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Subgroups Inclusions in 3-Factors Direct Product

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Abstract

The aim of this paper is to use a correspondent theorem to characterize containment of a degenerate 2-factor injective subdirect products. Namely, let Ω, Λ be degenerate 2-factor injective subdirect products of $M_1 \times M_2 \times M_3$, we provide necessary and sufficient conditions for $\Omega \leq \Lambda$. Based on a decomposition of the inclusion order on the subgroup lattice of a subdirect product as a relation product of three smaller partial orders, we induce a matrix product of three incidence matrices.

1 Introduction

The importance of Goursat lemma is self-evident. It is widely used in many fields, such as geometries, arithmetics, categories and many more. It is a very good mathematical tool to describe subobjects of direct product of two objects. It appears that there is no straightforward generalization to three factors. Indeed, Sen et al. [18] investigate a generalization to an arbitrary finite number of factors by devising a non-symmetric version of Goursat's lemma for two factors that can then be applied recursively. There are a number of interesting possibilities for generalizing this useful lemma. Anderson and Camillo [4] describe how Goursat's lemma can be stated in the context of rings, ideals, subrings and in modules. The most general category in which one can hope to have a Goursat lemma is an exact Goursat category, and for a proof of this fact confer ([7], Proposition 4.2).

Neuen and Schweitzer [8] investigate the structure of subdirect products of 3-factor direct products. The central observation in this structure theorem is that the dependencies

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among the group elements in the subdirect product that involve all three factors are of Abelian nature. They call a subdirect product of $M_1 \times M_2 \times M_3$ 2-factor injective if each of the three projections onto two factors is injective.

This dissertation provides a containment relation theorem between subgroups of a degenerate 2-factor injective subdirect products, as those who gives by Lewis [16, 6] in the case of 2-factor. In other words, let Ω, Λ be degenerate 2-factor injective subdirect products of $M_1 \times M_2 \times M_3$, we provide necessary and sufficient conditions for $\Omega \leq \Lambda$. We show that this induces a decomposition of the partial order $\leqslant$ as a product of three partial orders, which we denote by $\leqslant_t, \leqslant_{t/b}, \leqslant_b$ for reasons that will become clear in Section 2. Thus

$$\leqslant = \leqslant_t \circ \leqslant_{t/b} \circ \leqslant_b .$$

2 Preliminaries

Let $M = M_1 \times M_2 \times \cdots \times M_s$ be a direct product of groups. We define for $i \in \{1, \cdots, s\}$ the map π_i as the projection to the i-th coordinate and we define the homomorphism $\psi_i : \Lambda \to M_1 \times \cdots \times M_{i-1} \times M_{i+1} \times \cdots \times M_s : (m_1, m_2, \cdots, m_s) \mapsto (m_1, ..., m_{i-1}, m_{i+1}, ..., m_s)$. A group $\Lambda \leq M$ of the direct product is said to be a subdirect product if $\pi_i(\Lambda) = M_i$ for all $1 \leq i \leq s$. Goursat lemma is a classic statement concerned with the structure of subdirect products of direct products of two factors. We now focus on 3-factor subdirect products. We say $\Lambda \leq M_1 \times M_2 \times M_3$ is 2-factor surjective if ψ_i is surjective for all $1 \leq i \leq 3$. Note that the analogous definition of 1-factor surjectivity (i.e., all ψ_i are surjective) means then the same as being subdirect. Similarly, we say Λ is 2-factor injective if ψ_i is injective for all $1 \leq i \leq 3$. Note that this assumption is equivalent to saying that two components of an element of Λ determine the third. Analogously 1-factor injective then means that one component determines the other two. We argue that we can focus our attention on 2-factor injective degenerate subdirect products. In what follows assume that Λ is a 2-factor injective subdirect product of $M_1 \times M_2 \times M_3$. Let $L_i = \ker(\pi_i) \cap \Lambda = \{(m_1, m_2, m_3) \in \Lambda | \ m_i = 1\}$. Define $Z_i := \pi_i(L_k) \cap \pi_i(L_j)$, $D_i = \pi_i(L_{i+2})$, $E_i = \pi_i(L_{i+1})$ where j and k are chosen so that $\{i, j, k\} = \{1, 2, 3\}$.

