10).

$$\begin{aligned} &4m(1-\lambda)(1+2\delta m)d_{2m+1} - \big(2m(m+1)(1-\lambda)(1+2\delta m)\big)d_{m+1}^2 \\ &= \alpha(p_{2m}-q_{2m}) + \frac{\alpha(\alpha-1)}{2}(p_m^2-q_m^2) \end{aligned} \tag{2.18}$$

By using (2.13), (2.14) and (2.18), we get

$$d_{2m+1} = \frac{\alpha^2(m+1)(p_m^2+q_m^2)}{4m^2(1-\lambda)^2(1+\delta m)^2} + \frac{\alpha(p_{2m}-q_{2m})}{4m(1-\lambda)(1+2\delta m)}. \tag{2.19}$$

Stratifying Lemma 1.1 once again for the coefficients p_m, p_{2m} and q_m, q_{2m}, we get

$$|d_{2m+1}| \le \frac{2\alpha^2(m+1)}{m^2(1-\lambda)^2(1+\delta m)^2} + \frac{\alpha}{m(1-\lambda)(1+2\delta m)}.$$

This proves Theorem 2.1.

3. Coefficient Bounds for the Function Class $\mathcal{AR}_{\Sigma_m}(\delta, \lambda; \beta)$

Definition 3.1. A function $k(s) \in \Sigma_m$ given by (1.4) is told to be in the class $\mathcal{AR}_{\Sigma_m}(\delta, \lambda; \beta)$ if the following conditions are fulfilled:

$$Re\left((1-\delta)\left(\frac{sk'(s)}{\lambda sk'(s)+(1-\lambda)k(s)}\right)+\delta\left(\frac{sk''(s)+k'(s)}{\lambda sk''(s)+k'(s)}\right)\right) > \beta \ (s \in \Delta) \quad (3.1)$$

and

$$Re\left((1-\delta)\left(\frac{rh'(r)}{\lambda rh'(r)+(1-\lambda)h(r)}\right)+\delta\left(\frac{rh''(r)+h'(r)}{\lambda rh''(r)+h'(r)}\right)\right) > \beta \ \ (r \in \Delta), \quad (3.2)$$

where the function $h = k^{-1}$ is given by (1.5) and $(0 \leq \delta \leq 1; 0 \leq \lambda < 1; \ 0 \leq \beta < 1)$.

Theorem 3.1. *Let the function $k(s)$, given by (1.4), be in the class $\mathcal{AR}_{\Sigma_m}(\delta, \lambda; \beta)$. Then*

$$|d_{m+1}| \leq \frac{1}{m}\sqrt{\frac{2(1-\beta)}{|(1-\lambda)(1-\lambda+\delta m-2\delta\lambda m-\delta\lambda m^2)|}} \quad (3.3)$$

and

$$|d_{2m+1}| \leq \frac{2(1-\beta)^2(m+1)}{m^2(1-\lambda)^2(1+\delta m)^2}+\frac{(1-\beta)}{m(1-\lambda)(1+2\delta m)}. \quad (3.4)$$

Proof. It follows from (3.1) and (3.2) that there exists $p, q \in \mathcal{P}$ such that

$$(1-\delta)\left(\frac{sk'(s)}{\lambda sk'(s)+(1-\lambda)k(s)}\right)+\delta\left(\frac{sk''(s)+k'(s)}{\lambda sk''(s)+k'(s)}\right) = \beta+(1-\beta)p(s) \quad (3.5)$$

and

$$(1-\delta)\left(\frac{rh'(r)}{\lambda rh'(r)+(1-\lambda)h(r)}\right)+\delta\left(\frac{rh''(r)+h'(r)}{\lambda rh''(r)+h'(r)}\right) = \beta+\ (1-\beta)q(r), \quad (3.6)$$

where $p(s)$ and $q(r)$ have the forms (2.7) and (2.8). It follows from (3.5) and (3.6), we find

