Insertion of a Contra-α-continuous Function

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Abstract

A necessary and sufficient condition in terms of lower cut sets is given for the insertion of a contra-α-continuous function between two comparable real-valued functions.

1. Introduction

The concept of a preopen set in a topological space was introduced by Corson and Michael in 1964 [4]. A subset $A$ of a topological space $(X, \tau)$ is called preopen or locally dense or nearly open if $A \subseteq \text{Int} (\text{Cl}(A))$. A set $A$ is called preclosed if its complement is preopen or equivalently if $\text{Cl}(\text{Int}(A)) \subseteq A$. The term, preopen, was used for the first time by Mashhour et al. [20], while the concept of a locally dense set was introduced by Corson and Michael [4].

The concept of a semi-open set in a topological space was introduced by Levine in 1963 [17]. A subset $A$ of a topological space $(X, \tau)$ is called semi-open [10] if $A \subseteq \text{Cl}(\text{Int}(A))$. A set $A$ is called semi-closed if its complement is semi-open or equivalently if $\text{Int}(\text{Cl}(A)) \subseteq A$. 

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Recall that a subset $A$ of a topological space $(X, \tau)$ is called $\alpha$-open if $A$ is the difference of an open and a nowhere dense subset of $X$. A set $A$ is called $\alpha$-closed if its complement is $\alpha$-open or equivalently if $A$ is union of a closed and a nowhere dense set.

A set is $\alpha$-open if and only if it is semi-open and preopen.

A generalized class of closed sets was considered by Maki in [19]. He investigated the sets that can be represented as union of closed sets and called them $V$-sets. Complements of $V$-sets, i.e., sets that are intersection of open sets are called $\Lambda$-sets [19].

Recall that a real-valued function $f$ defined on a topological space $X$ is called $A$-continuous [25] if the preimage of every open subset of $\mathbb{R}$ belongs to $A$, where $A$ is a collection of subsets of $X$. Most of the definitions of function used throughout this paper are consequences of the definition of $A$-continuity. However, for unknown concepts the reader may refer to [5, 11]. In the recent literature many topologists had focused their research in the direction of investigating different types of generalized continuity.

Dontchev in [6] introduced a new class of mappings called contra-continuity. Jafari and Noiri in [12, 13] exhibited and studied among others a new weaker form of this class of mappings called contra-$\alpha$-continuous. A good number of researchers have also initiated different types of contra-continuous like mappings in the papers [1, 3, 8, 9, 10, 24].

Hence, a real-valued function $f$ defined on a topological space $X$ is called contra-$\alpha$-continuous (resp. contra-semi-continuous, contra-precontinuous) if the preimage of every open subset of $\mathbb{R}$ is $\alpha$-closed (resp. semi-closed, preclosed) in $X$ [6].

Results of Katětov [14, 15] concerning binary relations and the concept of an indefinite lower cut set for a real-valued function, which is due to Brooks [2], are used in order to give a necessary and sufficient conditions for the insertion of a contra-$\alpha$-continuous function between two comparable real-valued functions.

If $g$ and $f$ are real-valued functions defined on a space $X$, we write $g \leq f$ (resp. $g < f$) in case $g(x) \leq f(x)$ (resp. $g(x) < f(x)$) for all $x$ in $X$.

The following definitions are modifications of conditions considered in [16].

A property $P$ defined relative to a real-valued function on a topological space is a $c\alpha$-property provided that any constant function has property $P$ and provided that the
sum of a function with property $P$ and any contra-$\alpha$-continuous function also has property $P$. If $P_1$ and $P_2$ are $c\alpha$-property, the following terminology is used: (i) A space $X$ has the weak $c\alpha$-insertion property for $(P_1, P_2)$ if and only if for any functions $g$ and $f$ on $X$ such that $g \leq f$, $g$ has property $P_1$ and $f$ has property $P_2$, then there exists a contra-$\alpha$-continuous function $h$ such that $g \leq h \leq f$. (ii) A space $X$ has the $c\alpha$-insertion property for $(P_1, P_2)$ if and only if for any functions $g$ and $f$ on $X$ such that $g < f$, $g$ has property $P_1$ and $f$ has property $P_2$, then there exists a contra-$\alpha$-continuous function $h$ such that $g < h < f$. (iii) A space $X$ has the weakly $c\alpha$-insertion property for $(P_1, P_2)$ if and only if for any functions $g$ and $f$ on $X$ such that $g < f$, $g$ has property $P_1$, $f$ has property $P_2$ and $f - g$ has property $P_2$, then there exists a contra-$\alpha$-continuous function $h$ such that $g < h < f$.