Here, we write $A \overset{\beta}{\cong} B$ to denote that A and B are isomorphic via an isomorphism β, and $(A, B) \leqslant (C, D)$ if and only if $A \leqslant C$, and $B \leqslant D$. Let $(X, \leqslant)$ be a finite partially ordered set (poset) with incidence matrix

$$A(\leqslant) = (a_{xy})_{x,y \in X}, \quad \text{where} \quad a_{xy} = \begin{cases} 1 & \text{if } y \leqslant x \\ 0, & \text{else.} \end{cases}$$

Lemma 2.1. *Let i, j, k be integers such that $\{i, j, k\} = \{1, 2, 3\}$. Then there is a canonical isomorphism $\beta := \beta^i_{j,k}$ from $\pi_j(L_i)$ to $\pi_k(L_i)$ that maps Z_j to Z_k.*

Proof. Suppose without loss of generality that $i = 1, j = 2$ and $k = 3$. Define a map $\beta : \pi_2(L_1) \to \pi_3(L_1)$ such that $(1, m_2, \beta(m_2)^{-1}) \in \Lambda$ for all $m_2 \in \pi_2(L_1)$. Such a map exists and is well defined since Λ is a 2-factor injective subdirect product. Assume $m_2 \in Z_2$ then $(1, m_2, \beta(m_2)^{-1}) \in \Lambda$ and there is a m_1 such that $(m_1, m_2, 1) \in \Lambda$. Then $(1, m_2, \beta(m_2)^{-1})(m_1, m_2, 1)^{-1} = (m_1^{-1}, 1, \beta(m_2)^{-1})$ so $\beta(m_2) \in Z_3$. It follows by symmetry that all Z_i are isomorphic and that $\beta \restriction_{Z_2}$ is an isomorphism from Z_2 to Z_3. □

Definition 2.2. Let Λ be a subdirect product of $M_1 \times M_2 \times M_3$. We say Λ is degenerate if $\pi_i(\ker(\pi_{i+1})) \cap \pi_i(\ker(\pi_{i+2})) = \pi_i(\ker(\psi_i))$(i.e. $Z_i = 1$) for some, and thus every, $i \in \{1, 2, 3\}$.

Neuen and Schweitzer [8] investigate the possibility of having a correspondence theorem in the style of Goursat theorem [18] for 3-factors.

Theorem 2.3. *There is a natural one-to-one correspondence between degenerate 2-factor injective subdirect products of $M_1 \times M_2 \times M_3$ and tuples of the form $\kappa(\Lambda) = (D_1, D_2, D_3, E_1, E_2, E_3, \beta_1, \beta_2, \beta_3)$ for which for all $i \in \{1, 2, 3\}$ (indices taken modulo 3) we have*

1. $D_i, E_i \trianglelefteq M_i$,
2. $D_i \cap E_i = 1$,
3. $M_i/E_i \overset{\beta_i}{\cong} M_{i+1}/D_{i+1}$,
4. $[D_i, E_i] = 1$,

5. $\beta_i(D_iE_i) = E_{i+1}D_{i+1}$,

6. $\beta_3(\beta_2(\beta_1(m_1D_1E_1))) = m_1D_1E_1$ *for all* $m_1 \in M_1$,

7. $\Lambda = \{(m_1, m_2, m_3) \in M_1 \times M_2 \times M_3 \mid \beta_i(m_iE_i) = m_{i+1}D_{i+1}\}$.

Proof. For $i \in \{1, 2, 3\}$ define a homomorphism $\beta_i : M_i/E_i \to M_{i+1}/D_{i+1}$ by setting $\beta_i(m_iE_i) = m_{i+1}D_{i+1}$ if $(m_1, m_2, m_3) \in \Lambda$ for some $m_i \in M_i$. We first have to show that β_i is well-defined. Without loss of generality consider $i = 1$ and let (m_1, m_2, m_3),$(m'_1, m'_2, m'_3) \in \Lambda$ with $m_1E_1 = m'_1E_1$. Then there is a $(e, 1, l_2) \in \Lambda$ with $m'_1e = m_1$. We obtain $(m'_1, m'_2, m'_3)(e, 1, l_2)(m_1, m_2, m_3)^{-1} = (1, m'_2m_2^{-1}, m''_3)$ for some $m''_3 \in M_3$ and hence, $m_2D_2 = m'_2D_2$ .So β_i is well-defined. Since Λ is a subdirect product, β_i is a surjective homomorphism. Suppose $\beta_1(m_1E_1) = D_2$. Then $(m_1e_1, d_2, m_3) \in \Lambda$ for some $e_1 \in E_1, d_2 \in D_2$ and $m_3 \in M_3$. Also there is $l_3 \in M_3$ with $(1, d_2, l_3) \in \Lambda$ and hence, $(m_1e_1, 1, m_3l_3^{-1}) \in \Lambda$ implying that $m_1 \in E_1$. So $M_i/E_i \overset{\beta_i}{\cong} M_{i+1}/D_{i+1}$.

