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# Aspects of Free Actions Based on Dependent Elements in Group Rings

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## Abstract

This paper contains two directions of work. The first one gives material related to free action (an inner derivation) mappings on a group ring $\mathcal{R}[\mathcal{G}]$ which is a construction involving a group $\mathcal{G}$ and a ring $\mathcal{R}$ and the dependent elements related to those mappings in $\mathcal{R}[\mathcal{G}]$. The other direction deals with a generalization of the definition of dependent elements and free actions. We concentrate our study on dependent elements, free action mappings and those which satisfy $\mathcal{T}(x)\gamma = \delta x, x \in \mathcal{R}[\mathcal{G}]$ and some fixed $\gamma, \delta \in \mathcal{R}[\mathcal{G}]$. In the first part we work with one dependent element. In other words, there exists an element $\gamma \in \mathcal{R}[\mathcal{G}]$ such that $\mathcal{T}(x)\gamma = \gamma x, x \in \mathcal{R}[\mathcal{G}]$. In second one, we characterize the two elements $\gamma, \delta \in \mathcal{R}[\mathcal{G}]$ which have the property $\mathcal{T}(x)\gamma = \delta x, x \in \mathcal{R}[\mathcal{G}]$ and some fixed $\gamma, \delta \in \mathcal{R}[\mathcal{G}]$, when $\mathcal{T}$ is assumed to have additional properties like generalized a derivation mappings.

## 1. Introduction

A group ring was done by Mihalev and Zalesskii [1], denoted by $\mathcal{R}[\mathcal{G}]$, a construction involving a group $\mathcal{G}$ and a ring $\mathcal{R}$. Many references are given here, but the interested readers are invited to consult the book [1] or the surveys, [2, 3, 4]. Group rings have since found applications in many different branches of algebra, and there are naturally many open problems which are areas of active researches.

For any $x \in \mathcal{R}[\mathcal{G}]$ has the form sums $x = \sum_{g \in \mathcal{G}} a_g g \left(a_g \in \mathcal{R}, g \in \mathcal{G}\right)$, for which all but finitely many coefficients $a_g = 0$. Also, $\mathcal{R}[\mathcal{G}]$ with addition and multiplication occurred as follows:

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$$\sum_{g\in\mathcal{G}} a_g\, g + \sum_{g\in\mathcal{G}} b_g\, g = \sum_{g\in\mathcal{G}} (a_g + b_g)\, g$$

and

$$\left(\sum_{g\in\mathcal{G}} a_g\, g\right)\left(\sum_{h\in\mathcal{G}} b_h\, h\right) = \sum_{g,h\in\mathcal{G}} a_g b_h\, gh.$$

The above definitions make $\mathcal{R}[\mathcal{G}]$ into an associative and unital ring. The multiplicative identity element is $1_{\mathcal{R}} \cdot 1_{\mathcal{G}}$, where $1_{\mathcal{R}} \in \mathcal{R}$ and $1_{\mathcal{G}} \in \mathcal{G}$, since

$$1_{\mathcal{R}} \cdot 1_{\mathcal{G}}\left(\sum_{g\in\mathcal{G}} a_g\, g\right) = \sum_{g\in\mathcal{G}} (1_{\mathcal{R}} a_g)(1_{\mathcal{G}} g) = \sum_{g\in\mathcal{G}} a_g\, g = \sum_{g\in\mathcal{G}} (a_g 1_{\mathcal{R}})(g 1_{\mathcal{G}}) = \left(\sum_{g\in\mathcal{G}} a_g\, g\right) 1_{\mathcal{R}} \cdot 1_{\mathcal{G}}.$$

The inverse of an element $\sum_{g\in\mathcal{G}} a_g\, g \in \mathcal{R}[\mathcal{G}]$ has the form $\sum_{g\in\mathcal{G}} a_g\, g$. Occasionally, $\mathcal{R}[\mathcal{G}]$ is not commutative. It is commutative iff together $\mathcal{R}$ and $\mathcal{G}$ are commutative. We can also define an action of the ring $\mathcal{R}$ on $\mathcal{R}[\mathcal{G}]$ by $\mathfrak{r}(\sum_{g\in\mathcal{G}} a_g\, g) = \sum_{g\in\mathcal{G}} (\mathfrak{r} a_g) g$ $(\mathfrak{r} \in \mathcal{R})$.

Obviously, $\mathcal{R}[\mathcal{G}]$ is an extension of $\mathcal{R}$ and a ring embedding $\mathcal{R} \longrightarrow \mathcal{R}[\mathcal{G}]$ given by $\mathfrak{r} \longrightarrow \mathfrak{r} \cdot 1_{\mathcal{G}}$. If $\mathcal{R}$ is a commutative, then the image of $\mathcal{R}$ in $\mathcal{R}[\mathcal{G}]$ is contained in $\mathcal{C}(\mathcal{R}[\mathcal{G}])$ ($\mathcal{C}(\mathcal{R}[\mathcal{G}])$ center of $\mathcal{R}[\mathcal{G}]$) such that:

$$\mathcal{C}(\mathcal{R}[\mathcal{G}]) = \left\{\gamma = \sum_{g\in\mathcal{G}} a_g\, g \in \mathcal{R}[\mathcal{G}] : h\left(\sum_{g\in\mathcal{G}} a_g\, g\right) h^{-1} = \sum_{g\in\mathcal{G}} a_{hgh^{-1}}\, g \text{ for all } h \in \mathcal{G}\right\} \text{ [5].}$$