In this paper, it is given a sufficient condition for the weak $c\alpha$-insertion property. Also for a space with the weak $c\alpha$-insertion property, we give a necessary and sufficient condition for the space to have the $c\alpha$-insertion property. Several insertion theorems are obtained as corollaries of these results.

2. The Main Result

Before giving a sufficient condition for insertability of a contra-$\alpha$-continuous function, the necessary definitions and terminology are stated.

Let $(X, \tau)$ be a topological space. Then the family of all $\alpha$-open, $\alpha$-closed, semi-open, semi-closed, preopen and preclosed will be denoted by $\alpha O(X, \tau)$, $\alpha C(X, \tau)$, $sO(X, \tau)$, $sC(X, \tau)$, $pO(X, \tau)$ and $pC(X, \tau)$, respectively.

**Definition 2.1.** Let $A$ be a subset of a topological space $(X, \tau)$. We define the subsets $A^\Lambda$ and $A^V$ as follows:

$$A^\Lambda = \cap \{O : O \supseteq A, O \in (X, \tau)\} \quad \text{and} \quad A^V = \cup \{F : F \subseteq A, F^c \in (X, \tau)\}. $$

In [7, 18, 23], $A^\Lambda$ is called the kernel of $A$.

We define the subsets $\alpha(A^\Lambda)$, $\alpha(A^V)$, $p(A^\Lambda)$, $p(A^V)$, $s(A^\Lambda)$ and $s(A^V)$ as follows:
\[ \alpha(A^\Lambda) = \cap \{ O : O \supseteq A, O \in \alpha O(X, \tau) \}, \]
\[ \alpha(A^V) = \cup \{ F : F \subseteq A, F \in \alpha C(X, \tau) \}, \]
\[ p(A^\Lambda) = \cap \{ O : O \supseteq A, O \in pO(X, \tau) \}, \]
\[ p(A^V) = \cup \{ F : F \subseteq A, F \in pC(X, \tau) \}, \]
\[ s(A^\Lambda) = \cap \{ O : O \supseteq A, O \in sO(X, \tau) \} \]
and
\[ s(A^V) = \cup \{ F : F \subseteq A, F \in sC(X, \tau) \}. \]

\[ \alpha(A^\Lambda) \text{ (resp. } p(A^\Lambda), s(A^\Lambda)) \text{ is called the } \alpha \text{-kernel (resp. prekernel, semi-kernel) of } A. \]

The following first two definitions are modifications of conditions considered in [14, 15].

**Definition 2.2.** If \( \rho \) is a binary relation in a set \( S \), then \( \overline{\rho} \) is defined as follows: \( x \overline{\rho} y \) if and only if \( y \rho v \) implies \( x \rho v \) and \( u \rho x \) implies \( u \rho y \) for any \( u \) and \( v \) in \( S \).

**Definition 2.3.** A binary relation \( \rho \) in the power set \( P(X) \) of a topological space \( X \) is called a **strong binary relation** in \( P(X) \) in case \( \rho \) satisfies each of the following conditions:

1. If \( A_i \rho B_j \) for any \( i \in \{1, ..., m\} \) and for any \( j \in \{1, ..., n\} \), then there exists a set \( C \) in \( P(X) \) such that \( A_i \rho C \) and \( C \rho B_j \) for any \( i \in \{1, ..., m\} \) and any \( j \in \{1, ..., n\} \).

2. If \( A \subseteq B \), then \( A \overline{\rho} B \).

3. If \( A \rho B \), then \( \alpha(A^\Lambda) \subseteq B \) and \( A \subseteq \alpha(B^V) \).

The concept of a lower indefinite cut set for a real-valued function was defined by Brooks [2] as follows:

**Definition 2.4.** If \( f \) is a real-valued function defined on a space \( X \) and if \( \{ x \in X : f(x) < \ell \} \subseteq A(f, \ell) \subseteq \{ x \in X : f(x) \leq \ell \} \) for a real number \( \ell \), then \( A(f, \ell) \) is called a **lower indefinite cut set** in the domain of \( f \) at the level \( \ell \).
We now give the following main result:

**Theorem 2.1.** Let \( g \) and \( f \) be real-valued functions on the topological space \( X \), in which \( \alpha \)-kernel sets are \( \alpha \)-open, with \( g \leq f \). If there exists a strong binary relation \( \rho \) on the power set of \( X \) and if there exist lower indefinite cut sets \( A(f, t) \) and \( A(g, t) \) in the domain of \( f \) and \( g \) at the level \( t \) for each rational number \( t \) such that if \( t_1 < t_2 \), then \( A(f, t_1) \rho A(g, t_2) \), then there exists a contra-\( \alpha \)-continuous function \( h \) defined on \( X \) such that \( g \leq h \leq f \).