For every $d_1 \in D_1$ there is a $e_2 \in E_2$ with $(d_1, e_2, 1) \in \Lambda$ and $\beta_1(d_1E_1) = e_2D_2 \in E_2D_2$. By symmetry it follows that $\beta_i(D_iE_i) = E_{i+1}D_{i+1}$ for all $i \in \{1, 2, 3\}$. Now let Λ' be the group defined in item (7). Clearly $\Lambda \leq \Lambda'$ by the definition of β_i for $i \in \{1, 2, 3\}$. So let $(m'_1, m'_2, m'_3) \in \Lambda'$. Since Λ is subdirect there is a $(m'_1, m_2, m_3) \in \Lambda$ with $m_2D_2 = m'_2E_2$. So we can assume that $m_2 = m'_2$. But then, by 2-factor injectivity of Λ, we get that $m_3 = m'_3$. Finally for $(m_1, m_2, m_3) \in \Lambda$ we have that $\beta_i(m_iD_iE_i) = \beta_i(m_iD_i)\beta_i(D_iE_i) = m_{i+1}E_{i+1}D_{i+1} = m_{i+1}D_{i+1}E_{i+1}$. So $\beta_2(\beta_1(m_1D_1E_1)) = \beta_3^{-1}m_1D_1E_1$ for all $m_1 \in M_1$.

The converse is straightforward and will be omitted. □

Proposition 2.4. *There is a natural one-to-one correspondence between degenerate subdirect products of* $M_1 \times M_2 \times M_3$ *and tuples of the form* $\kappa(\Lambda) = (P_1, P_2, P_3, D_1, D_2, D_3, E_1, E_2, E_3, \beta_1, \beta_2, \beta_3)$ *for which for all* $i \in \{1, 2, 3\}$ *(indices taken modulo* 3*) we have*

1. $P_i \trianglelefteq M_i$,

2. $D_i, E_i \trianglelefteq M_i/P_i$,

3. $D_i \cap E_i = 1$,
4. $(M_i/P_i)/E_i \overset{\beta_i}{\cong} (M_{i+1}/P_{i+1})/D_{i+1}$,
5. $[D_i, E_i] = 1$,
6. $\beta_i(D_iE_i) = E_{i+1}D_{i+1}$,
7. $\beta_2(\beta_1(m_1D_1E_1)) = \beta_3^{-1}(m_1D_1E_1)$ *for all* $m_1 \in M_1/P_1$.

Proof. There is a natural one-to-one correspondence between subdirect products of $\Lambda' \leq M_1 \times M_2 \times M_3$ and the tuples (P_1, P_2, P_3, Λ), where $P_i = \pi_i(ker(\psi_i) \trianglelefteq M_i$ for every $i \in \{1, 2, 3\}$ and Λ is a 2-factor injective subdirect product of $M_1/P_1 \times M_2/P_2 \times M_3/P_3$. And we apply Theorem 2.3 then we have a correspondence as desired. □

Theorem 2.5. *There is a natural one-to-one correspondence between subdirect products* Λ *of* $M_1 \times M_2 \times M_3$ *which are 2-factor injective satisfying* $\Lambda = \langle L_1, L_2, L_3\rangle$ *and tuples of the form* $\kappa(\Lambda) = (D_1, D_2, D_3, E_1, E_2, E_3, \beta_1, \beta_2, \beta_3)$ *for which for all* $i \in \{1, 2, 3\}$ *(indices taken modulo* 3*) we have*

1. $D_i, E_i \trianglelefteq M_i$,
2. $E_iD_i = M_i$,
3. $E_i \overset{\beta_i}{\cong} D_{i+1}$,
4. $E_i/Z_i \overset{\beta_i}{\cong} D_{i+1}/Z_{i+1}$,
5. $[D_i, E_i] = 1$,
6. $\beta_i(D_i \cap E_i) = D_{i+1} \cap E_{i+1}$,
7. $\beta_3 |_{Z_3} \circ\beta_2 |_{Z_2} \circ\beta_1 |_{Z_1}= id$.

Proof. Define Λ to be the set of triples $(m_1, m_2, m_3) \in M_1 \times M_2 \times M_3$ that satisfy $m_i = d_ie_i^{-1}$ for $e_i \in E_i$, $d_i \in D_i$, $d_{i+1}Z_{i+1} = \beta_i(e_iZ_i)$, and $d_3\beta_2(e_2)^{-1}.\beta_3^{-1}(d_1)e_3^{-1}.\beta_2(d_2\beta_1(e_1^{-1})) = 1$. □

3 Correspondence Theorems

We now describe and analyze the partial order of subgroups of $M_1 \times M_2 \times M_3$ in terms of pairs of morphisms.

