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Insertion of a Contra-α-continuous Function

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Abstract

A necessary and sufficient condition in terms of lower cut sets is given for the insertion of a contra-α-continuous function between two comparable real-valued functions.

1. Introduction

The concept of a preopen set in a topological space was introduced by Corson and Michael in 1964 [4]. A subset A of a topological space (X, τ) is called *preopen* or *locally dense* or *nearly open* if $A \subseteq Int(Cl(A))$. A set A is called *preclosed* if its complement is preopen or equivalently if $Cl(Int(A)) \subseteq A$. The term, preopen, was used for the first time by Mashhour et al. [20], while the concept of a locally dense set was introduced by Corson and Michael [4].

The concept of a semi-open set in a topological space was introduced by Levine in 1963 [17]. A subset A of a topological space (X, τ) is called *semi-open* [10] if $A \subseteq Cl(Int(A))$. A set A is called *semi-closed* if its complement is semi-open or equivalently if $Int(Cl(A)) \subseteq A$.

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Recall that a subset A of a topological space (X, τ) is called α-*open* if A is the difference of an open and a nowhere dense subset of X. A set A is called α-*closed* if its complement is α-open or equivalently if A is union of a closed and a nowhere dense set.

A set is α-open if and only if it is semi-open and preopen.

A generalized class of closed sets was considered by Maki in [19]. He investigated the sets that can be represented as union of closed sets and called them *V-sets*. Complements of *V*-sets, i.e., sets that are intersection of open sets are called Λ-*sets* [19].

Recall that a real-valued function f defined on a topological space X is called *A-continuous* [25] if the preimage of every open subset of $\mathbb{R}$ belongs to A, where A is a collection of subsets of X. Most of the definitions of function used throughout this paper are consequences of the definition of A-continuity. However, for unknown concepts the reader may refer to [5, 11]. In the recent literature many topologists had focused their research in the direction of investigating different types of generalized continuity.

Dontchev in [6] introduced a new class of mappings called *contra-continuity*. Jafari and Noiri in [12, 13] exhibited and studied among others a new weaker form of this class of mappings called *contra*-α-*continuous*. A good number of researchers have also initiated different types of contra-continuous like mappings in the papers [1, 3, 8, 9, 10, 24].

Hence, a real-valued function f defined on a topological space X is called *contra*-α-*continuous* (resp. *contra-semi-continuous*, *contra-precontinuous*) if the preimage of every open subset of $\mathbb{R}$ is α-closed (resp. semi-closed, preclosed) in X [6].

Results of Katětov [14, 15] concerning binary relations and the concept of an indefinite lower cut set for a real-valued function, which is due to Brooks [2], are used in order to give a necessary and sufficient conditions for the insertion of a contra-α-continuous function between two comparable real-valued functions.

If g and f are real-valued functions defined on a space X, we write $g \le f$ (resp. $g < f$) in case $g(x) \le f(x)$ (resp. $g(x) < f(x)$) for all x in X.

The following definitions are modifications of conditions considered in [16].

A property P defined relative to a real-valued function on a topological space is a $c\alpha$-*property* provided that any constant function has property P and provided that the sum of a function with property *P* and any contra-α-continuous function also has property *P*. If P_1 and P_2 are *c*α-property, the following terminology is used: (i) A space *X* has the *weak c*α*-insertion property* for (P_1, P_2) if and only if for any functions *g* and *f* on *X* such that $g \leq f$, *g* has property P_1 and *f* has property P_2, then there exists a contra-α-continuous function *h* such that $g \leq h \leq f$. (ii) A space *X* has the *c*α*-insertion property* for (P_1, P_2) if and only if for any functions *g* and *f* on *X* such that $g < f$, *g* has property P_1 and *f* has property P_2, then there exists a contra-α-continuous function *h* such that $g < h < f$. (iii) A space *X* has the *weakly c*α*-insertion property* for (P_1, P_2) if and only if for any functions *g* and *f* on *X* such that $g < f$, *g* has property P_1, *f* has property P_2 and $f - g$ has property P_2, then there exists a contra-α-continuous function *h* such that $g < h < f$.