The mapping $g \longrightarrow 1_{\mathcal{R}} \cdot g$ is a group embedding of $\mathcal{G}$ in $\mathcal{R}[\mathcal{G}]$. Accordingly primness, $\mathcal{R}[\mathcal{G}]$ is prime iff $\mathcal{R}$ is prime ring and $\mathcal{G}$ has no finite normal sub group. Also, $\mathcal{R}[\mathcal{G}]$ is a semiprime iff $\mathcal{R}$ is semiprime ring and the order of each finite normal subgroup of $\mathcal{G}$ is regular in $\mathcal{R}$. In certain references (see [5, 6, 7]) were studied the properties of group rings.

Derivations of group rings have been a topic for studies by Smith in [8], was one of the first to study the derivations in group rings. On the other hand, motivated by the works in [9, 10, 11, 12, 13, 14, 15] to see more results about derivation mappings on a group rings.

In recent papers [16, 17, 18, 19, 20, 21, 22], properties of generalized derivations, central derivations and derivations of group rings over finite rings were studied.

In our study, we study generalized derivations of an $\mathcal{R}[\mathcal{G}]$, where $\mathcal{G}$ is a finite group over an integral domain $\mathcal{R}$ with 1. As an application, we determine the characterizations of dependent elements of generalized derivations in $\mathcal{R}[\mathcal{G}]$ by taking a certain assumptions, we also show that for a derivations are free action associated with an automorphism and give some results about that.

## 2. Main Results

In this section $\mathcal{R}[\mathcal{G}]$ is a prime group ring, unless otherwise stated. Also, $\mathfrak{D}(\mathcal{R}[\mathcal{G}])$ denotes the set of all derivation mappings on $\mathcal{R}[\mathcal{G}]$ in our main results.

**Definition 2.1.** An element $\gamma \in \mathcal{R}[\mathcal{G}]$ is a dependent of $\mathcal{T} \in \mathfrak{Der}(\mathcal{R}[\mathcal{G}])$, if $\mathcal{T}(x)\gamma = \gamma x \ \forall x \in \mathcal{R}[\mathcal{G}]$. The set of all dependent elements of $\mathcal{T}$ denotes $\mathcal{D}(\mathcal{T})$. If there exist no non-zero element $\gamma \in \mathcal{R}[\mathcal{G}]$ which satisfies that if $\mathcal{T}(x)\gamma = \gamma x$ for all $x \in \mathcal{R}[\mathcal{G}]$, then we say that $\mathcal{T}$ is a free action on $\mathcal{R}[\mathcal{G}]$, in other words, $\mathcal{D}(\mathcal{T}) = 0$. The set of all free action mappings on $\mathcal{R}[\mathcal{G}]$ denotes $\mathcal{F}(\mathcal{R}[\mathcal{G}])$.

**Example 2.2.** Take $\mathcal{R}$ be a field, $\mathcal{G} = \{1_{\mathcal{G}}, g, g^2, \ldots, g^{n-1}\}$ ($\mathcal{G}$ cyclic group of order $n$) and the group ring $\mathcal{R}[\mathcal{G}]$. Consider $\gamma \in \mathcal{R}[\mathcal{G}]$. We can define a mapping $\mathcal{T}: \mathcal{R}[\mathcal{G}] \longrightarrow \mathcal{R}[\mathcal{G}]$ by $\mathcal{T}(x) = \gamma x \gamma^{-1} \forall x \in \mathcal{R}[\mathcal{G}]$ with a fixed $\gamma$. Then $\mathcal{T} \in \mathfrak{Der}(\mathcal{R}[\mathcal{G}])$ such that $\gamma \in \mathcal{D}(\mathcal{T})$ (since $\mathcal{T}(x)\gamma = \gamma x \gamma^{-1}\gamma = \gamma x$).

**Theorem 2.3.** *Let* $\mathcal{T} \in \mathfrak{Der}(\mathcal{R}[\mathcal{G}])$. *Then* $\gamma \in \mathcal{D}(\mathcal{T})$ *if and only if* $\gamma \in \mathcal{C}(\mathcal{R}[\mathcal{G}])$ *and* $\mathcal{T}(\gamma) = \gamma$.

