Earthline Journal of Mathematical Sciences
E-ISSN: 2581-8147
Volume 10, Number 2, 2022, Pages 211-225
https://doi.org/10.34198/ejms.10222.211225

Estimates for a Generalized Class of Analytic and Bi-univalent Functions Involving Two q-Operators

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Abstract

By making use of q-derivative and q-integral operators, we define a class of analytic and bi-univalent functions in the unit disk $|z| < 1$. Subsequently, we investigate some properties such as some early coefficient estimates and then obtain the Fekete-Szegö inequality for both real and complex parameters. Further, some interesting corollaries are discussed.

1 Introduction

In what follows, let $\mathcal{A}$ represent the class of analytic functions normalized by the conditions $f(0) = 0 = f'(0) - 1$ so that $f(z)$ is of the Maclaurin series representation:

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m \quad (|z| < 1). \tag{1.1}$$

Also let $\mathcal{S}$ represent the subset of $\mathcal{A}$ which is the class of analytic and univalent functions in $|z| < 1$. In view of function class $\mathcal{S}$, the Koebe one-quarter theorem is a familiar theorem that asserts that the range of every function $f \in \mathcal{S}$ covers the disk

$$\mathbb{D} = \{w : |w| < 0.25\} \subset f(|z| < 1).$$

For this reason, $f \in \mathcal{S}$ of the form (1.1) has the inverse function f^{-1} such that

$$f^{-1}(f(z)) = z \quad (|z| < 1)$$

Received: June 9, 2022; Accepted: June 29, 2022
2020 Mathematics Subject Classification: 30C45, 30C50, 30C55.
Keywords and phrases: analytic function, bi-univalent function, q-calculus, Bernardi q-integral operator, Fekete-Szegö functional.

and

$$f(f^{-1}(w)) = w \quad (|w| < r_0(f),\ r_0(f) \geqq 0.25)$$

where by simple calculation we get

$$\mathcal{F}(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots. \quad (1.2)$$

A function $f \in \mathcal{S}$ is said to be *bi-univalent* if both $f(z)$ and $\mathcal{F}(w)$ are univalent in $|z| < 1$. We represent by $\varXi$ the class of analytic and bi-univalent functions in $|z| < 1$.

We thus remark that class $\varXi$ is a non-empty set because the functions:

$$f(z) = z, \quad f(z) = z(1-z)^{-1}, \quad f(z) = -\log(1-z),$$

and more are in $\varXi$. Note that the familiar functions:

$$f(z) = z(1-z)^{-2}, \quad f(\theta; z) = z(1-e^{i\theta}z)^{-2} \quad \text{and} \quad f(z) = z(1-z^2)^{-1}$$

that are in class $\mathcal{S}$ are non-members of $\varXi$.

Historically, Lewin [18] presented the class $\varXi$ of $\mathcal{A}$ and established that every function $f \in \varXi$ has coefficient estimate $|a_2| < 1.51$. Other established estimates for $f \in \varXi$ that improved that of Lewin [18] are $|a_2| \leqq \sqrt{2}$, $|a_2| \leqq 4/3$ and $|a_2| \leqq 1.485$ in [6, 24, 33] respectively. The estimates $|a_m|$ $(m = \{3, 4, \ldots\})$ are presumed yet unsolved. We refer interested readers to the works in [6, 7, 9, 15, 21, 22, 25, 26, 27, 29, 34, 35] for more information on history, properties and definitions of some existing subclasses of $\varXi$.

In recent times, the concept of q-calculus (q-difference, q-integral, q-series and q-numbers) has attracted the attention of theorists of geometric functions. The concept of q-analysis was first introduced in the works of Jackson [11, 12, 13] and since then many researchers (such as in [7, 15, 16, 17, 25, 32]) have used it in various ways to define and establish some properties of many classes of functions in Geometric Function Theory. In particular, Aral et al. [4], Annaby and Mansour [3], Kac and Cheung [14] and Srivastava [28] extensively discussed some applications of q-calculus in so many areas of (q-)analysis.