In this paper, it is given a sufficient condition for the weak *c*α-insertion property. Also for a space with the weak *c*α-insertion property, we give a necessary and sufficient condition for the space to have the *c*α-insertion property. Several insertion theorems are obtained as corollaries of these results.

2. The Main Result

Before giving a sufficient condition for insertability of a contra-α-continuous function, the necessary definitions and terminology are stated.

Let (X, τ) be a topological space. Then the family of all α-open, α-closed, semi-open, semi-closed, preopen and preclosed will be denoted by $\alpha O(X, \tau)$, $\alpha C(X, \tau)$, $sO(X, \tau)$, $sC(X, \tau)$, $pO(X, \tau)$ and $pC(X, \tau)$, respectively.

Definition 2.1. Let *A* be a subset of a topological space (X, τ). We define the subsets A^{Λ} and A^{V} as follows:

$$A^{\Lambda} = \bigcap\{O : O \supseteq A, O \in (X, \tau)\} \quad \text{and} \quad A^{V} = \bigcup\{F : F \subseteq A, F^{c} \in (X, \tau)\}.$$

In [7, 18, 23], A^{Λ} is called the *kernel* of *A*.

We define the subsets $\alpha(A^{\Lambda}), \alpha(A^{V}), p(A^{\Lambda}), p(A^{V}), s(A^{\Lambda})$ and $s(A^{V})$ as follows:

$$\alpha(A^{\Lambda}) = \bigcap\{O : O \supseteq A, O \in \alpha O(X, \tau)\},$$

$$\alpha(A^{V}) = \bigcup\{F : F \subseteq A, F \in \alpha C(X, \tau)\},$$

$$p(A^{\Lambda}) = \bigcap\{O : O \supseteq A, O \in pO(X, \tau)\},$$

$$p(A^{V}) = \bigcup\{F : F \subseteq A, F \in pC(X, \tau)\},$$

$$s(A^{\Lambda}) = \bigcap\{O : O \supseteq A, O \in sO(X, \tau)\}$$

and

$$s(A^{V}) = \bigcup\{F : F \subseteq A, F \in sC(X, \tau)\}.$$

$\alpha(A^{\Lambda})$ (resp. $p(A^{\Lambda})$, $s(A^{\Lambda})$) is called the α-*kernel* (resp. *prekernel*, *semi-kernel*) of A.

The following first two definitions are modifications of conditions considered in [14, 15].

Definition 2.2. If ρ is a binary relation in a set S, then $\overline{\rho}$ is defined as follows: $x \,\overline{\rho}\, y$ if and only if $y \,\rho\, v$ implies $x \,\rho\, v$ and $u \,\rho\, x$ implies $u \,\rho\, y$ for any u and v in S.

Definition 2.3. A binary relation ρ in the power set $P(X)$ of a topological space X is called a *strong binary relation* in $P(X)$ in case ρ satisfies each of the following conditions:

(1) If $A_i \,\rho\, B_j$ for any $i \in \{1, ..., m\}$ and for any $j \in \{1, ..., n\}$, then there exists a set C in $P(X)$ such that $A_i \,\rho\, C$ and $C \,\rho\, B_j$ for any $i \in \{1, ..., m\}$ and any $j \in \{1, ..., n\}$.

(2) If $A \subseteq B$, then $A \,\overline{\rho}\, B$.

(3) If $A \,\rho\, B$, then $\alpha(A^{\Lambda}) \subseteq B$ and $A \subseteq \alpha(B^{V})$.

The concept of a lower indefinite cut set for a real-valued function was defined by Brooks [2] as follows:

Definition 2.4. If f is a real-valued function defined on a space X and if $\{x \in X : f(x) < \ell\} \subseteq A(f, \ell) \subseteq \{x \in X : f(x) \leq \ell\}$ for a real number ℓ, then $A(f, \ell)$ is called a *lower indefinite cut set* in the domain of f at the level ℓ.