$$m(1-\lambda)(1+\delta m)d_{m+1} = (1-\beta)p_m, \quad (3.7)$$

$$2m(1-\lambda)(1+2\delta m)d_{2m+1} - m(1-\lambda)(1+\lambda m)(1+2\delta m+\delta m^2)d_{m+1}^2$$
$$= (1-\beta)p_{2m}, \quad (3.8)$$

$$-m(1-\lambda)(1+\delta m)d_{m+1}=(1-\beta)q_m, \tag{3.9}$$

and

$$m(1-\lambda)(1+2m-\lambda m+2\delta m+3\delta m^2-2\delta\lambda m^2-\delta\lambda m^3)d_{m+1}^2$$
$$-2m(1-\lambda)(1+2\delta m)d_{2m+1}$$
$$=(1-\beta)q_{2m}. \tag{3.10}$$

From (3.7) and (3.9), we get

$$p_m=-q_m \tag{3.11}$$

and

$$2m^2(1-\lambda)^2(1+\delta m)^2d_{m+1}^2=(1-\beta)^2(p_m^2+q_m^2). \tag{3.12}$$

Adding (3.8) and (3.10), we get

$$2m^2(1-\lambda)(1-\lambda+\delta m-2\delta\lambda m-\delta\lambda m^2)d_{m+1}^2=(1-\beta)(p_{2m}+q_{2m}). \tag{3.13}$$

Therefore, we obtain

$$d_{m+1}^2=\frac{(1-\beta)(p_{2m}+q_{2m})}{2m^2(1-\lambda)(1-\lambda+\delta m-2\delta\lambda m-\delta\lambda m^2)}. \tag{3.14}$$

Stratifying Lemma 1.1 for coefficients p_{2m} and q_{2m}, we readily get

$$|d_{m+1}|\le\frac{1}{m}\sqrt{\frac{2(1-\beta)}{|(1-\lambda)(1-\lambda+\delta m-2\delta\lambda m-\delta\lambda m^2)|}}.$$

This gives the desired estimate on $|d_{m+1}|$ given by (3.3).

Next, the bound on $|d_{2m+1}|$ is then found by subtracting (3.10) from (3.8).

$$4m(1-\lambda)(1+2\delta m)d_{2m+1}-\big(2m(m+1)(1-\lambda)(1+2\delta m)\big)d_{m+1}^2$$
$$=(1-\beta)(p_{2m}-q_{2m}). \tag{3.15}$$

Or equivalently

$$d_{2m+1}=\frac{(m+1)}{2}d_{m+1}^2+\frac{(1-\beta)(p_{2m}-q_{2m})}{4m(1-\lambda)(1+2\delta m)}. \tag{3.16}$$

By substituting the value of d_{m+1}^2 from (3.12), we find

$$d_{2m+1} = \frac{(1-\beta)^2(m+1)(p_m^2+q_m^2)}{4m^2(1-\lambda)^2(1+\delta m)^2} + \frac{(1-\beta)(p_{2m}-q_{2m})}{4m(1-\lambda)(1+2\delta m)}. \tag{3.17}$$

By using Lemma 1.1 once again for the coefficients p_{2m}, p_m, q_{2m} *and* q_m, we get

$$|d_{2m+1}| \leq \frac{2(1-\beta)^2(m+1)}{m^2(1-\lambda)^2(1+\delta m)^2} + \frac{(1-\beta)}{m(1-\lambda)(1+2\delta m)}.$$

This proves Theorem 3.1.

4. Corollaries and Consequences

If we set $\delta = 1$ in Definition 2.1 and Definition 3.1, then the classes $\mathcal{AR}_{\Sigma_m}(\delta,\lambda;\alpha)$ and $\mathcal{AR}_{\Sigma_m}(\delta,\lambda;\beta)$ shorten to the classes $\mathcal{AR}_{\Sigma_m}(\lambda;\alpha)$ and $\mathcal{AR}_{\Sigma_m}(\lambda;\beta)$ and thus, Theorem 2.1 and Theorem 3.1 shorten to Corollary 4.1 and Corollary 4.2, respectively.