Definition 1.1 ([11, 12]). For function $f(z) \in \mathcal{A}$ of the form (1.1) and $q \in (0,1)$, the q-derivative operator $\mathcal{D}_q : \mathcal{A} \longrightarrow \mathcal{A}$ of $f(z)$ is defined by

$$\left.\begin{array}{l} \mathcal{D}_q f(z) = \frac{f(z)-f(qz)}{z(1-q)} = 1 + \sum\limits_{m=2}^{\infty} [m]_q a_m z^{m-1} \quad (z \neq 0) \\ \\ \mathcal{D}_q f(0) = f'(0) = 1 \quad (z = 0) \quad \text{if it exists} \\ \\ \mathcal{D}_q^2 f(z) = \mathcal{D}_q(\mathcal{D}_q f(z)) = \sum\limits_{m=2}^{\infty} [m-1]_q [m]_q a_m z^{m-2} \\ \\ \text{where } [m]_q = \frac{1-q^m}{1-q} = 1 + q + q^2 + \cdots + q^{m-1} \implies \lim\limits_{q \uparrow 1} [m]_q = m. \end{array}\right\} \quad (1.3)$$

Using the idea of q-integration introduced by Jackson [13], Aldweby and Darus [1] defined the Bernardi q-integral operator of $f \in \mathcal{A}$ as follows.

Definition 1.2 (Bernardi q-Integral Operator). Let $f(z) \in \mathcal{A}$, then the Bernardi q-integral operator $\mathscr{L}_{q,k} : \mathcal{A} \longrightarrow \mathcal{A}$ $(q \in (0,1),\ k > -1)$ is defined by

$$\mathscr{L}_{q,k} f(z) = \frac{[1+k]_q}{z^k} \int_0^z t^{k-1} f(t) d_q t = z + \sum_{m=2}^{\infty} \frac{[1+k]_q}{[m+k]_q} a_m z^m. \quad (1.4)$$

Remark 1.3. The following properties hold for the function in (1.4).

1. $\lim\limits_{q \uparrow 1} \mathscr{L}_{q,0} f(z) = \int_0^z t^{-1} f(t) dt = z + \sum\limits_{m=2}^{\infty} \left(\frac{1}{m}\right) a_m z^m$ is the Alexander integral operator in [2].
2. $\lim\limits_{q \uparrow 1} \mathscr{L}_{q,1}(z) = \frac{2}{z} \int_0^z f(t) dt = z + \sum\limits_{m=2}^{\infty} \left(\frac{2}{m+1}\right) a_m z^m$ is the Libera integral operator in [19].
3. $\lim\limits_{q \uparrow 1} \mathscr{L}_{q,k} f(z) = \frac{1+k}{z^k} \int_0^z t^{k-1} f(t) dt = z + \sum\limits_{m=2}^{\infty} \left(\frac{1+k}{m+k}\right) a_m z^m$ is the Bernardi integral operator in [5].
4. $\mathscr{L}_{q,0} f(z) = \int_0^z t^{-1} f(t) d_q t = z + \sum\limits_{m=2}^{\infty} \frac{1}{[m]_q} a_m z^m$ is the q-analogue of Alexander integral operator.

5. $\mathscr{L}_{q,1}(z) = \frac{[2]_q}{z}\int_0^z f(t)d_qt = z + \sum\limits_{m=2}^{\infty} \frac{[2]_q}{[m+1]_q} a_m z^m$ is the q-analogue of Libera integral operator.

6. $z\mathcal{D}_q(\mathscr{L}_{q,0}f(z)) = f(z) = z\lim\limits_{q\uparrow 1}[\mathcal{D}_q(\mathscr{L}_{q,0}f(z))]$.

Motivated by the works of Lasode and Opoola [15] and Srivastava and Bansal [29]; the q-derivative operator in (1.3) and the q-integral operator in (1.4), we hereby present our new class as follows.