**Proof.** Theorem 2.1 in [22].

**Theorem 2.2.** Let \( P_1 \) and \( P_2 \) be \( \alpha \)-property and \( X \) be a space that satisfies the weak \( \alpha \)-insertion property for \( (P_1, P_2) \). Also assume that \( g \) and \( f \) are functions on \( X \) such that \( g < f \), \( g \) has property \( P_1 \) and \( f \) has property \( P_2 \). The space \( X \) has the \( \alpha \)-insertion property for \( (P_1, P_2) \) if and only if there exist lower cut sets \( A(f - g, 3^{-n+1}) \) and there exists a decreasing sequence \( \{D_n\} \) of subsets of \( X \) with empty intersection and such that for each \( n \), \( X \setminus D_n \) and \( A(f - g, 3^{-n+1}) \) are completely separated by contra-\( \alpha \)-continuous functions.

**Proof.** Theorem 2.1 in [21].

### 3. Applications

The abbreviations \( c\alpha c \), \( cpc \) and \( csc \) are used for contra-\( \alpha \)-continuous, contra-precontinuous and contra-semi-continuous, respectively.

Before stating the consequences of Theorems 2.1, 2.2, we suppose that \( X \) is a topological space whose \( \alpha \)-kernel sets are \( \alpha \)-open.

**Corollary 3.1.** If for each pair of disjoint preopen (resp. semi-open) sets \( G_1, G_2 \) of \( X \), there exist \( \alpha \)-closed sets \( F_1 \) and \( F_2 \) of \( X \) such that \( G_1 \subseteq F_1 \), \( G_2 \subseteq F_2 \) and \( F_1 \cap F_2 = \emptyset \), then \( X \) has the weak \( \alpha \)-insertion property for \( (cpc, cpc) \) (resp. \( (csc, csc) \)).

**Proof.** Corollary 3.1 in [22].
Corollary 3.2. If for each pair of disjoint preopen (resp. semi-open) sets $G_1, G_2$, there exist $\alpha$-closed sets $F_1$ and $F_2$ such that $G_1 \subseteq F_1$, $G_2 \subseteq F_2$ and $F_1 \cap F_2 = \emptyset$, then every contra-precontinuous (resp. contra-semi-continuous) function is contra-$\alpha$-continuous.

**Proof.** Corollary 3.2 in [22].

Corollary 3.3. If for each pair of disjoint preopen (resp. semi-open) sets $G_1, G_2$ of $X$, there exist $\alpha$-closed sets $F_1$ and $F_2$ of $X$ such that $G_1 \subseteq F_1$, $G_2 \subseteq F_2$ and $F_1 \cap F_2 = \emptyset$, then $X$ has the $c\alpha$-insertion property for (cpc, cpc) (resp. (csc, csc)).

**Proof.** Let $g$ and $f$ be real-valued functions defined on the $X$, such that $f$ and $g$ are cpc (resp. csc), and $g < f$. Set $h = (f + g)/2$, thus $g < h < f$, and by Corollary 3.2, since $g$ and $f$ are contra-$\alpha$-continuous functions hence $h$ is a contra-$\alpha$-continuous function.

Corollary 3.4. If for each pair of disjoint subsets $G_1, G_2$ of $X$, such that $G_1$ is preopen and $G_2$ is semi-open, there exist $\alpha$-closed subsets $F_1$ and $F_2$ of $X$ such that $G_1 \subseteq F_1$, $G_2 \subseteq F_2$ and $F_1 \cap F_2 = \emptyset$, then $X$ have the weak $c\alpha$-insertion property for (cpc, csc) and (csc, cpc).

**Proof.** Corollary 3.4 in [22].

Before stating consequences of Theorem 2.2, we state and prove the necessary lemmas.

Lemma 3.1. The following conditions on the space $X$ are equivalent:

(i) For each pair of disjoint subsets $G_1, G_2$ of $X$, such that $G_1$ is preopen and $G_2$ is semi-open, there exist $\alpha$-closed subsets $F_1, F_2$ of $X$ such that $G_1 \subseteq F_1$, $G_2 \subseteq F_2$ and $F_1 \cap F_2 = \emptyset$.