The classes $\mathcal{AR}_{\Sigma_m}(\lambda;\alpha)$ and $\mathcal{AR}_{\Sigma_m}(\lambda;\beta)$ are respectively defined as follows:

Definition 4.1. A function $k(s) \in \Sigma_m$ given by (1.4) is told to be in the class $\mathcal{AR}_{\Sigma_m}(\lambda;\alpha)$ if the following conditions are fulfilled:

$$\left|arg\left(\frac{sk''(s)+k'(s)}{\lambda sk''(s)+k'(s)}\right)\right| < \frac{\alpha\pi}{2} \quad \text{and} \quad \left|arg\left(\frac{rh''(r)+h'(r)}{\lambda rh''(r)+h'(r)}\right)\right| < \frac{\alpha\pi}{2} \quad (s,r \in \Delta),$$

where the function $h = k^{-1}$ is given by (1.5) and $(0 \leq \lambda < 1;\ 0 < \alpha \leq 1)$.

Definition 4.2. A function $k(s) \in \Sigma_m$ given by (1.4) is told to be in the class $\mathcal{AR}_{\Sigma_m}(\lambda;\beta)$ if the following conditions are fulfilled:

$$Re\left(\frac{sk''(s)+k'(s)}{\lambda sk''(s)+k'(s)}\right) > \beta \quad \text{and} \quad Re\left(\frac{rh''(r)+h'(r)}{\lambda rh''(r)+h'(r)}\right) > \beta \qquad (s,r \in \Delta),$$

where the function $h = k^{-1}$ is given by (1.5) and $(0 \leq \lambda < 1;\ 0 \leq \beta < 1)$.

Corollary 4.1. *Let* $k(s)$ *given by* (1.4) *be in the class* $\mathcal{AR}_{\Sigma_m}(\lambda;\alpha)$. *Then*

$$|d_{m+1}| \leq \frac{2\alpha}{m\sqrt{|2\alpha(1-\lambda)[(m+1)-\lambda(m+1)^2]+(1-\alpha)(1-\lambda)^2(m+1)^2|}}$$

and

$$|d_{2m+1}| \leq \frac{2\alpha^2}{m^2(1-\lambda)^2(m+1)} + \frac{\alpha}{m(1-\lambda)(1+2m)}.$$

Corollary 4.2. *Let* $k(s)$ *given by* (1.4) *be in the class* $\mathcal{AR}_{\Sigma_m}(\lambda;\beta)$. *Then*

$$|d_{m+1}| \leq \frac{1}{m}\sqrt{\frac{2(1-\beta)}{|(1-\lambda)[(m+1)-\lambda(m+1)^2]|}}$$

and

$$|d_{2m+1}| \leq \frac{2(1-\beta)^2}{m^2(1-\lambda)^2(m+1)} + \frac{(1-\beta)}{m(1-\lambda)(1+2m)}.$$

For one-fold symmetric holomorphic bi-univalent functions, the classes $\mathcal{AR}_{\Sigma_m}(\delta,\lambda;\alpha)$ and $\mathcal{AR}_{\Sigma_m}(\delta,\lambda;\beta)$ shorten to the classes $\mathcal{AR}_{\Sigma}(\delta,\lambda;\alpha)$ and $\mathcal{AR}_{\Sigma}(\delta,\lambda;\beta)$ and thus, Theorem 2.1 and Theorem 3.1 shorten to Corollary 4.2 and Corollary 4.3, respectively.