Definition 1.4. Let $q \in (0,1)$, $\gamma \in \mathbb{C}\setminus\{0\}$, $\lambda \in [0,1]$ and $\delta \in [0,1)$. A function $f \in \Xi$ is said to be a member of class $\Xi_q(k,\gamma,\lambda,\delta)$ if the conditions

$$\mathcal{R}e\left\{1 + \frac{1}{\gamma}\left[\mathcal{D}_q(\mathscr{L}_{q,k}f(z)) + \lambda z\mathcal{D}_q^2(\mathscr{L}_{q,k}f(z)) - 1\right]\right\} > \delta \quad (|z|<1) \tag{1.5}$$

and

$$\mathcal{R}e\left\{1 + \frac{1}{\gamma}\left[\mathcal{D}_q(\mathscr{L}_{q,k}\mathcal{F}(w)) + \lambda w\mathcal{D}_q^2(\mathscr{L}_{q,k}\mathcal{F}(w)) - 1\right]\right\} > \delta \quad (|w|<1) \tag{1.6}$$

hold where $\mathcal{F}(w) = f^{-1}(w)$ is defined in (1.2).

Remark 1.5. The following itemized subclasses hold.

1. $\lim\limits_{q\uparrow 1}\Xi_q(0,1,0,\delta) = \Xi(\delta)$ is the function class investigated by Srivastava et al. [31].

2. $\lim\limits_{q\uparrow 1}\Xi_q(0,1,\lambda,\delta) = \Xi(\lambda,\delta)$ is the function class investigated by Frasin [9], see also Srivastava et al. [30].

3. $\Xi_q(0,1,0,\delta) = \Xi_q(\delta)$ is the function class investigated by Bulut [7].

4. $\Xi_q(0,1,\lambda,\delta) = \Xi_q(\lambda,\delta)$ is the function class investigated by Sabil et al. [25].

5. $\Xi_q(0,1,\lambda,\delta) = \Xi_q(\lambda,\delta)$ is the function class investigated by Motamednezhad and Salehian for $p=1$ in [23].

The purpose of our present paper is to investigate a subclass of bi-univalent functions with positive real parts in $|z|<1$. The coefficient estimates $|a_2|$, $|a_3|$, $|a_4|$ are discussed, and the upper bound for the Fekete-Szegö functional $|a_3 - \alpha a_2^2|$ for real and complex parameters are established.

2 Applicable Lemmas

Let $\mathcal{P}$ be the class of analytic functions of the form

$$p(z) = 1 + \sum_{m=1}^{\infty} p_m z^m \quad (p(0) = 1,\ \mathcal{R}e\, p(z) > 0,\ |z| < 1). \tag{2.1}$$

Lemma 2.1 ([10]). *If $p \in \mathcal{P}$, then $|p_m| \leqq 2$ $(m \in \mathbb{N})$.*

Lemma 2.2 ([20]). *If $p \in \mathcal{P}$, then $2p_2 = p_1^2 + (4 - p_1^2)x$ for some x with $|x| \leqq 1$.*

3 Main Results

Unless otherwise declared, we assume henceforth in this paper that $q \in (0,1)$, $\gamma \in \mathbb{C} \setminus \{0\}$, $\lambda \in [0,1]$, $\delta \in [0,1)$, $k > -1$ and $f \in \Xi$. With these background, we establish our main results.

3.1 Coefficient estimates

Theorem 3.1. *Let $f(z) \in \Xi_q(k, \gamma, \lambda, \delta)$. Then*

$$|a_2| \leqq \frac{\sqrt{2|\gamma|(1-\delta)}}{\sqrt{\frac{[1+k]_q}{[3+k]_q}[3]_q(1+[2]_q\lambda)}} \tag{3.1}$$

$$|a_3| \leqq \frac{2|\gamma|(1-\delta)}{\frac{[1+k]_q}{[3+k]_q}[3]_q(1+[2]_q\lambda)} + \frac{4|\gamma|^2(1-\delta)^2}{\left\{\frac{[1+k]_q}{[2+k]_q}[2]_q(1+[1]_q\lambda)\right\}^2} \tag{3.2}$$

$$|a_4| \leqq \frac{2|\gamma|(1-\delta)}{\frac{[1+k]_q}{[4+k]_q}[4]_q(1+[3]_q\lambda)} + \frac{10|\gamma|^2(1-\delta)^2}{\frac{[1+k]_q}{[2+k]_q}[2]_q(1+[1]_q\lambda)\frac{[1+k]_q}{[3+k]_q}[3]_q(1+[2]_q\lambda)}. \tag{3.3}$$