(ii) If $G$ is a semi-open (resp. preopen) subset of $X$ which is contained in a preclosed (resp. semi-closed) subset $F$ of $X$, then there exists an $\alpha$-closed subset $H$ of $X$ such that $G \subseteq H \subseteq \alpha(H^\wedge) \subseteq F$.

**Proof.** Lemma 3.1 in [22].
Lemma 3.2. Suppose that $X$ is a topological space. If each pair of disjoint subsets $G_1, G_2$ of $X$, where $G_1$ is preopen and $G_2$ is semi-open, can be separated by $\alpha$-closed subsets of $X$, then there exists a contra-$\alpha$-continuous function $h : X \to [0, 1]$ such that $h(G_2) = \{0\}$ and $h(G_1) = \{1\}$.

Proof. Lemma 3.2 in [22].

Lemma 3.3. Suppose that $X$ is a topological space such that every two disjoint semi-open and preopen subsets of $X$ can be separated by $\alpha$-closed subsets of $X$. The following conditions are equivalent:

(i) Every countable covering of semi-closed (resp. preclosed) subsets of $X$ has a refinement consisting of preclosed (resp. semi-closed) subsets of $X$ such that for every $x \in X$, there exists an $\alpha$-closed subset of $X$ containing $x$ such that it intersects only finitely many members of the refinement.

(ii) Corresponding to every decreasing sequence $\{G_n\}$ of semi-open (resp. preopen) subsets of $X$ with empty intersection there exists a decreasing sequence $\{F_n\}$ of preclosed (resp. semi-closed) subsets of $X$ such that $\bigcap_{n=1}^{\infty} F_n = \emptyset$ and for every $n \in \mathbb{N}$, $G_n \subseteq F_n$.

Proof. (i) $\Rightarrow$ (ii) Suppose that $\{G_n\}$ is a decreasing sequence of semi-open (resp. preopen) subsets of $X$ with empty intersection. Then $\{G_n^c : n \in \mathbb{N}\}$ is a countable covering of semi-closed (resp. preclosed) subsets of $X$. By hypothesis (i) and Lemma 3.1, this covering has a refinement $\{V_n : n \in \mathbb{N}\}$ such that every $V_n$ is an $\alpha$-closed subset of $X$ and $\alpha(V_n^\Lambda) \subseteq G_n^c$. By setting $F_n = \alpha((V_n^\Lambda)^c)$, we obtain a decreasing sequence of $\alpha$-closed subsets of $X$ with the required properties.

(ii) $\Rightarrow$ (i) Now if $\{H_n : n \in \mathbb{N}\}$ is a countable covering of semi-closed (resp. preclosed) subsets of $X$, we set for $n \in \mathbb{N}$, $G_n = \left(\bigcup_{i=1}^{\infty} H_i\right)^c$. Then $\{G_n\}$ is a decreasing sequence of semi-open (resp. preopen) subsets of $X$ with empty intersection. By (ii) there exists a decreasing sequence $\{F_n\}$ consisting of preclosed (resp. semi-closed) subsets of $X$ such that $\bigcap_{n=1}^{\infty} F_n = \emptyset$ and for every $n \in \mathbb{N}$, $G_n \subseteq F_n$. Now we define the subsets
\( W_n \) of \( X \) in the following manner:

\( W_1 \) is an \( \alpha \)-closed subset of \( X \) such that \( F_1^c \subseteq W_1 \) and \( \alpha(W_1^\Lambda) \cap G_1 = \emptyset \).

\( W_2 \) is an \( \alpha \)-closed subset of \( X \) such that \( \alpha(W_1^\Lambda) \cup F_2^c \subseteq W_2 \) and \( \alpha(W_2^\Lambda) \cap G_2 = \emptyset \), and so on. (By Lemma 3.1, \( W_n \) exists).

Then since \( \{ F_n^c : n \in \mathbb{N} \} \) is a covering for \( X \), hence \( \{ W_n : n \in \mathbb{N} \} \) is a covering for \( X \) consisting of \( \alpha \)-closed sets. Moreover, we have

(i) \( \alpha(W_n^\Lambda) \subseteq W_{n+1} \).

(ii) \( F_n^c \subseteq W_n \).

(iii) \( W_n \subseteq \bigcup_{i=1}^{n} H_i \).

Now setting \( S_1 = W_1 \) and for \( n \geq 2 \), we set \( S_n = W_{n+1} \setminus \alpha(W_{n-1}^\Lambda) \).