We now give the following main result:

Theorem 2.1. *Let g and f be real-valued functions on the topological space X, in which α-kernel sets are α-open, with* $g \leq f$. *If there exists a strong binary relation ρ on the power set of X and if there exist lower indefinite cut sets* $A(f, t)$ *and* $A(g, t)$ *in the domain of f and g at the level t for each rational number t such that if* $t_1 < t_2$, *then* $A(f, t_1)\ \rho\ A(g, t_2)$, *then there exists a contra-α-continuous function h defined on X such that* $g \leq h \leq f$.

Proof. Theorem 2.1 in [22].

Theorem 2.2. *Let* P_1 *and* P_2 *be cα-property and X be a space that satisfies the weak cα-insertion property for* (P_1, P_2). *Also assume that g and f are functions on X such that* $g < f$, *g has property* P_1 *and f has property* P_2. *The space X has the cα-insertion property for* (P_1, P_2) *if and only if there exist lower cut sets* $A(f - g, 3^{-n+1})$ *and there exists a decreasing sequence* $\{D_n\}$ *of subsets of X with empty intersection and such that for each n,* $X \backslash D_n$ *and* $A(f - g, 3^{-n+1})$ *are completely separated by contra-α-continuous functions.*

Proof. Theorem 2.1 in [21].

3. Applications

The abbreviations *cαc*, *cpc* and *csc* are used for contra-α-continuous, contra-precontinuous and contra-semi-continuous, respectively.

Before stating the consequences of Theorems 2.1, 2.2, we suppose that *X* is a topological space whose α-kernel sets are α-open.

Corollary 3.1. *If for each pair of disjoint preopen (resp. semi-open) sets* G_1, G_2 *of X, there exist α-closed sets* F_1 *and* F_2 *of X such that* $G_1 \subseteq F_1$, $G_2 \subseteq F_2$ *and* $F_1 \cap F_2 = \varnothing$, *then X has the weak cα-insertion property for (cpc, cpc) (resp. (csc, csc)).*

Proof. Corollary 3.1 in [22].

Corollary 3.2. *If for each pair of disjoint preopen (resp. semi-open) sets* G_1, G_2, *there exist* α*-closed sets* F_1 *and* F_2 *such that* $G_1 \subseteq F_1, G_2 \subseteq F_2$ *and* $F_1 \cap F_2 = \varnothing$, *then every contra-precontinuous (resp. contra-semi-continuous) function is contra-*α*-continuous*.

Proof. Corollary 3.2 in [22].

Corollary 3.3. *If for each pair of disjoint preopen (resp. semi-open) sets* G_1, G_2 *of X, there exist* α*-closed sets* F_1 *and* F_2 *of X such that* $G_1 \subseteq F_1, G_2 \subseteq F_2$ *and* $F_1 \cap F_2 = \varnothing$, *then X has the c*$\alpha$*-insertion property for* (cpc, cpc) *(resp.* (csc, csc)*).*

Proof. Let g and f be real-valued functions defined on the X, such that f and g are cpc (resp. csc), and $g < f$. Set $h = (f + g)/2$, thus $g < h < f$, and by Corollary 3.2, since g and f are contra-α-continuous functions hence h is a contra-α-continuous function.

Corollary 3.4. *If for each pair of disjoint subsets* G_1, G_2 *of X, such that* G_1 *is preopen and* G_2 *is semi-open, there exist* α*-closed subsets* F_1 *and* F_2 *of X such that* $G_1 \subseteq F_1, G_2 \subseteq F_2$ *and* $F_1 \cap F_2 = \varnothing$, *then X have the weak c*α*-insertion property for* (cpc, csc) *and* (csc, cpc).

Proof. Corollary 3.4 in [22].

Before stating consequences of Theorem 2.2, we state and prove the necessary lemmas.

Lemma 3.1. *The following conditions on the space X are equivalent*:

(i) *For each pair of disjoint subsets* G_1, G_2 *of X, such that* G_1 *is preopen and* G_2 *is semi-open, there exist* α*-closed subsets* F_1, F_2 *of X such that* $G_1 \subseteq F_1, G_2 \subseteq F_2$ *and* $F_1 \cap F_2 = \varnothing$.

(ii) *If G is a semi-open (resp. preopen) subset of X which is contained in a preclosed (resp. semi-closed) subset F of X, then there exists an* α*-closed subset H of X such that* $G \subseteq H \subseteq \alpha(H^{\Lambda}) \subseteq F$.