Proof. Consider the functions

$$\mathcal{B}(z) = 1 + \sum_{m=1}^{\infty} b_m z^m,\ \mathcal{C}(z) = 1 + \sum_{m=1}^{\infty} c_m z^m \in \mathcal{P}, \tag{3.4}$$

so that from (1.5), (1.6), (2.1) and (3.4) we define the equations

$$1 + \frac{1}{\gamma}\Big[\mathcal{D}_q(\mathscr{L}_{q,k}f(z)) + \lambda z \mathcal{D}_q^2(\mathscr{L}_{q,k}f(z)) - 1\Big] = \delta + (1-\delta)\mathcal{B}(z) \quad (|z| < 1) \tag{3.5}$$

and

$$1+\frac{1}{\gamma}\Big[\mathcal{D}_q(\mathscr{L}_{q,k}\mathcal{F}(w))+\lambda w\mathcal{D}_q^2(\mathscr{L}_{q,k}\mathcal{F}(w))-1\Big]=\delta+(1-\delta)\mathcal{C}(w)\quad(|w|<1). \tag{3.6}$$

Comparing coefficients in (3.5) leads to

$$\frac{[1+k]_q}{[2+k]_q}[2]_q(1+[1]_q\lambda)a_2=\gamma(1-\delta)b_1, \tag{3.7}$$

$$\frac{[1+k]_q}{[3+k]_q}[3]_q(1+[2]_q\lambda)a_3=\gamma(1-\delta)b_2, \tag{3.8}$$

$$\frac{[1+k]_q}{[4+k]_q}[4]_q(1+[3]_q\lambda)a_4=\gamma(1-\delta)b_3 \tag{3.9}$$

and comparing coefficients in (3.6) in view of (1.2) leads to

$$-\frac{[1+k]_q}{[2+k]_q}[2]_q(1+[1]_q\lambda)a_2=\gamma(1-\delta)c_1, \tag{3.10}$$

$$\frac{[1+k]_q}{[3+k]_q}[3]_q(1+[2]_q\lambda)(2a_2^2-a_3)=\gamma(1-\delta)c_2, \tag{3.11}$$

$$-\frac{[1+k]_q}{[4+k]_q}[4]_q(1+[3]_q\lambda)(5a_2^3-5a_2a_3+a_4)=\gamma(1-\delta)c_3. \tag{3.12}$$

Adding (3.7) and (3.10) leads to

$$\frac{[1+k]_q}{[2+k]_q}[2]_q(1+[1]_q\lambda)a_2-\frac{[1+k]_q}{[2+k]_q}[2]_q(1+[1]_q\lambda)a_2$$
$$=\gamma(1-\delta)b_1+\gamma(1-\delta)c_1 \implies \begin{cases} b_1=-c_1,\\ b_1^2=c_1^2.\end{cases} \tag{3.13}$$

Now if we square (3.7) and (3.10) and add the results together we obtain

$$2\left\{\frac{[1+k]_q}{[2+k]_q}[2]_q(1+[1]_q\lambda)\right\}^2 a_2^2=\gamma^2(1-\delta)^2(b_1^2+c_1^2). \tag{3.14}$$

Adding (3.8) and (3.11) leads to

$$a_2^2=\frac{\gamma(1-\delta)(b_2+c_2)}{2\frac{[1+k]_q}{[3+k]_q}[3]_q(1+[2]_q\lambda)} \tag{3.15}$$

and applying Lemma 2.1 yields inequality (3.1).