The classes $\mathcal{AR}_{\Sigma}(\delta,\lambda;\alpha)$ and $\mathcal{AR}_{\Sigma}(\delta,\lambda;\beta)$ are defined in the following way:

Definition 4.3. A function $k(s) \in \Sigma$ given by (1.1) is told to be in the class $\mathcal{AR}_{\Sigma}(\delta,\lambda;\alpha)$ if the following conditions are fulfilled:

$$\left|arg\left[(1-\delta)\left(\frac{rk'(s)}{\lambda sk'(s)+(1-\lambda)k(s)}\right)+\delta\left(\frac{sk''(s)+k'(s)}{\lambda sk''(s)+k'(s)}\right)\right]\right| < \frac{\alpha\pi}{2} \quad (s \in \Delta)$$

and

$$\left|arg\left[(1-\delta)\left(\frac{rh'(r)}{\lambda rh'(r)+(1-\lambda)h(r)}\right)+\delta\left(\frac{rh''(r)+h'(r)}{\lambda rh''(r)+h'(r)}\right)\right]\right| < \frac{\alpha\pi}{2} \quad (r \in \Delta),$$

where the function $h = k^{-1}$ is given by (1.2) and $(0 \leq \delta \leq 1; 0 \leq \lambda < 1;\ 0 < \alpha \leq 1)$.

Definition 4.4. A function $k(s) \in \Sigma$ given by (1.1) is told to be in the class $\mathcal{AR}_{\Sigma}(\delta,\lambda;\beta)$ if the following conditions are fulfilled:

$$Re\left((1-\delta)\left(\frac{sk'(s)}{\lambda sk'(s)+(1-\lambda)k(s)}\right)+\delta\left(\frac{sk''(s)+k'(s)}{\lambda sk''(s)+k'(s)}\right)\right) > \beta \qquad (s \in \Delta)$$

and

$$Re\left((1-\delta)\left(\frac{rh'(r)}{\lambda rh'(r)+(1-\lambda)h(r)}\right)+\delta\left(\frac{rh''(r)+h'(r)}{\lambda rh''(r)+h'(r)}\right)\right) > \beta \qquad (r \in \Delta),$$

where the function $h = k^{-1}$ is given by (1.2) and $(0 \leq \delta \leq 1; 0 \leq \lambda < 1;\ 0 \leq \beta < 1)$.

Corollary 4.3. *Let* $k(s)$ *given by (1.1) be in the class* $\mathcal{AR}_{\Sigma}(\delta,\lambda;\alpha)$. *Then*

$$|d_2| \leq \frac{2\alpha}{\sqrt{|2\alpha(1-\lambda)((1-\lambda)(1+\delta)-2\delta\lambda)+(1-\alpha)(1-\lambda)^2(1+\delta)^2|}}$$

and

$$|d_3| \leq \frac{4\alpha^2}{(1-\lambda)^2(1+\delta)^2}+\frac{\alpha}{(1-\lambda)(1+2\delta)}.$$

Corollary 4.4. *Let* $k(s)$ *given by (1.1) be in the class* $\mathcal{AR}_{\Sigma}(\delta,\lambda;\beta)$. *Then*

$$|d_2| \leq \sqrt{\frac{2(1-\beta)}{|(1-\lambda)[(1-\lambda)(1+\delta)-2\delta\lambda\,]|}}$$

and

$$|d_3| \leq \frac{4(1-\beta)^2}{(1-\lambda)^2(1+\delta)^2}+\frac{(1-\beta)}{(1-\lambda)(1+2\delta)}.$$

If we set $\delta = 1$ and $m = 1$ in Definition 2.1 and Definition 3.1, then the classes $\mathcal{AR}_{\Sigma_m}(\delta,\lambda;\alpha)$ and $\mathcal{AR}_{\Sigma_m}(\delta,\lambda;\beta)$ shorten to the classes $\mathcal{AR}_{\Sigma}(\lambda;\alpha)$ and $\mathcal{AR}_{\Sigma}(\lambda;\beta)$ and thus, Theorem 2.1 and Theorem 3.1 shorten to Corollary 4.5 and Corollary 4.6, respectively.