Let $\Lambda, \Omega \leq M_1 \times M_2 \times M_3$ be degenerate 2-factor injective subdirect products and the tuples of the form $\kappa(\Lambda) = (D_1, D_2, D_3, E_1, E_2, E_3, \beta_1, \beta_2, \beta_3)$ and $\kappa(\Omega) = (B_1, B_2, B_3, A_1, A_2, A_3, \beta'_1, \beta'_2, \beta'_3)$ with the assumptions of Proposition 2.3. Given morphisms

$$M_i/E_i \overset{\alpha_i}{\cong} U_i \overset{\theta_i^{-1}}{\cong} M_{i+1}/D_{i+1},$$
$$M_i/A_i \overset{\alpha'_i}{\cong} U'_i \overset{\theta_i'^{-1}}{\cong} M_{i+1}/B_{i+1}.$$

Then $\beta_i = \theta_i^{-1}\alpha_i = \Pi(\theta_i, \alpha_i) : M_i/E_i \longrightarrow M_{i+1}/D_{i+1}$ and $\beta'_i = \theta_i'^{-1}\alpha'_i = \Pi(\theta'_i, \alpha'_i)$ whose graphs are subgroups $\Omega, \Lambda \leqslant M_1 \times M_2 \times M_3$.

Proposition 3.1. *Let $\alpha_i : M_i/E_i \overset{\sim}{\longrightarrow} U_i$ and $\theta_i : M_{i+1}/D_{i+1} \overset{\sim}{\longrightarrow} U_i$ be isomorphisms for $i \in \{1, 2, 3\}$ (indices taken modulo 3), let $\beta_i = \Pi(\theta_i, \alpha_i), \beta'_i = \Pi(\theta'_i, \alpha'_i)$ with corresponding subgroups Λ, Ω of $M_1 \times M_2 \times M_3$. Then $\Omega \leqslant \Lambda$ if and only if*

(a) $(E_i, D_i) \leqslant (A_i, B_i)$;

(b) $\lambda_i = \omega_i$ *for* $\lambda_i = \alpha_i\varphi_i\alpha_i'^{-1}$, $\omega_i = \theta_i\varphi_i'^{-1}\theta_i'^{-1}$, *and φ_i, φ'_i are the homomorphisms defined by* $\varphi_i(m_iA_i) = m_iE_i$, $\varphi'_i(m_{i+1}B_{i+1}) = m_{i+1}D_{i+1}$;

(c) $\beta_i\varphi_i = \varphi'_i\beta'_i$.

$$\begin{array}{ccccc} M_i/E_i & \xrightarrow{\alpha_i} & U_i & \xleftarrow{\theta_i} & M_{i+1}/D_{i+1} \\ \uparrow{\scriptstyle \varphi_i} & & \lambda_i \uparrow\uparrow \omega_i & & \uparrow{\scriptstyle \varphi'_i} \\ M_i/A_i & \xrightarrow{\alpha'_i} & U'_i & \xleftarrow{\theta'_i} & M_{i+1}/B_{i+1} \end{array} \qquad (1)$$

Proof. Write

$$\Omega = \{(m_1, m_2, m_3) \in M_1 \times M_2 \times M_3 | \alpha'_i(m_iA_i) = \theta'_i(m_{i+1}B_{i+1})\}$$

and

$$\Lambda = \{(m_1, m_2, m_3) \in M_1 \times M_2 \times M_3 | \alpha_i(m_iE_i) = \theta_i(m_{i+1}D_{i+1})\}.$$

Then $\Omega \leqslant \Lambda$ if and only if $(E_i, D_i) \leqslant (A_i, B_i)$, $i = 1, 2, 3$, and, for $m_i \in M_i$; $(m_1, m_2, m_3) \in \Omega \leq \lambda$ we have $(m_i E_i)^{\alpha_i} = (m_{i+1} D_{i+1})^{\theta_i}$, but if $(m_i) \in \Omega$, then

$$\begin{aligned}(m_i E_i)^{\alpha_i} &= \alpha_i \varphi_i((m_i A_i)\\ &= \lambda_i \alpha_i'((m_i A_i)\end{aligned}$$

So,

$$\begin{aligned}(m_i E_i)^{\alpha_i} &= \theta_i((m_{i+1} D_{i+1})\\ &= \theta_i \varphi_i'((m_{i+1} B_{i+1})\\ &= \omega_i \theta_i'((m_{i+1} B_{i+1})\end{aligned}$$

if and only if $\lambda_i \alpha_i'(m_i A_i) = w_i \theta_i'(m_{i+1} B_{i+1})$ and $(m_1, m_2, m_3) \in \Omega$. Then $\lambda_i = \omega_i$.