Also, subtracting (3.8) from (3.11) leads to

$$a_3 = a_2^2 + \frac{\gamma(1-\delta)(b_2-c_2)}{2\frac{[1+k]_q}{[3+k]_q}[3]_q(1+[2]_q\lambda)} \tag{3.16}$$

so that by applying (3.13) in (3.14) and putting the result in (3.16) leads to

$$a_3 = \frac{\gamma^2(1-\delta)^2 b_1^2}{\left\{\frac{[1+k]_q}{[2+k]_q}[2]_q(1+[1]_q\lambda)\right\}^2} + \frac{\gamma(1-\delta)(b_2-c_2)}{2\frac{[1+k]_q}{[3+k]_q}[3]_q(1+[2]_q\lambda)} \tag{3.17}$$

and applying Lemma 2.1 yield inequality (3.2).

Likewise, subtracting (3.9) from (3.12) leads to

$$2a_4 = \frac{\gamma(1-\delta)(b_3-c_3)}{\frac{[1+k]_q}{[4+k]_q}[4]_q(1+[3]_q\lambda)} - 5(a_2^3 - a_2a_3) \tag{3.18}$$

and observe that from (3.7) and (3.16) we obtain

$$a_2^3 - a_2a_3 = -\frac{\gamma^2(1-\delta)^2(b_2-c_2)b_1}{2\frac{[1+k]_q}{[2+k]_q}[2]_q(1+[1]_q\lambda)\frac{[1+k]_q}{[3+k]_q}[3]_q(1+[2]_q\lambda)} \tag{3.19}$$

so that by putting (3.19) into (3.18) leads to

$$a_4 = \frac{\gamma(1-\delta)(b_3-c_3)}{2\frac{[1+k]_q}{[4+k]_q}[4]_q(1+[3]_q\lambda)} + \frac{5\gamma^2(1-\delta)^2(b_2-c_2)b_1}{4\frac{[1+k]_q}{[2+k]_q}[2]_q(1+[1]_q\lambda)\frac{[1+k]_q}{[3+k]_q}[3]_q(1+[2]_q\lambda)} \tag{3.20}$$

and applying Lemma 2.1 yields inequality (3.3). □

Corollary 3.2. *Let* $f(z) \in \lim_{q\uparrow 1} \varXi_q(k,\gamma,\lambda,\delta)$. *Then*

$$|a_2| \leqq \frac{\sqrt{2|\gamma|(1-\delta)}}{\sqrt{3(\frac{1+k}{3+k})(1+2\lambda)}}$$

$$|a_3| \leqq \frac{|\gamma|^2(1-\delta)^2}{(\frac{1+k}{2+k})^2(1+\lambda)^2} + \frac{2|\gamma|(1-\delta)}{3(\frac{1+k}{3+k})(1+2\lambda)}$$

$$|a_4| \leqq \frac{5|\gamma|^2(1-\delta)^2}{3(\frac{1+k}{2+k})(\frac{1+k}{3+k})(1+\lambda)(1+2\lambda)} + \frac{|\gamma|(1-\delta)}{2(\frac{1+k}{4+k})(1+3\lambda)}.$$

Remark 3.3. The estimates in Theorem 3.1 will reduce to the results of the authors mentioned in Remark 1.5 when some involving parameters are varied accordingly.

3.2 The Fekete-Szegö Functional

Fekete and Szegö [8] released a classical theorem which states that for all $f(z) = z + a_2z^2 + a_3z^3 + \cdots \in \mathcal{S}$, the coefficient functional

$$|a_3 - \alpha a_2^2| \leqq \begin{cases} 3 - 4\alpha & \text{if} \quad \alpha \leqq 0, \\ 1 + 2e^{-(2\alpha)/(1-\alpha)} & \text{if} \quad 0 \leqq \alpha \leqq 1, \\ 4\alpha - 3 & \text{if} \quad \alpha \geqq 1, \end{cases}$$

is satisfied. This became a great consideration when Fekete and Szegö [8] proved the Littlewood-Parley conjunction to be negative. This inequality is known to be sharp since there is always a function in $\mathcal{S}$ such that the equality holds for each $\alpha \in \mathbb{R}$. For some recent works on Fekete-Szegö problem for some subclasses of $\varXi$ see [15, 21, 22].

