Earthline Journal of Mathematical Sciences
E-ISSN: 2581-8147
Volume 10, Number 1, 2022, Pages 169-181
https://doi.org/10.34198/ejms.10122.169181

Homomorphic Relations and Goursat Lemma

Brice Réné Amougou Mbarga

Laboratory of Algebra, Geometry and Application, University of Yaoundé I,
P.O.Box: 812, Yaoundé, Cameroon; e-mail: renebrice3@gmail.com

Abstract

Over the past years various authors have investigated the famous elementary result in group theory called Goursat's lemma for characterizing the subgroups of the direct product $A \times B$ of two groups A, B. Given a homomorphic relation $\rho = (R, A, B)$ where A and B are groups and R is a subgroup of $A \times B$. What can one say about the structure of ρ. In 1950 Riguet proved a theorem that allows us to obtain a characterization of ρ induces by examining the sections of the direct factors. The purpose of this paper is two-fold. A first and more concrete aim is to provide a containment relation property between homomorphic relation. Indeed if ρ, σ are homomorphic relations, we provide necessary and sufficient conditions for $\sigma \leq \rho$. A second and more abstract aim is to introduce a generalization of some notions in homological algebra. We define the concepts of θ-exact. We also obtain some interesting results. We use these results to find a generalization of Lambek Lemma.

1 Introduction

In 1889 Goursat proved that every subgroup of the direct product of two groups is determined by an isomorphism between factor groups of subgroups of the given groups. A like result is here shown for a general class of algebras, by a method due to Riguet [10]. Categories of algebras called Mal'cev varieties were investigated in [7], where it was pointed out that they should be suitable for developing some basic tools of homological algebra, thus serving as a non-additive generalization of the usual category of modules. A Mal'cev variety is a variety of algebras equipped with a ternary operation $m(x, y, z)$

Received: May 30, 2021; Accepted: June 14, 2022
2020 Mathematics Subject Classification: 20E07.
Keywords and phrases: Goursat, variety, relation, homomorphic, Mal'cev.

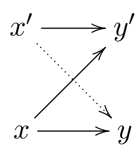
satisfying the equations $m(x, x, z) = z$ and $m(x, z, z) = x$. A famous result by Mal'cev asserts that this syntactical condition is in fact equivalent to a semantical one, namely that in the category of algebras any two congruence relations permute. Equivalent conditions were contained in [10], asserting that every homomorphic relation is difunctional and that every reflexive homomorphic relation is already a congruence. Examples are modules, groups, and many more. To presented our notation, we briefly review some notions from the calculus of binary relations. A binary relation between two sets A and B is a triple $\rho = (R, A, B)$, where R is a subset of the Cartesian product $A \times B$, called the graph of ρ. One usually writes $x\rho y$ to mean $(x, y) \in R$. Relations of special interest are the identity relation 1_A on A, the converse $\rho^- = (R^-, B, A)$ of ρ and the relative product $\rho\sigma = (RS, A, C)$ of ρ and $\sigma = (S, B, C)$. These are defined by

$$\begin{aligned} x1_Ax' &\Leftrightarrow x = x' \\ y\rho^- x &\Leftrightarrow x\rho y \\ x\rho\sigma z &\Leftrightarrow x\rho y \text{ and } y\sigma z \text{ for some } y \in B. \end{aligned}$$

We write $\rho \leq \rho' = (R', A, B)$ if R is a subset of R'. If $\rho = (R, A, A)$, one says that ρ is symmetric if $\rho^- \leq \rho$, ρ is reflexive if $1_A \leq \rho$, and transitive if $\rho\rho \leq \rho$. An equivalence relation satisfies all of these three. A relation $\rho = (R, A, B)$ is difunctional if $\rho\rho^-\rho = \rho$ and means that

$$(x\rho y \text{ and } x\rho y' \text{ and } x'\rho y') \Rightarrow x'\rho y$$

this implication is illustrated by the following diagram



We shall write $x\rho = \{y \mid x\rho y\}$; more generally, for any subset A' of A, $A'\rho = \rho A' = \{y \in B | x\rho y \text{ for some } x \in A'\}$ and $B\rho^- = \rho^- B = \{x | x\rho y \text{ for some } y \in B\}$. In particular, $A'\rho$ is the range of ρ, $B\rho^-$ is its domain. The following rules are well known and will be used freely:

$$\begin{aligned}
\rho(\sigma\tau) &= (\rho\sigma)\tau \\
\rho 1_B &= \rho = 1_A\rho \\
(\rho\sigma)^- &= \sigma^-\rho^- \\
A'(\rho\sigma) &= (A'\rho)\sigma
\end{aligned}$$

We often take advantage of the first and last of these to write without brackets $\rho\sigma\tau$ and $A'\rho\sigma$. Let A, B be groups the neutral element of each group A and B, with slight abuse of notation, will be written $'\, e\, '$. To generalize the notion of a homomorphism of a group A into a group B, we call the binary relation $\rho = (R, A, B)$ homomorphic if and only if

(i) $e\rho e$,

(ii) if $x\rho y$, then $x^{-1}\rho y^{-1}$,

(iii) if $x\rho y$ and $z\rho t$, then $xz\rho yt$.