**Proof.** Let $\gamma \in \mathcal{D}(\mathcal{T})$. Then,

$$\mathcal{T}(x)\gamma = \gamma x \ \ \forall x = \sum_{g \in \mathcal{G}} a_g g \in \mathcal{R}[\mathcal{G}]. \tag{1}$$

Replacing $x$ by $xy$ in (1), we get:

$$\mathcal{T}(xy)\gamma = \gamma xy \ \text{ for all } x, y \in \mathcal{R}[\mathcal{G}]. \tag{2}$$

Multiplying (2) by $z$ on the right, we get:

$$\mathcal{T}(x)y\gamma z = \gamma xyz \ \text{ for all } x, y, z \in \mathcal{R}[\mathcal{G}]. \tag{3}$$

Replacing $y$ by $yz$ in (2), we get:

$$\mathcal{T}(x)yz\gamma = \gamma xyz \ \text{ for all } x, y, z \in \mathcal{R}[\mathcal{G}]. \tag{4}$$

Subtracting (4) from (3), we get:

$$\mathcal{T}(x)y\gamma z - \mathcal{T}(x)yz\gamma = 0, \ \ \forall x, y, z \in \mathcal{R}[\mathcal{G}].$$

So,

$$\mathcal{T}(x)y(\gamma z - z\gamma) = 0, \ \ \forall x, y, z \in \mathcal{R}[\mathcal{G}].$$

Replacing $y$ by $\gamma y$ and then:

$$\mathcal{T}(x)\gamma y[\gamma , z] = 0, \ \ \forall x, y, z \in \mathcal{R}[\mathcal{G}].$$

From (1), implies that:

$$\gamma x y[\gamma , z] = 0, \ \ \forall x, y, z \in \mathcal{R}[\mathcal{G}].$$

By using semi-primeness of $\mathcal{R}[\mathcal{G}]$, we have:

$$\mathcal{T}(x)\gamma[\gamma , z] = 0, \ \forall x, z \in \mathcal{R}[\mathcal{G}].$$

That is, $[\gamma , z] = 0$, $\forall x, z \in \mathcal{R}[\mathcal{G}]$. From semi-primeness of $\mathcal{R}[\mathcal{G}]$ again, we have $\gamma[\gamma , z] = 0$, $\forall z = \sum_{g\in\mathcal{G}} c_g\, g \in \mathcal{R}[\mathcal{G}]$. This implies that, $\gamma \in \mathcal{C}(\mathcal{R}[\mathcal{G}])$.

Now, $\gamma = \sum_{g\in\mathcal{G}} d_g\, g \in \mathcal{C}(\mathcal{R}[\mathcal{G}])$, we have $\sum_{g\in\mathcal{G}} d_{hgh^{-1}}\, g = \sum_{g\in\mathcal{G}} d_g\, g$ for all $z \in \mathcal{R}[\mathcal{G}]$. This implies that:

$$\left(\sum_{g\in\mathcal{G}} d_g\, g\right)\left(\sum_{h\in\mathcal{G}} b_h\, h\right) = \sum_{g\in\mathcal{G}, h\in\mathcal{G}} a_g\, d_{g^{-1}h} g.$$

Thus, $\mathcal{T}(\gamma y) = \mathcal{T}(y\gamma)$ for all $y \in \mathcal{R}[\mathcal{G}]$ and hence, $\mathcal{T}(\gamma y) = \mathcal{T}(y\gamma) = \gamma y$. This implies that:

$$\mathcal{T}(\gamma)y = \mathcal{T}(y)\gamma = \gamma y = (\mathcal{T}(\gamma) - \gamma)y = 0.$$

Thus, $\mathcal{T}(\gamma) = \gamma$.

Conversely, let $\mathcal{T}(\gamma) = \gamma$ and $\gamma = \sum_{g\in\mathcal{G}} a_g\, g \in \mathcal{C}(\mathcal{R}[\mathcal{G}])$. Therefore,

$$h\left(\sum_{g\in\mathcal{G}} a_g\, g\right)h^{-1} = \sum_{g\in\mathcal{G}} a_{hgh^{-1}}\, g \ \text{ for all } h \in \mathcal{G}.$$

Since $\mathcal{T} \in \mathfrak{D}er(\mathcal{R}[\mathcal{G}])$,

$$\mathcal{T}(\gamma) = \mathcal{T}\left(h\left(\sum_{g\in\mathcal{G}} a_g\, g\right)h^{-1}\right) = \mathcal{T}\left(\sum_{g\in\mathcal{G}} a_{hgh^{-1}}\, g\right).$$

That is,

$$\mathcal{T}(x\gamma) = \mathcal{T}(x)\gamma = \gamma x = \mathcal{T}(\gamma x) = \mathcal{T}(\gamma)x = \gamma x \text{ for all } x \in \mathcal{R}[\mathcal{G}].$$

This implies that $\gamma \in \mathcal{D}(\mathcal{T})$.

**Theorem 2.4.** *Let* $\mathcal{T} \in \mathfrak{Der}(\mathcal{R}[\mathcal{G}])$, ($\mathcal{T} \neq \mathcal{J}$, $\mathcal{J}$ *an identity derivation*). *Then* $\mathcal{T} \in \mathcal{F}(\mathcal{R}[\mathcal{G}])$.

**Proof.** Let $\gamma \in \mathcal{D}(\mathcal{T})$. Then, $\mathcal{T}(x\gamma) = \gamma x$ for all $x \in \mathcal{R}[\mathcal{G}]$. By using Theorem 2.3, it follows that, $\gamma \in \mathcal{C}(\mathcal{R}[\mathcal{G}])$ and $\mathcal{T}(\gamma) = \gamma$. Then, $\mathcal{T}(x)\gamma = \gamma x$ for all $x \in \mathcal{R}[\mathcal{G}]$.