□

Let $\Lambda, \Omega \leq M_1 \times M_2 \times M_3$ be degenerate subdirect products and the tuples of the form $\kappa(\Lambda) = (P_1, P_2, P_3, D_1, D_2, D_3, E_1, E_2, E_3, \beta_1, \beta_2, \beta_3)$ and $\kappa(\Omega) = (Q_1, Q_2, Q_3, B_1, B_2, B_3, A_1, A_2, A_3, \beta_1', \beta_2', \beta_3')$ with the assumptions of Proposition 2.4. Given morphisms

$$\begin{aligned}(M_i/P_i)/E_i &\overset{\alpha_i}{\cong} U_i \overset{\theta_i^{-1}}{\cong} (M_{i+1}/P_{i+1})/D_{i+1},\\ (M_i/Q_i)/A_i &\overset{\alpha_i'}{\cong} U_i' \overset{\theta_i'^{-1}}{\cong} (M_{i+1}/Q_{i+1})/B_{i+1}.\end{aligned}$$

We are now in a position to state the following corollary, in analogy to Proposition 3.1 .

Corollary 3.2. *Let* $\alpha_i : (M_i/P_i)/E_i \overset{\sim}{\longrightarrow} U_i$ *and* $\theta_i : (M_{i+1}/P_{i+1})/D_{i+1} \overset{\sim}{\longrightarrow} U_i$ *be isomorphisms for* $i \in \{1, 2, 3\}$ *(indices taken modulo* 3*), let* $\beta_i = \Pi(\theta_i, \alpha_i), \beta_i' = \Pi(\theta_i', \alpha_i')$ *with corresponding subgroups* Λ, Ω *of* $M_1 \times M_2 \times M_3$*. Then* $\Omega \leqslant \Lambda$ *if and only if*

(a) $(M_i/P_i, E_i, D_i) \leqslant (M_i/Q_i, A_i, B_i)$;

(b) $\lambda_i = \omega_i$ *for* $\lambda_i = \alpha_i \varphi_i \alpha_i'^{-1},\ \omega_i = \theta_i \varphi_i'^{-1} \theta_i'^{-1}$ *;*

(c) $\beta_i\varphi_i = \varphi_i'\beta_i'$.

$$\begin{array}{ccccc} (M_i/P_i)/E_i & \xrightarrow{\alpha_i} & U_i & \xleftarrow{\theta_i} & (M_{i+1}/P_{i+1})/D_{i+1} \\ \varphi_i \uparrow & & \lambda_i \uparrow\uparrow \omega_i & & \uparrow \varphi_i' \\ (M_i/Q_i)/A_i & \xrightarrow{\alpha_i'} & U_i' & \xleftarrow{\theta_i'} & (M_{i+1}/Q_{i+1})/B_{i+1} \end{array} \qquad (2)$$

Definition 3.3. Let $\Lambda = (\beta_i : (M_i/P_i)/E_i \xrightarrow{\sim} (M_{i+1}/P_{i+1})/D_{i+1})$ and $\Omega = (\beta_i' : (M_i/Q_i)/A_i \xrightarrow{\sim} (M_{i+1}/Q_{i+1})/B_{i+1})$ be degenerate subdirect products of $M_1 \times M_2 \times M_3$ and suppose that $\Omega \leqslant \Lambda$. We write

(i) $\Omega \leqslant_t \Lambda$, if $M_i/P_i = M_i/Q_i, i = 1,2,3$, i.e., if both Λ, Ω have same top groups,

(ii) $\Omega \leqslant_b \Lambda$, if $(E_i, D_i) = (A_i, B_i), i = 1,2,3$, i.e., if both Λ, Ω have same bottom groups,

(iii) $\Omega \leqslant_{t/b} \Lambda$, if the canonical homomorphisms φ_i, φ_i' (see (2)) are isomorphisms.

All three relations are obviously partial orders. Moreover, they decompose the partial order $\leqslant$ on the subgroups of $M_1 \times M_2 \times M_3$.

Theorem 3.4. *Let* $\Lambda = (\beta_i : (M_i/P_i)/E_i \xrightarrow{\sim} (M_{i+1}/P_{i+1})/D_{i+1})$ *and* $\Omega = (\beta_i' : (M_i/Q_i)/A_i \xrightarrow{\sim} (M_{i+1}/Q_{i+1})/B_{i+1})$ *degenerate subdirect products of* $M_1 \times M_2 \times M_3$ *be such that* $\Omega \leqslant \Lambda$ *. Define a map*

$$\begin{aligned} \widehat{\beta}_i' : (M_i/Q_i)/((M_i/Q_i) \cap E_i) &\longrightarrow (M_{i+1}/Q_{i+1})/((M_{i+1}/Q_{i+1}) \cap D_{i+1}) \\ g_i((M_i/Q_i) \cap E_i) &\longmapsto g_{i+1}((M_{i+1}/Q_{i+1}) \cap D_{i+1}), \end{aligned}$$

and a map

$$\begin{aligned} \widetilde{\beta}_i : (M_i/Q_i)E_i/E_i &\longrightarrow (M_{i+1}/Q_{i+1})D_{i+1}/D_{i+1} \\ g_iE_i &\longmapsto g_{i+1}D_{i+1}. \end{aligned}$$

Then

(i) $\widehat{\beta}_i'$ *and* $\widetilde{\beta}_i$ *are isomorphisms with corresponding graphs* $\Omega_{\widehat{\beta}'}$ *and* $\Lambda_{\widetilde{\beta}} \leqslant M_1 \times M_2 \times M_3$.