Motivated by the works of the aforementioned authors, we now obtain the Fekete-Szegö inequalities for the class $\varXi_q(k, \gamma, \lambda, \delta)$.

Proposition 3.4. From (3.4) and Lemma 2.2, we obtain

$$\left.\begin{aligned} 2b_2 &= b_1^2 + x(4 - b_1^2) \\ 2c_2 &= c_1^2 + y(4 - c_1^2) \end{aligned}\right\} \implies 2(b_2 - c_2) = (4 - b_1^2)(x - y)$$

for some x, y such that $|x|, |y| \leqq 1$.

Theorem 3.5. *Let $f(z) \in \varXi_q(k, \gamma, \lambda, \delta)$ and $\alpha \in \mathbb{R}$. Then*

$$|a_3 - \alpha a_2^2| \leqq \begin{cases} \frac{|\gamma|(1-\delta)}{\frac{[1+k]_q}{[3+k]_q}[3]_q(1+[2]_q\lambda)}|\phi(\alpha)| & for \quad |\phi(\alpha)| \geqq 1 \\ \\ \frac{2|\gamma|(1-\delta)}{\frac{[1+k]_q}{[3+k]_q}[3]_q(1+[2]_q\lambda)} & for \quad 0 \leqq |\phi(\alpha)| \leqq 1 \end{cases} \tag{3.21}$$

where $\phi(\alpha) = 1 - \alpha$.

Proof. Consider (3.15) and (3.16), and using (3.13) we obtain

$$\begin{aligned} a_3 - \alpha a_2^2 &= a_2^2 + \frac{\gamma(1-\delta)(b_2-c_2)}{2\frac{[1+k]_q}{[3+k]_q}[3]_q(1+[2]_q\lambda)} - \alpha a_2^2 \\ &= \frac{\gamma(1-\delta)(b_2-c_2)}{2\frac{[1+k]_q}{[3+k]_q}[3]_q(1+[2]_q\lambda)} + (1-\alpha)a_2^2 \\ &= \frac{\gamma(1-\delta)(b_2-c_2)}{2\frac{[1+k]_q}{[3+k]_q}[3]_q(1+[2]_q\lambda)} + (1-\alpha)\frac{\gamma(1-\delta)(b_2+c_2)}{2\frac{[1+k]_q}{[3+k]_q}[3]_q(1+[2]_q\lambda)} \\ &= \frac{\gamma(1-\delta)}{2\frac{[1+k]_q}{[3+k]_q}[3]_q(1+[2]_q\lambda)}\{(\phi(\alpha)+1)b_2 + (\phi(\alpha)-1)c_2\} \end{aligned}$$

for $\phi(\alpha) = (1-\alpha)$. Now applying triangle inequality and Lemma 2.1 leads to

$$|a_3 - \alpha a_2^2| \leqq \frac{2|\gamma|(1-\delta)}{\frac{[1+k]_q}{[3+k]_q}[3]_q(1+[2]_q\lambda)}\{|\phi(\alpha)|+1\}$$

from where we can conclude that inequality (3.21) holds. □

Theorem 3.6. *Let $f(z) \in \Xi_q(k,\gamma,\lambda,\delta)$ and $\beta \in \mathbb{C}$. Then*