The classes $\mathcal{AR}_{\Sigma}(\lambda;\alpha)$ and $\mathcal{AR}_{\Sigma}(\lambda;\beta)$, are respectively defined as follows:

Definition 4.5. A function $k(s) \in \Sigma$ given by (1.1) is told to be in the class $\mathcal{AR}_{\Sigma}(\lambda;\alpha)$ if the following conditions are fulfilled:

$$\left|arg\left(\frac{sk''(s)+k'(s)}{\lambda sk''(s)+k'(s)}\right)\right| < \frac{\alpha\pi}{2} \quad \text{and} \quad \left|arg\left(\frac{rh''(r)+h'(r)}{\lambda rh''(r)+h'(r)}\right)\right| < \frac{\alpha\pi}{2} \quad (s,r \in \Delta),$$

where the function $h = k^{-1}$ is given by (1.2) and $(0 \leq \lambda < 1;\ 0 < \alpha \leq 1)$.

Definition 4.6. A function $k(s) \in \Sigma$ given by (1.1) is told to be in the class $\mathcal{AR}_{\Sigma}(\lambda;\beta)$ if the following conditions are fulfilled:

$$Re\left(\frac{sk''(s)+k'(s)}{\lambda sk''(s)+k'(s)}\right) > \beta \quad \text{and} \quad Re\left(\frac{rh''(r)+h'(r)}{\lambda rh''(r)+h'(r)}\right) > \beta \qquad (s,r \in \Delta),$$

where the function $h = k^{-1}$ is given by (1.2) and $(0 \leq \lambda < 1;\ 0 \leq \beta < 1)$.

Corollary 4.5. *Let* $k(s)$ *given by (1.1) be in the class* $\mathcal{AR}_{\Sigma}(\lambda;\alpha)$. *Then*

$$|d_2| \leq \frac{\alpha}{\sqrt{|\alpha(1-\lambda)(1-2\lambda)+(1-\alpha)(1-\lambda)^2|}} \quad and \quad |d_3| \leq \frac{\alpha^2}{(1-\lambda)^2}+\frac{\alpha}{3(1-\lambda)}.$$

Corollary 4.6. *Let* $k(s)$ *given by (1.1) be in the class* $\mathcal{AR}_{\Sigma}(\lambda;\beta)$. *Then*

$$|d_2| \leq \sqrt{\frac{(1-\beta)}{|(1-\lambda)(1-2\lambda)|}} \quad and \quad |d_3| \leq \frac{(1-\beta)^2}{(1-\lambda)^2}+\frac{(1-\beta)}{3(1-\lambda)}.$$

Remark 4.1. For m-fold symmetric holomorphic bi-univalent functions:

1. Putting $\delta = 0$, in Theorems 2.1 and 3.1, we get the corresponding outcomes given by Altinkaya and Yalçin [1].

2. Putting $\delta = 0$ and $\lambda = 0$, in Theorems 2.1 and 3.1, we get the corresponding outcomes given by Altinkaya and Yalçin [1].

3. Putting $\delta = 1$ and $\lambda = 0$, in Theorems 2.1 and 3.1, we get the corresponding outcomes given by Kumar et al. [6].

4. Putting $\lambda = 0$, in Theorems 2.1 and 3.1, we get the corresponding outcomes given by Sivasubramanlan and Sivakumar [12].

Remark 4.2. For 1-fold symmetric holomorphic bi-univalent functions:

1. Putting $\delta = 1$ and $\lambda = 0$, in Theorems 2.1 and 3.1, we get the corresponding outcomes given by Kumar et al. [6].

2. Putting $\delta = 0$ and $\lambda = 0$, in Theorems 2.1 and 3.1, we get the corresponding outcomes given by Murugusundaramoorthy et al. [9].

3. Putting $\delta = 0$, in Theorems 2.1 and 3.1, we get the corresponding outcomes given by Murugusundaramoorthy et al. [9].