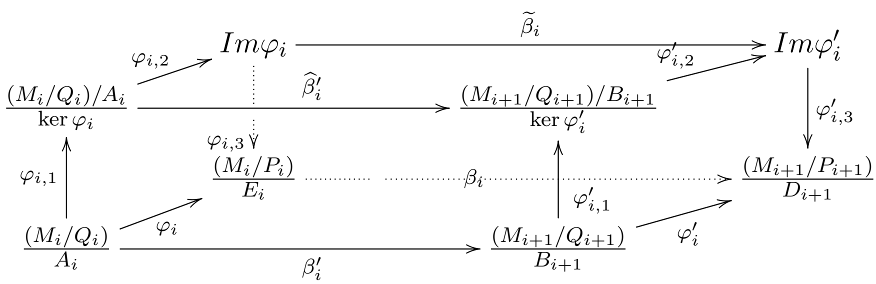
(ii) $\Omega_{\widehat{\beta'}}$ *and* $\Lambda_{\widetilde{\beta}}$ *are the unique degenerate subdirect products of* $M_1 \times M_2 \times M_3$ *with*

$$\Omega \leqslant_t \Omega_{\widehat{\beta'}} \leqslant_{t/b} \Lambda_{\widetilde{\beta}} \leqslant_b \Lambda.$$

Proof. Define by φ'_i, φ_i as (2). According to the homomorphism theorem φ_i can be decomposed into a surjective, bijective and injective part, that is $\varphi_i = \varphi_{i,3}\varphi_{i,2}\varphi_{i,1}$, we have a commutative diagram:

$$\begin{array}{ccc} (M_i/Q_i)/A_i & \xrightarrow{\varphi_i} & (M_i/P_i)/E_i \\ \downarrow{\scriptstyle \varphi_{i,1}} & & \uparrow{\scriptstyle \varphi_{i,3}} \\ ((M_i/Q_i)/A_i)/\ker\varphi_i & \xrightarrow[\varphi_{i,2}]{} & Im\varphi_i \end{array}$$

and $\varphi'_i = \varphi'_{i,3}\varphi'_{i,2}\varphi'_{i,1}$. By Corollary 4.2, $\beta_i\varphi_i = \varphi'_i\beta'_i$. It follows that $(Im\varphi_i)^{\beta_i} = Im\varphi'_i$ and $(\ker\varphi_i)^{\beta'_i} = \ker\varphi'_i$. Thus β_i restricts to an isomorphism $\widetilde{\beta}_i$ from $Im\varphi_i$ to $Im\varphi'_i$ and β'_i induces an isomorphism $\widehat{\beta}'_i$ from $((M_i/Q_i)/A_i)/\ker\varphi_i$ to $((M_{i+1}/Q_{i+1})/B_{i+1})/\ker\varphi'_i$ and the following diagram commutes:



$$Im\varphi_i = (M_i/Q_i)E_i/E_i \quad ; \quad Im\varphi'_i = (M_{i+1}/Q_{i+1})D_{i+1}/D_{i+1}.$$

And

$$\begin{aligned} \frac{(M_i/Q_i)/A_i}{\ker\varphi_i} &\cong (M_i/Q_i)/((M_i/Q_i) \cap E_i), \\ \frac{(M_{i+1}/Q_{i+1})/B_{i+1}}{\ker\varphi'_i} &\cong (M_{i+1}/Q_{i+1})/((M_{i+1}/Q_{i+1}) \cap D_{i+1}). \end{aligned}$$

□

We denote by $S_{M_1\times M_2\times M_3}$ the set of all finite degenerate subdirect products of $M_1 \times M_2 \times M_3$.

Corollary 3.5. *The partial order $\leqslant$ on $S_{M_1\times M_2\times M_3}$ is a product of three relations:*

$$\leqslant = \leqslant_t \circ \leqslant_{t/b} \circ \leqslant_b .$$

Moreover, if $A(R)$ denotes the incidence matrix of the relation R, the stronger property

$$A(\leqslant) = A(\leqslant_t) \cdot A(\leqslant_{t/b}) \cdot A(\leqslant_b)$$

also holds.