That is, $(\mathcal{T}(x) - x)\gamma = 0, \forall x \in \mathcal{R}[\mathcal{G}]$. Since $\gamma \in \mathcal{C}(\mathcal{R}[\mathcal{G}])$, we get $(\mathcal{T}(x) - x)z\gamma = 0$, $\forall z \in \mathcal{R}[\mathcal{G}]$. Since $\mathcal{T} \in \mathfrak{Der}(\mathcal{R}[\mathcal{G}])$ with the condition above, and from primeness of $\mathcal{R}[\mathcal{G}]$, so $\gamma = 0$. So, $\mathcal{T} \in \mathcal{F}(\mathcal{R}[\mathcal{G}])$.

**Theorem 2.5.** *Let* $\mathcal{T} \in \mathfrak{Der}(\mathcal{R}[\mathcal{G}])$, $\mathcal{T}$ *be an injective. Then* $\mathcal{T} + \mathcal{J} \in \mathcal{F}(\mathcal{R}[\mathcal{G}])$.

**Proof.** It is clear that $\mathcal{T} + \mathcal{J} \in \mathfrak{Der}(\mathcal{R}[\mathcal{G}])$. Let $\gamma \in \mathcal{D}(\mathcal{T} + \mathcal{J})$. By using Theorem 2.3, it follows that $\gamma \in \mathcal{C}(\mathcal{R}[\mathcal{G}])$ and $(\mathcal{T} + \mathcal{J})(\gamma) = \gamma$. So, $\mathcal{T}(\gamma) + \gamma = \gamma$ and hence $\mathcal{T}(\gamma) = 0$. Consequently, $\gamma \in ker(\mathcal{T})$. Since $\mathcal{T}$ be an injective, we have $ker(\mathcal{T}) = 0$ and hence $\gamma = 0$. This implies that $\mathcal{T} + \mathcal{J} \in \mathcal{F}(\mathcal{R}[\mathcal{G}])$.

**Theorem 2.6.** *Let* $\mathcal{T} \in \mathfrak{Der}(\mathcal{R}[\mathcal{G}])$. *If* $\varphi \in \mathfrak{Der}(\mathcal{R}[\mathcal{G}])$ *defined by* $\varphi(x) = [\mathcal{T}(x), x], \forall x \in \mathcal{R}[\mathcal{G}]$, *then* $\varphi \in \mathcal{F}(\mathcal{R}[\mathcal{G}])$.

**Proof.** Let $\gamma \in \mathcal{D}(\mathcal{T})$. Then,

$$\varphi(x\gamma) = [\mathcal{T}(x), x]\gamma = \gamma x \text{ for all } x = \textstyle\sum_{g \in \mathcal{G}} a_g\, g \in \mathcal{R}[\mathcal{G}]. \quad (1)$$

Since $\varphi \in \mathfrak{Der}(\mathcal{R}[\mathcal{G}])$ and replacing $x$ by $x + y$ and from properties of the commutators, we have:

$$\varphi(x + y) = [\mathcal{T}(x + y), x + y]\gamma = [\mathcal{T}(x), y]\gamma + [\mathcal{T}(y), x]\gamma = 0, \forall x, y \in \mathcal{R}[\mathcal{G}]. \quad (2)$$

Now, replacing $y$ by $x$ in (2) and from properties of the commutators again, implies that:

$$0 = [\mathcal{T}(x), x]\gamma + [\mathcal{T}(x), x]\gamma = 2[\mathcal{T}(x), x]\gamma, \ \forall x \in \mathcal{R}[\mathcal{G}]. \quad (3)$$

Replacing $y$ by $xy$ in(2), we have:

$$\begin{aligned} 0 &= [\mathcal{T}(x), xy]\gamma + [\mathcal{T}(xy), x]\gamma \\ &= x[\mathcal{T}(x), y]\gamma + [\mathcal{T}(x), x]y\gamma + [\mathcal{T}(x)y + x\mathcal{T}(y), x]\gamma \\ &= x[\mathcal{T}(x), y]\gamma + [\mathcal{T}(x), x]y\gamma + \mathcal{T}(x)[y, x]\gamma + [\mathcal{T}(x), x]y\gamma + x[\mathcal{T}(y), x]\gamma, \end{aligned}$$

for all $x, y \in \mathcal{R}[\mathcal{G}]$. That is,

$$0 = x([\mathcal{T}(x), y]\gamma + [\mathcal{T}(y), x]\gamma) + 2[\mathcal{T}(x), x]y\gamma + \mathcal{T}(x)[y, x]\gamma, \forall x, y \in \mathcal{R}[\mathcal{G}]. \quad (4)$$

Consequently, from (2) and (4), we have:

$$2[\mathcal{T}(x), x]y\gamma + \mathcal{T}(x)[y, x]\gamma = 0 \text{ for all } x, y \in \mathcal{R}[\mathcal{G}]. \quad (5)$$

Replacing $y$ by $y\gamma$ in (5), we have:

$$\begin{aligned} 0 &= 2[\mathcal{T}(x), x]y\gamma^2 + \mathcal{T}(x)[y\gamma, x]\gamma \\ &= 2[\mathcal{T}(x), x]y\gamma^2 + \mathcal{T}(x)[y, x]\gamma^2 + \mathcal{T}(x)y[\gamma, x]\gamma^2 \end{aligned}$$

for all $x, y \in \mathcal{R}[\mathcal{G}]$.