$$|a_3 - \beta a_2^2| \leqq \begin{cases} \frac{2|\gamma|(1-\delta)}{\frac{[1+k]_q}{[3+k]_q}[3]_q(1+[2]_q\lambda)} & \text{for} \quad |1-\beta| \in \left[0, \frac{\left\{\frac{[1+k]_q}{[2+k]_q}[2]_q(1+[1]_q\lambda)\right\}^2}{2|\gamma|(1-\delta)\frac{[1+k]_q}{[3+k]_q}[3]_q(1+[2]_q\lambda)}\right) \\ \frac{4|\gamma|^2(1-\delta)^2}{\left\{\frac{[1+k]_q}{[2+k]_q}[2]_q(1+[1]_q\lambda)\right\}^2}|1-\beta| & \text{for} \quad |1-\beta| \in \left[\frac{\left\{\frac{[1+k]_q}{[2+k]_q}[2]_q(1+[1]_q\lambda)\right\}^2}{2|\gamma|(1-\delta)\frac{[1+k]_q}{[3+k]_q}[3]_q(1+[2]_q\lambda)}, \infty\right) \end{cases}. \tag{3.22}$$

Proof. Consider (3.15) and (3.16), and using (3.13) we obtain

$$\begin{aligned} a_3 - \beta a_2^2 &= a_2^2 + \frac{\gamma(1-\delta)(b_2-c_2)}{2\frac{[1+k]_q}{[3+k]_q}[3]_q(1+[2]_q\lambda)} - \beta a_2^2 \\ &= \frac{\gamma^2(1-\delta)^2(1-\beta)b_1^2}{\left\{\frac{[1+k]_q}{[2+k]_q}[2]_q(1+[1]_q\lambda)\right\}^2} + \frac{\gamma(1-\delta)(b_2-c_2)}{2\frac{[1+k]_q}{[3+k]_q}[3]_q(1+[2]_q\lambda)}. \end{aligned} \tag{3.23}$$

Applying Preposition 3.4 leads to

$$a_3 - \beta a_2^2 = (1-\beta)\frac{\gamma^2(1-\delta)^2 b_1^2}{\left\{\frac{[1+k]_q}{[2+k]_q}[2]_q(1+[1]_q\lambda)\right\}^2} + \frac{\gamma(1-\delta)(4-b_1^2)}{4\frac{[1+k]_q}{[3+k]_q}[3]_q(1+[2]_q\lambda)}(x-y). \tag{3.24}$$

Recall that for $\mathcal{B}(z) \in \mathcal{P}$ in (3.4), $|b_1| \leqq 2$ by Lemma 2.1 and for simplicity, let $b = b_1 \leqq 2$ so that we may assume without any restriction that $b \in [0, 2]$. Now, using triangle inequality and letting $X = |x| \leqq 1$ and $Y = |y| \leqq 1$, then (3.24) becomes

$$\begin{aligned} |a_3 - \beta a_2^2| &= \left| (1-\beta)\frac{\gamma^2(1-\delta)^2 b^2}{\left\{\frac{[1+k]_q}{[2+k]_q}[2]_q(1+[1]_q\lambda)\right\}^2} + \frac{\gamma(1-\delta)(4-b^2)}{4\frac{[1+k]_q}{[3+k]_q}[3]_q(1+[2]_q\lambda)}(x-y) \right| \\ &\leqq |1-\beta|\frac{|\gamma|^2(1-\delta)^2 b^2}{\left\{\frac{[1+k]_q}{[2+k]_q}[2]_q(1+[1]_q\lambda)\right\}^2} + \frac{|\gamma|(1-\delta)(4-b^2)}{4\frac{[1+k]_q}{[3+k]_q}[3]_q(1+[2]_q\lambda)}(X+Y) \\ &= \varphi(X, Y). \end{aligned} \tag{3.25}$$