Proof. This follows from the uniqueness of the intermediate subgroups in Theorem 3.4. □

4 Subgroups of a Direct Product

The goal of this section is to give another type of characterization of containment of subgroups in a product of groups. This is accomplished in Theorem 4.1.

Theorem 4.1. *Let $\Lambda, \Omega \leq M_1 \times M_2 \times M_3$ be 2-factor injective subdirect products and the tuples of the form $\kappa(\Lambda) = (D_1, D_2, D_3, E_1, E_2, E_3, \beta_1, \beta_2, \beta_3)$ with $Z_i = D_i \cap E_i$, and $\kappa(\Omega) = (A_1, A_2, A_3, B_1, B_2, B_3, \gamma_1, \gamma_2, \gamma_3)$, $Y_i = A_i \cap B_i$ with the assumptions of Theorem 2.5. Then $\Omega \leq \Lambda$ if and only if*

(1) $A_i \leq D_i, B_i \leq E_i$,

(2) $\left(\frac{B_i Z_i}{Z_i}\right)^{\beta_i} = \frac{A_{i+1} Z_{i+1}}{Z_{i+1}}$,

(3) $\left(\dfrac{B_i \cap Z_i}{Y_i}\right)^{\gamma_i} = \dfrac{A_{i+1} \cap Z_{i+1}}{Y_{i+1}}$,

(4) $(B_i)^{\beta_i} = A_{i+1}$,

(5) $\theta_i \circ \widetilde{\beta}_i = \widetilde{\gamma}_i \circ \alpha_i$ *where $\widetilde{\beta}_i$ and $\widetilde{\gamma}_i$ are restrictions of β_i and γ_i respectively.*

An equivalent statement for (5) *is that the following diagram is a commutative diagram.*

$$\begin{array}{ccc} \frac{B_iZ_i}{Z_i} & \xrightarrow{\widetilde{\beta}_i} & \frac{A_{i+1}Z_{i+1}}{Z_{i+1}} \\ \alpha_i \downarrow & & \downarrow \theta_i \\ \frac{B_i}{B_i\cap Z_i} & \xrightarrow{\widetilde{\gamma}_i} & \frac{A_{i+1}}{A_{i+1}\cap Z_{i+1}} \end{array}$$

Proof. ($\Rightarrow$) Assume that $\Omega \leq \Lambda$. It is clearly that $A_i \leq D_i, B_i \leq E_i$, $(\frac{B_iZ_i}{Z_i})^{\beta_i} = \frac{A_{i+1}Z_{i+1}}{Z_{i+1}}$, $(\frac{B_i\cap Z_i}{Y_i})^{\gamma_i} = \frac{A_{i+1}\cap Z_{i+1}}{Y_{i+1}}$. The diagram below is a commutative diagram.

$$\begin{array}{ccc} (\frac{B_i}{Y_i})/(\frac{B_i\cap Z_i}{Y_i}) & \xrightarrow{\widehat{\gamma}_i} & (\frac{A_{i+1}}{Y_{i+1}})/(\frac{A_{i+1}\cap Z_{i+1}}{Y_{i+1}}) \\ \varepsilon_i \downarrow & & \downarrow \rho_i \\ \frac{B_i}{B_i\cap Z_i} & \xrightarrow{\widetilde{\gamma}_i} & \frac{A_{i+1}}{A_{i+1}\cap Z_{i+1}} \end{array}$$

Where ε_i and ρ_i are isomorphisms and $\widetilde{\gamma}_i = \rho_i\widehat{\gamma}_i\varepsilon_i^{-1}$. Therefore, it suffices to demonstrate that the diagram commutes. More explicitly that $\theta_i \circ \widetilde{\beta}_i = \widetilde{\gamma}_i \circ \alpha_i$. Let $(m_1, m_2, m_3) \in \Omega$ such that $m_i = a_ib_i^{-1}$ for $b_i \in B_i, a_i \in A_i, a_{i+1}Y_{i+1} = \gamma_i(b_iY_i)$. Since $\Omega \leq \Lambda$ we know $a_{i+1}Z_{i+1} = \beta_i(b_iZ_i)$ and $\widetilde{\beta}_i$ is a restriction of β. Then $\big((b_iZ_i)^{\widetilde{\beta}_i}\big)^{\theta_i} = (a_{i+1}Z_{i+1})^{\theta_i} = a_{i+1}(A_{i+1} \cap Z_{i+1})$. On the other side, we obtain

$$\big((b_iZ_i)^{\alpha_i}\big)^{\widetilde{\gamma}_i} = \big(b_i(B_i \cap Z_i)\big)^{\widetilde{\gamma}_i} = a_{i+1}(A_{i+1} \cap Z_{i+1}).$$

Therefore, $\theta_i \circ \widetilde{\beta}_i = \widetilde{\gamma}_i \circ \alpha_i$, and the diagram commutes.