That is

$$(2[\mathcal{T}(x), x]y\gamma + \mathcal{T}(x)[y, x]\gamma)\gamma + \mathcal{T}(x)y[\gamma, x]\gamma^2 = 0 \text{ for all } x, y \in \mathcal{R}[\mathcal{G}]. \quad (6)$$

From (5) and (6), we have

$$\mathcal{T}(x)y[\gamma, x]\gamma = 0 \text{ for all } x, y \in \mathcal{R}[\mathcal{G}]. \quad (7)$$

Replacing $y$ by $xy$ in (7), we have

$$\mathcal{T}(x)xy[\gamma, x]\gamma = 0 \text{ for all } x, y \in \mathcal{R}[\mathcal{G}]. \quad (8)$$

Multiplying (7) by $x$, we have

$$x\mathcal{T}(x)y[\gamma, x]\gamma = 0 \text{ for all } x, y \in \mathcal{R}[\mathcal{G}]. \quad (9)$$

Subtracting (9) from (8), we obtain

$$[\mathcal{T}(x), x]y[\gamma, x]\gamma = 0 \forall x, y \in \mathcal{R}[\mathcal{G}]. \quad (10)$$

Replacing $y$ by $\gamma y$ and from (1) and (10), we have:

$$\gamma xy[\gamma, x]\gamma = 0 \text{ for all } x, y \in \mathcal{R}[\mathcal{G}]. \quad (11)$$

Replacing $y$ by $\gamma^2 y$ in(11), we have:

$$\gamma x\gamma^2 y[\gamma, x]\gamma = 0 \text{ for all } x, y \in \mathcal{R}[\mathcal{G}]. \quad (12)$$

Multiplying (11) by $\gamma$ and replacing $y$ by $\gamma y$, we have:

$$\gamma^2 x\gamma y[\gamma, x]\gamma = 0 \text{ for all } x, y \in \mathcal{R}[\mathcal{G}]. \quad (13)$$

Subtracting (12) from (13), we have:

$$\gamma(\gamma x - x\gamma)\gamma y[\gamma, x]\gamma = 0 \text{ for all } x, y \in \mathcal{R}[\mathcal{G}]. \quad (14)$$

Replacing $y$ by $y\gamma$ in (14), we have:

$$\gamma[\gamma, x]\gamma y\gamma[\gamma, x]\gamma = 0 \text{ for all } x, y \in \mathcal{R}[\mathcal{G}].$$

From semi-primeness of $\mathcal{R}[\mathcal{G}]$, it implies that $\gamma[\gamma\,,x]\gamma = 0$ for all $x \in \mathcal{R}[\mathcal{G}]$, but $\gamma[\mathcal{T}(\gamma),\gamma]\gamma = 0$. From (1), it implies that $\gamma^3 = 0$, hence $\gamma = 0$. Consequently, $\mathcal{D}(\mathcal{T}) = 0$, so $\varphi \in \mathcal{F}(\mathcal{R}[\mathcal{G}])$.

**Theorem 2.7.** *If* $\gamma\mathcal{R}[\mathcal{G}] \subseteq \mathcal{R}[\mathcal{G}]\gamma$ *for some* $\gamma \in \mathcal{R}[\mathcal{G}]$. *Then there exists* $\mathcal{T} \in \mathfrak{Der}(\mathcal{R}[\mathcal{G}])$ *such that* $\gamma \in \mathcal{D}(\mathcal{T})$.

**Proof.** Let $\gamma = \sum_{g\in\mathcal{G}} a_g\, g \in \mathcal{R}[\mathcal{G}]$. Given any $x = \sum_{h\in\mathcal{G}} b_h\, h \in \mathcal{R}[\mathcal{G}]$. Then, there exists $y = \sum_{\ell\in\mathcal{G}} b_\ell\, \ell \in \mathcal{R}[\mathcal{G}]$ such that

$$\sum_{g\in\mathcal{G},h\in\mathcal{G}} \sum_{g\in\mathcal{G},h\in\mathcal{G}} a_g\, b_{g^{-1}h}\, g = \sum_{\ell\in\mathcal{G},g\in\mathcal{G}} \sum_{g\in\mathcal{G},h\in\mathcal{G}} b_\ell\, a_{\ell^{-1}h}\, \ell.$$

Define, $\mathcal{T} : \mathcal{R}[\mathcal{G}] \longrightarrow \mathcal{R}[\mathcal{G}]$ by $\mathcal{T}(x) = y$, $\forall x = \sum_{h\in\mathcal{G}} b_h\, h \in \mathcal{R}[\mathcal{G}]$. Then $\mathcal{T} \in \mathfrak{Der}(\mathcal{R}[\mathcal{G}])$ with $\gamma \in \mathcal{D}(\mathcal{T})$, since $\gamma(\mathcal{T}(x)\gamma = y\gamma = \gamma x)$.