Proof. Lemma 3.1 in [22].

Lemma 3.2. *Suppose that X is a topological space. If each pair of disjoint subsets* G_1, G_2 *of X, where* G_1 *is preopen and* G_2 *is semi-open, can be separated by α-closed subsets of X, then there exists a contra-α-continuous function* $h : X \to [0, 1]$ *such that* $h(G_2) = \{0\}$ *and* $h(G_1) = \{1\}$.

Proof. Lemma 3.2 in [22].

Lemma 3.3. *Suppose that X is a topological space such that every two disjoint semi-open and preopen subsets of X can be separated by α-closed subsets of X. The following conditions are equivalent*:

(i) *Every countable covering of semi-closed* (*resp. preclosed*) *subsets of X has a refinement consisting of preclosed* (*resp. semi-closed*) *subsets of X such that for every* $x \in X$, *there exists an α-closed subset of X containing x such that it intersects only finitely many members of the refinement.*

(ii) *Corresponding to every decreasing sequence* $\{G_n\}$ *of semi-open* (*resp. preopen*) *subsets of X with empty intersection there exists a decreasing sequence* $\{F_n\}$ *of preclosed* (*resp. semi-closed*) *subsets of X such that* $\bigcap_{n=1}^{\infty} F_n = \varnothing$ *and for every* $n \in \mathbb{N}, G_n \subseteq F_n$.

Proof. (i) $\Rightarrow$ (ii) Suppose that $\{G_n\}$ is a decreasing sequence of semi-open (resp. preopen) subsets of X with empty intersection. Then $\{G_n^c : n \in \mathbb{N}\}$ is a countable covering of semi-closed (resp. preclosed) subsets of *X*. By hypothesis (i) and Lemma 3.1, this covering has a refinement $\{V_n : n \in \mathbb{N}\}$ such that every V_n is an α-closed subset of X and $\alpha(V_n^{\Lambda}) \subseteq G_n^c$. By setting $F_n = \alpha((V_n^{\Lambda})^c)$, we obtain a decreasing sequence of α-closed subsets of *X* with the required properties.

(ii) $\Rightarrow$ (i) Now if $\{H_n : n \in \mathbb{N}\}$ is a countable covering of semi-closed (resp. preclosed) subsets of *X*, we set for $n \in \mathbb{N}$, $G_n = \left(\bigcup_{i=1}^{n} H_i\right)^c$. Then $\{G_n\}$ is a decreasing sequence of semi-open (resp. preopen) subsets of X with empty intersection. By (ii) there exists a decreasing sequence $\{F_n\}$ consisting of preclosed (resp. semi-closed) subsets of X such that $\bigcap_{n=1}^{\infty} F_n = \varnothing$ and for every $n \in \mathbb{N}, G_n \subseteq F_n$. Now we define the subsets

W_n of X in the following manner:

W_1 is an α-closed subset of X such that $F_1^c \subseteq W_1$ and $\alpha(W_1^\Lambda) \cap G_1 = \varnothing$.

W_2 is an α-closed subset of X such that $\alpha(W_1^\Lambda) \cup F_2^c \subseteq W_2$ and $\alpha(W_2^\Lambda) \cap G_2 = \varnothing$, and so on. (By Lemma 3.1, W_n exists).

Then since $\{F_n^c : n \in \mathbb{N}\}$ is a covering for X, hence $\{W_n : n \in \mathbb{N}\}$ is a covering for X consisting of α-closed sets. Moreover, we have

(i) $\alpha(W_n^\Lambda) \subseteq W_{n+1}$.

(ii) $F_n^c \subseteq W_n$.

(iii) $W_n \subseteq \bigcup_{i=1}^n H_i$.

Now setting $S_1 = W_1$ and for $n \geq 2$, we set $S_n = W_{n+1} \backslash \alpha(W_{n-1}^\Lambda)$.