($\Leftarrow$) Conversely, suppose the containments hold and the diagram commutes. Our goal is to show $\Omega \leq \Lambda$. Let $(m_1, m_2, m_3) \in \Omega$. Then $\gamma_i(b_iY_i) = a_{i+1}Y_{i+1}$. We have

$$\begin{aligned} (b_iZ_i)^{\widetilde{\beta}_i} &= (b_iZ_i)^{\alpha_i\widetilde{\gamma}_i\theta_i^{-1}} \\ &= (b_i(B_i \cap Z_i))^{\widetilde{\gamma}_i\theta_i^{-1}} \\ &= (a_{i+1}(A_{i+1} \cap Z_{i+1}))^{\theta_i^{-1}} \\ &= a_{i+1}Z_{i+1}. \end{aligned}$$

And $\beta_i \mid_{B_i} = \gamma_i$. We conclude that $(m_1, m_2, m_3) \in \Lambda$, and $\Omega \leq \Lambda$. □

Corollary 4.2. *Let $\Lambda, \Omega \leq M_1 \times M_2 \times M_3$ be degenerate subdirect products and the tuples of the form $\kappa(\Lambda) = (P_1, P_2, P_3, D_1, D_2, D_3, E_1, E_2, E_3, \beta_1, \beta_2, \beta_3)$ and $\kappa(\Omega) = (Q_1, Q_2, Q_3, B_1, B_2, B_3, A_1, A_2, A_3, \gamma_1, \gamma_2, \gamma_3)$ with the assumptions of Proposition 2.4. Then $\Omega \leq \Lambda$ if and only if*

(a) $A_i \leq E_i, B_i \leq D_i, M_i/Q_i \leq M_i/P_i$,

(b) $\left(\frac{(M_i/Q_i)E_i}{E_i}\right)^{\beta_i} = \frac{(M_{i+1}/Q_{i+1})D_{i+1}}{D_{i+1}}$,

(c) $\left(\dfrac{(M_i/Q_i) \cap E_i}{A_i}\right)^{\gamma_i} = \dfrac{(M_{i+1}/Q_{i+1}) \cap D_{i+1}}{B_{i+1}}$,

(d) *the following diagram is a commutative diagram*

$$\begin{array}{ccc} \frac{(M_i/Q_i)E_i}{E_i} & \xrightarrow{\widetilde{\beta}_i} & \frac{(M_{i+1}/Q_{i+1})D_{i+1}}{D_{i+1}} \\ \alpha_i \downarrow & & \downarrow \theta_i \\ \frac{(M_i/Q_i)}{(M_i/Q_i)\cap E_i} & \xrightarrow{\widetilde{\gamma}_i} & \frac{(M_{i+1}/Q_{i+1})}{(M_{i+1}/Q_{i+1})\cap D_{i+1}} \end{array}$$

Proof. The proof of this corollary come immediately from Theorem 4.1 and Proposition 2.4. □

5 Application

The next application, that of determining the cyclic subgroups of $M_1 \times M_2 \times M_3$ will involve more substantial use of Theorem 2.3. Cyclic subgroups are not closed under products.We shall henceforth use additive notation since M_1,M_2,M_3 will be abelian.

Theorem 5.1. *Let Λ be a degenerate 2-factor injective subdirect products of $M_1 \times M_2 \times M_3$ with the assumptions of Theorem 2.3, and tuples of the form*

$$\kappa(\Lambda) = (D_1, D_2, D_3, E_1, E_2, E_3, \beta_1, \beta_2, \beta_3).$$

(a) *The subgroup Λ is finite cyclic if and only if M_1, M_2, M_3 are finite cyclic and each of the pairs of integers $(|D_2|, |E_1|), (|D_1|, |E_3|), (|D_3|, |E_2|)$ is coprime. In this case one also has*

$$|\Lambda| = lcm(|M_1|, |M_2|, |M_3|).$$

(b) *The subgroup Λ is infinite cyclic if and only if one of the following three cases (up to obvious permutation of indices) occur:*

(i) $M_1 \approx \mathbb{Z}$, *M_2 and M_3 are finite cyclic,* $D_2 = E_3 = \{0\}$, *and* $(|D_3|, |E_2|)$ *are coprime.*

(ii) $M_1 \approx M_2 \approx \mathbb{Z}$, *$M_3$ finite cyclic, and* $D_2 = E_1 = E_3 = D_3 = \{0\}$.

(iii) $M_i \approx \mathbb{Z}$ *and* $D_i = E_i = \{0\}$ *for* $i = 1, 2, 3$.

Proof. This is an immediate consequence of Theorem 4.5 in [18]. □

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