**Definition 2.8.** Let $\mathcal{T} \in \mathfrak{Der}(\mathcal{R}[\mathcal{G}])$. An element $\gamma \in \mathcal{R}[\mathcal{G}]$ is said to be associated with an element $\eta \in \mathcal{R}[\mathcal{G}]$, if $\mathcal{T}(x)\gamma = \eta x$ for all $x \in \mathcal{R}[\mathcal{G}]$. We shall denote $\mathcal{A}(\mathcal{T}) = \{(\gamma\,,\eta) : \mathcal{T}(x)\gamma = \eta x \text{ for all } x \in \mathcal{R}[\mathcal{G}]\}$ the set associated pairs of $\mathcal{T}$.

**Example 2.9.** Take $\mathcal{R} = \mathbb{Z}$ and $\mathcal{G} = D_8 = \{\mathfrak{r}\,,\mathfrak{s} : \operatorname{ord}(\mathfrak{r}) = 8,\ \operatorname{ord}(\mathfrak{s}) = 2\,, \mathfrak{s}\mathfrak{r}\mathfrak{s} = \mathfrak{r}^{-1}\}$, that is, $\mathcal{R}[\mathcal{G}]$ has the form the dihedral group ring of order 8. Consider elements $\gamma\,,\eta \in \mathcal{R}[\mathcal{G}]$. Consider $\mathcal{T} : \mathcal{R}[\mathcal{G}] \longrightarrow \mathcal{R}[\mathcal{G}]$ as a mapping defined by $\mathcal{T}(x) = \eta x \gamma^{-1}$ for all $x \in \mathcal{R}[\mathcal{G}]$. Then $(\gamma, \eta) \in \mathcal{A}(\mathcal{T})$.

**Theorem 2.10.** *Let* $\mathcal{T} \in \mathfrak{Der}(\mathcal{R}[\mathcal{G}])$. *Then* $\mathcal{D}(\mathcal{T})$ *leads to* $\mathcal{A}(\mathcal{T})$.

**Proof.** According to Definition 2.8, and we put $\gamma = \eta$ it is trivial to see that any $\gamma \in \mathcal{D}(\mathcal{T})$ leads to $\gamma \in \mathcal{A}(\mathcal{T})$.

**Remark 3.11.** $\mathcal{A}(\mathcal{T})$ does not lead to $\mathcal{D}(\mathcal{T})$, see Example 2.9.

**Theorem 3.12.** *Let* $\mathcal{T} \in \mathfrak{Der}(\mathcal{R}[\mathcal{G}])$. *If* $(\gamma\,,\eta\,) \in \mathcal{A}(\mathcal{T})$, *then* $(\gamma^n\,,\eta^n) \in \mathcal{A}(\mathcal{T}^n)$ *for every* $n \in \mathbb{N}$.

**Proof.** We need to prove that $\mathcal{T}^n(x)\gamma^n = \eta^n x$ for all $x = \sum_{\ell\in\mathcal{G}} b_\ell\, \ell \in \mathcal{R}[\mathcal{G}]$ and for every $n \in \mathbb{N}$. Using the induction principal: For $n = 1$, we already have $\mathcal{T}(x)\gamma = \eta x$. For $n = 2$, $\mathcal{T}^2(x)\gamma^2 = \mathcal{T}(\mathcal{T}(x)\gamma)\gamma = \eta\mathcal{T}(x)\gamma = \eta^2 x$. Now, if $\mathcal{T}^k(x)\gamma^k = \eta^k x$, then $\mathcal{T}^{k+1}(x)\gamma^{k+1} = \mathcal{T}(\mathcal{T}^k(x)\gamma^k)\gamma = \mathcal{T}(\eta^k x)\gamma = \eta^{k+1}x$.

**Definition 2.13.** Let $\mathcal{T}, \mathcal{S} \in \mathfrak{Der}(\mathcal{R}[\mathcal{G}])$ and $\gamma = \sum_{g\in\mathcal{G}} a_g\, g \in \mathcal{R}[\mathcal{G}]$. Then, $\mathcal{T}$ and $\mathcal{S}$ is said to be *dependently equivalent*, if $\mathcal{T}(x)\gamma = \mathcal{S}(x)\gamma$ for all $x \in \mathcal{R}[\mathcal{G}]$.

Recall that, $\mathcal{R}[\mathcal{G}]$ is a simple iff $\mathcal{R}$ is simple ring and $\mathcal{G}$ is finite with $|\mathcal{G}|$ is invertible in $\mathcal{R}$, [6].