Then since \( \alpha(W_n^\Lambda) \subseteq W_n \) and \( S_n \supseteq W_{n+1} \setminus W_n \), it follows that \( \{ S_n : n \in \mathbb{N} \} \) consists of \( \alpha \)-closed sets and covers \( X \). Furthermore, \( S_i \cap S_j \neq \emptyset \) if and only if \( |i - j| \leq 1 \). Finally, consider the following sets:

\[
\begin{align*}
S_1 \cap H_1, & \quad S_1 \cap H_2 \\
S_2 \cap H_1, & \quad S_2 \cap H_2, & \quad S_2 \cap H_3 \\
S_3 \cap H_1, & \quad S_3 \cap H_2, & \quad S_3 \cap H_3, & \quad S_3 \cap H_4 \\
& \vdots \\
S_i \cap H_1, & \quad S_i \cap H_2, & \quad S_i \cap H_3, & \quad S_i \cap H_4, & \quad \ldots, & \quad S_i \cap H_{i+1} \\
& \vdots
\end{align*}
\]

These sets are \( \alpha \)-closed sets, cover \( X \) and refine \( \{ H_n : n \in \mathbb{N} \} \). In addition, \( S_i \cap H_j \) can intersect at most the sets in its row, immediately above, or immediately below row.

Hence if \( x \in X \) and \( x \in S_n \cap H_m \), then \( S_n \cap H_m \) is an \( \alpha \)-closed set containing \( x \) that intersects at most finitely many of sets \( S_i \cap H_j \). Consequently, \( \{ S_i \cap H_j : i \in \mathbb{N} \} \).
$j = 1, \ldots, i + 1$ refines $\{H_n : n \in \mathbb{N}\}$ such that its elements are $\alpha$-closed sets, and for every point in $X$ we can find an $\alpha$-closed set containing the point that intersects only finitely many elements of that refinement.

**Corollary 3.5.** If every two disjoint semi-open and preopen subsets of $X$ can be separated by $\alpha$-closed subsets of $X$, and in addition, every countable covering of semi-closed (resp. preclosed) subsets of $X$ has a refinement that consists of preclosed (resp. semi-closed) subsets of $X$ such that for every point of $X$ we can find an $\alpha$-closed subset containing that point such that it intersects only a finite number of refining members, then $X$ has the weakly $c\alpha$-insertion property for $(cpc, csc)$ (resp. $(csc, cpc)$).

**Proof.** Since every two disjoint semi-open and preopen sets can be separated by $\alpha$-closed subsets of $X$, therefore by Corollary 3.4, $X$ has the weak $c\alpha$-insertion property for $(cpc, csc)$ and $(csc, cpc)$. Now suppose that $f$ and $g$ are real-valued functions on $X$ with $g < f$, such that $g$ is $cpc$ (resp. $csc$), $f$ is $csc$ (resp. $cpc$) and $f - g$ is $csc$ (resp. $cpc$). For every $n \in \mathbb{N}$, set

$$A(f - g, 3^{-n+1}) = \{x \in X : (f - g)(x) \leq 3^{-n+1}\}.$$ 

Since $f - g$ is $csc$ (resp. $cpc$), hence $A(f - g, 3^{-n+1})$ is a semi-open (resp. preopen) subset of $X$. Consequently, $\{A(f - g, 3^{-n+1})\}$ is a decreasing sequence of semi-open (resp. preopen) subsets of $X$ and furthermore since $0 < f - g$, it follows that

$$\bigcap_{n=1}^{\infty} A(f - g, 3^{-n+1}) = \emptyset.$$ 

Now by Lemma 3.3, there exists a decreasing sequence $\{D_n\}$ of preclosed (resp. semi-closed) subsets of $X$ such that $A(f - g, 3^{-n+1}) \subseteq D_n$ and

$$\bigcap_{n=1}^{\infty} D_n = \emptyset.$$ 

But by Lemma 3.2, the pair $A(f - g, 3^{-n+1})$ and $X \setminus D_n$ of semi-open (resp. preopen) and preopen (resp. semi-open) subsets of $X$ can be completely separated by contra-$\alpha$-continuous functions. Hence by Theorem 2.2, there exists a contra-$\alpha$-continuous function $h$ defined on $X$ such that $g < h < f$, i.e., $X$ has the weakly $c\alpha$-insertion property for $(cpc, csc)$ (resp. $(csc, cpc)$).

**References**


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