For $X, Y \in [0, 1]$,

$$\begin{aligned} &\max\{\varphi(X, Y)\} \\ =&\varphi(1,1) = |1-\beta|\frac{|\gamma|^2(1-\delta)^2 b^2}{\left\{\frac{[1+k]_q}{[2+k]_q}[2]_q(1+[1]_q\lambda)\right\}^2} + \frac{|\gamma|(1-\delta)(4-b^2)}{2\frac{[1+k]_q}{[3+k]_q}[3]_q(1+[2]_q\lambda)} \\ =&|1-\beta|\frac{|\gamma|^2(1-\delta)^2 b^2}{\left\{\frac{[1+k]_q}{[2+k]_q}[2]_q(1+[1]_q\lambda)\right\}^2} + \frac{2|\gamma|(1-\delta)}{\frac{[1+k]_q}{[3+k]_q}[3]_q(1+[2]_q\lambda)} \\ &- \frac{|\gamma|(1-\delta)b^2}{2\frac{[1+k]_q}{[3+k]_q}[3]_q(1+[2]_q\lambda)} \\ =&\frac{|\gamma|^2(1-\delta)^2}{\left\{\frac{[1+k]_q}{[2+k]_q}[2]_q(1+[1]_q\lambda)\right\}^2}\left\{|1-\beta| - \frac{\Theta_2^2}{2|\gamma|(1-\delta)\frac{[1+k]_q}{[3+k]_q}[3]_q(1+[2]_q\lambda)}\right\}b^2 \\ &+ \frac{2|\gamma|(1-\delta)}{\frac{[1+k]_q}{[3+k]_q}[3]_q(1+[2]_q\lambda)} = \psi(b). \end{aligned}$$

For $b \in [0, 2]$,

$$\psi'(b) = \frac{2|\gamma|^2(1-\delta)^2}{\left\{\frac{[1+k]_q}{[2+k]_q}[2]_q(1+[1]_q\lambda)\right\}^2}\left\{|1-\beta| - \frac{\left\{\frac{[1+k]_q}{[2+k]_q}[2]_q(1+[1]_q\lambda)\right\}^2}{2|\gamma|(1-\delta)\frac{[1+k]_q}{[3+k]_q}[3]_q(1+[2]_q\lambda)}\right\}b \tag{3.26}$$

implies that there is a critical point at $\psi'(b) = 0$. Clearly,

$$\psi'(b) < 0, \quad \text{if} \quad |1-\beta| \in \left[0, \frac{\left\{\frac{[1+k]_q}{[2+k]_q}[2]_q(1+[1]_q\lambda)\right\}^2}{2|\gamma|(1-\delta)\frac{[1+k]_q}{[3+k]_q}[3]_q(1+[2]_q\lambda)}\right) \tag{3.27}$$

thus, the function $\psi(b)$ is strictly a decreasing function of $|1-\beta| \in \left[0, \frac{\left\{\frac{[1+k]_q}{[2+k]_q}[2]_q(1+[1]_q\lambda)\right\}^2}{2|\gamma|(1-\delta)\frac{[1+k]_q}{[3+k]_q}[3]_q(1+[2]_q\lambda)}\right)$, therefore,

$$\max\{\psi(b) : b \in [0,2]\} = \psi(0) = \frac{2|\gamma|(1-\delta)}{\frac{[1+k]_q}{[3+k]_q}[3]_q(1+[2]_q\lambda)}. \tag{3.28}$$

Likewise for

$$\psi'(b) \geqq 0, \ |1-\beta| \in \left[\frac{\left\{\frac{[1+k]_q}{[2+k]_q}[2]_q(1+[1]_q\lambda)\right\}^2}{2|\gamma|(1-\delta)\frac{[1+k]_q}{[3+k]_q}[3]_q(1+[2]_q\lambda)}, \ 0\right) \tag{3.29}$$

implies that function $\psi(b)$ is an increasing function of $|1-\beta| \in \left[\frac{\left\{\frac{[1+k]_q}{[2+k]_q}[2]_q(1+[1]_q\lambda)\right\}^2}{2|\gamma|(1-\delta)\frac{[1+k]_q}{[3+k]_q}[3]_q(1+[2]_q\lambda)}, \ 0\right)$, therefore,

$$\max\{\psi(b) : b \in [0,2]\} = \psi(2) = \frac{4|\gamma|^2(1-\delta)^2|1-\beta|}{\left\{\frac{[1+k]_q}{[2+k]_q}[2]_q(1+[1]_q\lambda)\right\}^2} \tag{3.30}$$

hence the proof is complete. □

Acknowledgements. The author is most thankful to the reviewer(s) for their valuable suggestions.

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