**Example 2.14.** For the same $\mathcal{R}[\mathcal{G}]$ data in Example 2.2 and $\mathcal{T} \in \mathfrak{Der}(\mathcal{R}[\mathcal{G}])$, where $\mathcal{T}(x)\gamma = \gamma x$ for all $x \in \mathcal{R}[\mathcal{G}]$ with a fixed $\gamma$. Therefore, $\mathcal{T}$ and $\mathcal{S}$ are dependently equivalent for any $\mathcal{S} \in \mathfrak{Der}(\mathcal{R}[\mathcal{G}])$ such that $\mathcal{S}(x) = \gamma x \eta$ for some $\eta \in \mathcal{R}[\mathcal{G}]$.

That is, if $\gamma = 0$, then $\mathcal{T}(x)\gamma = 0 = \gamma x$ and hence $\mathcal{T}(x)\gamma = \mathcal{S}(x)\gamma = 0$. If $\gamma \neq 0$, from simpleness of $\mathcal{R}$, we obtain $\mathcal{R}[\mathcal{G}] = \mathcal{R}[\mathcal{G}]\gamma\mathcal{R}[\mathcal{G}]$.

Now, since $y\gamma x \in \mathcal{R}[\mathcal{G}]\mathcal{T}(x)\gamma \subseteq \mathcal{R}[\mathcal{G}]\gamma$, we have that $\mathcal{R}[\mathcal{G}]\gamma = \mathcal{R}[\mathcal{G}]$. Then, there exists $\eta \in \mathcal{R}[\mathcal{G}]$ such that $\eta\gamma = 1$. Then $\mathcal{S}(x) = \gamma x \eta$. So, $\mathcal{S}(x)\gamma = (\gamma x \eta)\gamma = \gamma x$, but $\big(\mathcal{T}(x) - \mathcal{S}(x)\big)\gamma = \gamma x - \gamma x = 0$. Therefore, $\mathcal{T}$ and $\mathcal{S}$ are dependently equivalent.

**Remark 2.15.** It might not be true that $\mathcal{T}(x) = \mathcal{S}(x)$ for all $x = \sum_{h \in \mathcal{G}} b_h h \in \mathcal{R}[\mathcal{G}]$, but then at least $(\mathcal{T}(x) - \mathcal{S}(x))\gamma = 0$, see Example 2.9.

**Theorem 2.16.** *Let* $\mathcal{T} \in \mathfrak{Der}(\mathcal{R}[\mathcal{G}])$. *If* $(\gamma, \eta) \in \mathcal{A}(\mathcal{T})$, *then* $\gamma \in \mathcal{C}(\mathcal{R}[\mathcal{G}])$.

**Proof.** Suppose that $(\gamma, \eta) \in \mathcal{A}(\mathcal{T})$. Then we have

$$\mathcal{T}(x)\gamma = \eta x \text{ for all } x \in \mathcal{R}[\mathcal{G}]. \tag{1}$$

Replacing $x$ by $xy$ in (1), where $y \in \mathcal{R}[\mathcal{G}]$, we get

$$\mathcal{T}(xy)\gamma = \mathcal{T}(x)y\gamma = \eta xy. \tag{2}$$

Right multiplication of (2) by $z \in \mathcal{R}[\mathcal{G}]$, we get

$$\mathcal{T}(xy)\gamma z = \mathcal{T}(x)y\gamma z = \eta xyz. \tag{3}$$

Replacing $y$ by $yz$ in (2), we get

$$\mathcal{T}(x)yz\gamma = \eta xyz. \tag{4}$$

Subtracting (4) from (3), we get

$$\mathcal{T}(x)y[\gamma, z] = 0 \text{ for all } x, y, z \in \mathcal{R}[\mathcal{G}]. \tag{5}$$

From primeness of $\mathcal{R}[\mathcal{G}]$, $\mathcal{T} \neq \Theta$ and (5) gives $[\gamma, z] = 0$ and this means $\gamma \in \mathcal{C}(\mathcal{R}[\mathcal{G}])$.

**Corollary 2.17.** *Let* $\mathcal{T} \in \mathfrak{Der}(\mathcal{R}[\mathcal{G}])$. *If* $(\gamma, \eta) \in \mathcal{A}(\mathcal{T})$, *then* $\gamma \in \mathcal{C}(\mathcal{R}[\mathcal{G}])$.

**Proof.** The result is obtained according to Theorem 2.16.

For the next theorem we need the following well known, a mapping $\mathcal{T} : \mathcal{R}[\mathcal{G}] \longrightarrow \mathcal{R}[\mathcal{G}]$ is a $(\sigma, \tau)$-derivation, if $\mathcal{T}(xy) = \mathcal{T}(x)\sigma(y) + \tau(x)\mathcal{T}(y)$ for all $x, y \in \mathcal{R}[\mathcal{G}]$, where $\sigma$ and $\tau$ are two an automorphisms on $\mathcal{R}[\mathcal{G}]$. The set of all $(\sigma,\tau)$-derivation mappings on a $\mathcal{R}[\mathcal{G}]$ will be denoted by $\mathfrak{D}_{(\sigma,\tau)}(\mathcal{R}[\mathcal{G}])$, [22].

**Theorem 2.18.** *Let* $\mathcal{T} \in \mathfrak{D}_{(\sigma,\tau)}(\mathcal{R}[\mathcal{G}])$. *Then* $\mathcal{T} \in \mathcal{F}(\mathcal{R}[\mathcal{G}])$.