Then since $\alpha(W_{n-1}^\Lambda) \subseteq W_n$ and $S_n \supseteq W_{n+1} \backslash W_n$, it follows that $\{S_n : n \in \mathbb{N}\}$ consists of α-closed sets and covers X. Furthermore, $S_i \cap S_j \neq \varnothing$ if and only if $|i - j| \leq 1$. Finally, consider the following sets:

$$
\begin{array}{llllll}
S_1 \cap H_1, & S_1 \cap H_2 & & & & \\
S_2 \cap H_1, & S_2 \cap H_2, & S_2 \cap H_3 & & & \\
S_3 \cap H_1, & S_3 \cap H_2, & S_3 \cap H_3, & S_3 \cap H_4 & & \\
\vdots & & & & & \\
S_i \cap H_1, & S_i \cap H_2, & S_i \cap H_3, & S_i \cap H_4, & \ldots, & S_i \cap H_{i+1} \\
\vdots & & & & &
\end{array}
$$

These sets are α-closed sets, cover X and refine $\{H_n : n \in \mathbb{N}\}$. In addition, $S_i \cap H_j$ can intersect at most the sets in its row, immediately above, or immediately below row.

Hence if $x \in X$ and $x \in S_n \cap H_m$, then $S_n \cap H_m$ is an α-closed set containing x that intersects at most finitely many of sets $S_i \cap H_j$. Consequently, $\{S_i \cap H_j : i \in \mathbb{N},$

$j = 1, \ldots, i+1\}$ refines $\{H_n : n \in \mathbb{N}\}$ such that its elements are α-closed sets, and for every point in X we can find an α-closed set containing the point that intersects only finitely many elements of that refinement.

Corollary 3.5. *If every two disjoint semi-open and preopen subsets of X can be separated by α-closed subsets of X, and in addition, every countable covering of semi-closed (resp. preclosed) subsets of X has a refinement that consists of preclosed (resp. semi-closed) subsets of X such that for every point of X we can find an α-closed subset containing that point such that it intersects only a finite number of refining members, then X has the weakly cα-insertion property for (cpc, csc) (resp. (csc, cpc)).*

Proof. Since every two disjoint semi-open and preopen sets can be separated by α-closed subsets of X, therefore by Corollary 3.4, X has the weak $c\alpha$-insertion property for (cpc, csc) and (csc, cpc). Now suppose that f and g are real-valued functions on X with $g < f$, such that g is cpc (resp. csc), f is csc (resp. cpc) and $f - g$ is csc (resp. cpc). For every $n \in \mathbb{N}$, set

$$A(f - g, 3^{-n+1}) = \{x \in X : (f - g)(x) \leq 3^{-n+1}\}.$$

Since $f - g$ is csc (resp. cpc), hence $A(f - g, 3^{-n+1})$ is a semi-open (resp. preopen) subset of X. Consequently, $\{A(f - g, 3^{-n+1})\}$ is a decreasing sequence of semi-open (resp. preopen) subsets of X and furthermore since $0 < f - g$, it follows that $\bigcap_{n=1}^{\infty} A(f - g, 3^{-n+1}) = \varnothing$. Now by Lemma 3.3, there exists a decreasing sequence $\{D_n\}$ of preclosed (resp. semi-closed) subsets of X such that $A(f - g, 3^{-n+1}) \subseteq D_n$ and $\bigcap_{n=1}^{\infty} D_n = \varnothing$. But by Lemma 3.2, the pair $A(f - g, 3^{-n+1})$ and $X \setminus D_n$ of semi-open (resp. preopen) and preopen (resp. semi-open) subsets of X can be completely separated by contra-α-continuous functions. Hence by Theorem 2.2, there exists a contra-α-continuous function h defined on X such that $g < h < f$, i.e., X has the weakly $c\alpha$-insertion property for (cpc, csc) (resp. (csc, cpc)).

References

[1] A. Al-Omari and M. S. Md Noorani, Some properties of contra-b-continuous and almost contra-b-continuous functions, *Eur. J. Pure. Appl. Math.* 2(2) (2009), 213-230.

[2] F. Brooks, Indefinite cut sets for real functions, *Amer. Math. Monthly* 78 (1971), 1007-1010. https://doi.org/10.1080/00029890.1971.11992929

[3] M. Caldas and S. Jafari, Some properties of contra-β-continuous functions, *Mem. Fac. Sci. Kochi Univ.* 22 (2001), 19-28.