4. Putting $\lambda = 0$, in Theorems 2.1 and 3.1, we get the corresponding outcomes given by Li and Wang [8].

References

[1] Ş. Altinkaya and S. Yalçin, Coefficient bounds for certain subclasses of m-fold symmetric bi-univalent functions, *Journal of Mathematics* 2015 (2015), Article ID 241683, 5 pp. https://doi.org/10.1155/2015/241683

[2] W. G. Atshan and N. A. J. Al-Ziadi, Coefficients bounds for a general subclasses of m-fold symmetric bi-univalent functions, *J. Al-Qadisiyah Comput. Sci. Math.* 9(2) (2017), 33-39. https://doi.org/10.29304/jqcm.2017.9.2.141

[3] D. Brannan and J. G. Clunie (Eds), Aspects of contemporary complex analysis, (Proceedings of the NATO advanced study institute held at the Univ. of Durham, Durham; July 1-20, 1979), New York, London: Academic Press, 1980.

[4] P. L. Duren, *Univalent Functions*, Vol. 259 of Grundlehren der Mathematischen Wissenschaften, Springer, New York, NY, USA, 1983.

[5] W. Koepf, Coefficients of symmetric functions of bounded boundary rotations, *Proc. Amer. Math. Soc.* 105 (1989), 324-329. https://doi.org/10.1090/S0002-9939-1989-0930244-7

[6] T. R. K. Kumar, S. Karthikeyan, S. Vijayakumar and G. Ganapathy, Initial coefficient estimates for certain subclasses of m-fold symmetric bi-univalent functions, *Advances in Dynamical Systems and Applications* 16(2) (2021), 789-800.

[7] M. Lewin, On a coefficient problem for bi-univalent functions, *Proc. Amer. Math. Soc.* 18 (1967), 63-68. https://doi.org/10.1090/S0002-9939-1967-0206255-1

[8] X. Li and A. Wang, Two new subclasses of bi-univalent functions, *International Math. Forum* 7(30) (2012), 1495-1504.

[9] G. Murugusundaramoorthy, N. Magesh and V. Prameela, Coefficient bounds for certain subclasses of bi-univalent functions, *Abstract and Applied Analysis* 2013 (2013), Article ID 573017, 3 pp. https://doi.org/10.1155/2013/573017

[10] C. Pommerenke, On the coefficients of close-to-convex functions, *Michigan Math. J.* 9 (1962), 259-269. https://doi.org/10.1307/mmj/1028998726

[11] T. G. Shaba and A. B. Patil, Coefficient estimates for certain subclasses of m-fold symmetric bi-univalent functions associated with pseudu-starlike functions, *Earthline Journal of Mathematical Sciences* 6(2) (2021), 209-223. https://doi.org/10.34198/ejms.6221.209223

[12] S. Sivasubramanian and R. Sivakumar, Initial coefficient bound for m-fold symmetric bi-λ-convex functions, *J. Math. Inequalities* 10(3) (2016), 783-791. https://doi.org/10.7153/jmi-10-63

[13] H. M. Srivastava, S. Gaboury and F. Ghanim, Coefficient estimates for some subclasses of m-fold symmetric bi-univalent functions, *Acta Universitatis Apulensis* 41 (2015), 153-164. https://doi.org/10.17114/j.aua.2015.41.12

[14] H. M. Srivastava, S. Gaboury and F. Ghanim, Initial coefficient estimates for some subclasses of m-fold symmetric bi-univalent functions, *Acta Mathematica Scientia* 36(3) (2016), 863-871. https://doi.org/10.1016/S0252-9602(16)30045-5

[15] H. M. Srivastava, A. K. Mishra and P. Gochhayat, Certain subclasses of analytic and bi-univalent functions, *Appl. Math. Lett.* 23 (2010), 1188-1192. https://doi.org/10.1016/j.aml.2010.05.009

[16] H. M. Srivastava, S. Sivasubramanian and R. Sivakumar, Initial coefficient bounds for a subclass of m-fold symmetric bi-univalent functions, *Tbilisi Mathematical J.* 7(2) (2014), 1-10. https://doi.org/10.2478/tmj-2014-0011

[17] A. K. Wanas and H. Tang, Initial coefficient estimates for a classes of m-fold symmetric bi-univalent functions involving Mittag-Leffler function, *Mathematica Moravica* 24(2) (2020), 51-61. https://doi.org/10.5937/MatMor2002051K