**Proof.** Easily we get, $\sigma + \mathcal{T} \in \mathfrak{D}_{(\sigma,\tau)}(\mathcal{R}[\mathcal{G}])$ and $\sigma + \mathcal{T}$ has dependent element $\gamma \in \mathcal{R}[\mathcal{G}]$ that gives the facts, $(\sigma + \mathcal{T})(x)\gamma = \sigma(x)\gamma + \mathcal{T}(x)\gamma = \gamma x$ for all $x \in \mathcal{R}[\mathcal{G}]$. From dependentness of $\gamma$, we have $\sigma(x)\gamma = 0, \forall x =\in \mathcal{R}[\mathcal{G}]$. Consequently, $\sigma(x) = 0, \forall x \in \mathcal{R}[\mathcal{G}]$. Thus, $\gamma x \gamma = 0$ and hence $\gamma = 0$ and so $\mathcal{T} \in \mathcal{F}(\mathcal{R}[\mathcal{G}])$.

**Theorem 2.19.** *Let* $\mathcal{T}, \tau \in \mathfrak{D}_{(\sigma)}(\mathcal{R}[\mathcal{G}])$ *such that* $\mathcal{T}(x) = \sigma(x) + \tau(x), \forall x \in \mathcal{R}[\mathcal{G}]$. *If* $(\gamma,\eta) \in \mathcal{A}(\mathcal{T})$, *then* $(\gamma,\eta) \in \mathcal{A}(\tau)$ *or* $\gamma\eta = \eta\gamma$.

**Proof.** Assume that $(\gamma,\eta) \in \mathcal{A}(\mathcal{T})$, then we have $\mathcal{T}(x)\gamma = \eta x$ for all $x \in \mathcal{R}[\mathcal{G}]$. This gives:

$$\mathcal{T}(x)\gamma = \sigma(x)\gamma + \tau(x)\gamma = \eta x \text{ for all } x \in \mathcal{R}[\mathcal{G}]. \quad (1)$$

Replacing $x$ by $xy$ in (1), we have $\mathcal{T}(xy)\gamma = \sigma(x)y\gamma + \tau(x)y\gamma + x\tau(y)\gamma = \eta xy$. So we have:

$$\sigma(x)y\gamma + \tau(x)\sigma(y)\gamma + \tau(x)\tau(y)\gamma = \eta xy \text{ for all } x, y \in \mathcal{R}[\mathcal{G}].$$

That is,

$$\begin{aligned}\sigma(x)y\gamma + \tau(x)\big(\sigma(y) + \tau(y)\big)\gamma &= \sigma(x)y\gamma + \tau(x)\mathcal{T}(y)\gamma \\ &= \sigma(x)y\gamma + \tau(x)\eta x = \eta xy\end{aligned}$$

for all $x, y \in \mathcal{R}[\mathcal{G}]$.

$$\sigma(x)y\gamma + \tau(x)\eta x - \eta xy = 0, \ \forall x, y \in \mathcal{R}[\mathcal{G}]. \quad (2)$$

Replacing $y$ by $\gamma y$ in (2) and from (1) using $\tau(x)\gamma = \eta x - \sigma(x)\gamma$, we get

$$\eta xy\gamma - \sigma(x)\gamma y\gamma + \sigma(x)\eta\gamma y - \eta x\gamma y = 0 \text{ for all } x, y \in \mathcal{R}[\mathcal{G}] \quad (3)$$

In (3), replace $y$ by $y\eta$ to get

$$\eta xy\eta\gamma - \sigma(x)\gamma y\eta\gamma + \sigma(x)\eta\gamma y\eta - \eta x\gamma y\eta = 0 \text{ for all } x, y \in \mathcal{R}[\mathcal{G}]. \quad (4)$$

Right multiplication of (3) by $\eta$, we get

$$\eta xy\gamma\eta - \sigma(x)\gamma y\gamma\eta + \sigma(x)\eta\gamma y\eta - \eta x\gamma y\eta = 0, \forall x, y \in \mathcal{R}[\mathcal{G}] \quad (5)$$

Subtracting (4) from (5), we get

$$\eta x y[\gamma, \eta] - \sigma(x)\gamma y[\gamma, \eta] = 0, \ \forall x, y \in \mathcal{R}[\mathcal{G}]. \quad (6)$$

That is

$$(\eta x - \sigma(x)\gamma) y[\gamma, \eta] = 0, \ \forall x, y \in \mathcal{R}[\mathcal{G}]. \quad (7)$$

From primeness of $\mathcal{R}[\mathcal{G}]$, we get $\sigma(x)\gamma = \eta x$ or $[\gamma, \eta] = 0$ for all $x \in \mathcal{R}[\mathcal{G}]$.

## 3. Conclusion

In this paper, we presented the concept of dependent elements of a derivation mappings on a prime group rings and the concept of free action mappings. We chose the center of a prime group ring as a tool to obtain the necessary and sufficient condition of dependent elements and study of the special characteristics of its approved elements and a statement.

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