[4] H. H. Corson and E. Michael, Metrizability of certain countable unions, *Illinois J. Math.* 8 (1964), 351-360. https://doi.org/10.1215/ijm/1256059678

[5] J. Dontchev, Characterization of some peculiar topological space via *A*- and *B*-sets, *Acta Math. Hungar.* 69(1-2) (1995), 67-71. https://doi.org/10.1007/BF01874608

[6] J. Dontchev, Contra-continuous functions and strongly S-closed spaces, *Internat. J. Math. Math. Sci.* 19(2) (1996), 303-310. https://doi.org/10.1155/S0161171296000427

[7] J. Dontchev and H. Maki, On sg-closed sets and semi-λ-closed sets, *Questions Answers Gen. Topology* 15(2) (1997), 259-266.

[8] E. Ekici, On contra-continuity, *Ann. Univ. Sci. Bodapest* 47 (2004), 127-137.

[9] E. Ekici, New forms of contra-continuity, *Carpathian J. Math*. 24(1) (2008), 37-45.

[10] A. I. El-Magbrabi, Some properties of contra-continuous mappings, *Int. J. General Topol*. 3(1-2) (2010), 55-64.

[11] M. Ganster and I. Reilly, A decomposition of continuity, *Acta Math. Hungar*. 56(3-4) (1990), 299-301. https://doi.org/10.1007/BF01903846

[12] S. Jafari and T. Noiri, Contra-α-continuous functions between topological spaces, *Iran. Int. J. Sci. 2* (2001), 153-167.

[13] S. Jafari and T. Noiri, On contra-precontinuous functions, *Bull. Malays. Math. Sci. Soc.* 25 (2002), 115-128.

[14] M. Katětov, On real-valued functions in topological spaces, *Fund. Math.* 38 (1951), 85-91. https://doi.org/10.4064/fm-38-1-85-91

[15] M. Katětov, Correction to “On real-valued functions in topological spaces” (Fund. Math. 38 (1951), pp. 85-91), *Fund. Math.* 40 (1953), 203-205. https://doi.org/10.4064/fm-40-1-203-205

[16] E. Lane, Insertion of a continuous function, *Pacific J. Math.* 66 (1976), 181-190. https://doi.org/10.2140/pjm.1976.66.181

[17] N. Levine, Semi-open sets and semi-continuity in topological space, *Amer. Math. Monthly* 70 (1963), 36-41. https://doi.org/10.1080/00029890.1963.11990039

[18] S. N. Maheshwari and R. Prasad, On $(R_0)_s$ -spaces, *Portugal. Math.* 34 (1975), 213-217.

[19] H. Maki, Generalized Λ-sets and the associated closure operator, *The Special Issue in Commemoration of Prof. Kazusada IKEDA's Retirement* (1986), 139-146.

[20] A. S. Mashhour, M. E. Abd El-Monsef and S. N. El-Deeb, On precontinuous and weak precontinuous mappings, *Proc. Math. Phys. Soc. Egypt* 53 (1982), 47-53.

[21] M. Mirmiran, Insertion of a function belonging to a certain subclass of $\mathbb{R}^x$, *Bull. Iranian Math. Soc.* 28(2) (2002), 19-27.

[22] M. Mirmiran and B. Naderi, Strong insertion of a contra-α-continuous function between two comparable real-valued functions, *Earthline J. Math. Sci.* 2(1) (2019), 223-239. https://doi.org/10.34198/ejms.2119.223239

[23] M. Mršević, On pairwise R_0 and pairwise R_1 bitopological spaces, *Bull. Math. Soc. Sci. Math. R. S. Roumanie* 30 (1986), 141-148.

[24] A. A. Nasef, Some properties of contra-γ-continuous functions, *Chaos Solitons Fractals* 24 (2005), 471-477. https://doi.org/10.1016/j.chaos.2003.10.033

[25] M. Przemski, A decomposition of continuity and α-continuity, *Acta Math. Hungar.* 61(1-2) (1993), 93-98. https://doi.org/10.1007/BF01872101