Clearly then, ρ is homomorphic if and only if its graph R is a subgroup of the direct product $A \times B$. It is easily verified that the identity relation, the converse of a homomorphic relation and the relative product of two homomorphic relations are all homomorphic. Our general approach to giving a characterization of containment of homomorphic relations and to provide applications of it is given .

2 Generalizing Some Theorems of Group Theory

Riguet has used homomorphic relations to proved a theorem which describes the subgroup structure of a direct product in terms of the sections of the factor groups. One also verifies for any homomorphic $\rho = (R, A, B)$ that if A' is a subgroup of A then $A'\rho$ is a subgroup of B. A homomorphic equivalence relation is usually called a congruence relation. We shall call subcongruence any homomorphic relation which is transitive and symmetric without necessarily being reflexive. If $\kappa = (K, A, A)$ is such a subcongruence on A, it induces a congruence relation $(K, A\kappa, A\kappa)$ on its range $A\kappa$. The factor group of $A\kappa$ modulo κ is usually written $A\kappa/\kappa$, we shall call it a subfactor of

A. We denote by $\mathrm{Con}(A)$ the set of congruence of A. We define $\bar{\kappa} = (\bar{K}, A, A\kappa/\kappa)$ by $(2,1)$ $a\bar{\kappa}(a'\kappa)$ iff $a\kappa a'$, so that a $a\bar{\kappa} = a\kappa$. A simple calculation shows that $(2,2)$ $\bar{\kappa}\bar{\kappa}^{-} = \kappa, \bar{\kappa}^{-}\bar{\kappa} = 1_{A\kappa/\kappa}$, whence (2.3) $\bar{\kappa}^{-}\kappa\bar{\kappa} = 1_{A\kappa/\kappa}$. Note that $\bar{\kappa}$ induces the well-known natural homomorphism $(\bar{K}, A\kappa, A\kappa/\kappa)$.

Theorem 2.1. *(Riguet) If $\rho = (R, A_1, A_2)$ is a difunctional homomorphic relation (between two groups), then*

(i) $\kappa_1 = \rho\rho^{-}$ *is a subcongruence of* A_1 *with range* $A_2\rho^{-}$,

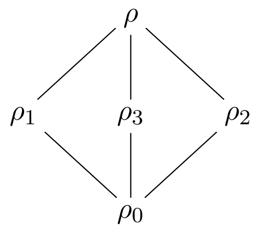
(ii) $\kappa_2 = \rho^{-}\rho$ *is a subcongruence of* A_2 *with range* $A_1\rho$,

(iii) ρ *induces an isomorphism* μ *between subfactors* $\dfrac{A_1\kappa_1}{\kappa_1}$ *and* $\dfrac{A_2\kappa_2}{\kappa_2}$ *such that* $(a\kappa_1) = \mu(b\kappa_2)$ *if and only if* $a\rho b$.

Conversely, every isomorphism between subfactors $A_1\kappa_1/\kappa_1$ and $A_2\kappa_2/\kappa_2$ of (groups) A_1 and A_2 respectively are isomorphic if there exists a difunctional homomorphic relation $\rho = (R, A_1, A_2)$ such that $\rho\rho^{-} = \kappa_1$ and $\rho^{-}\rho = \kappa_2$.

Theorem 2.1 give Goursat's characterization of the subgroups of the direct product of two groups, since all such subgroups are graphs of homomorphic relations between the groups.

Example 2.2. Let $\rho = (R, S_2, S_2)$ homomorphic relations. We want to describe all relation of ρ. It suffices to determine all subgroups of $S_2 \times S_2$. First, the subgroups of S_2 are $\langle(1)\rangle$,$\langle(12)\rangle$. Consider the subnormal quotient groups A/B where $B \trianglelefteq A \subseteq S_2$. If $|A/B| = 1$, one has $\langle(1)\rangle/\langle(1)\rangle; \langle(12)\rangle/\langle(12)\rangle$. It has only the identity maps between the 2 different quotients; so there are 4 different isomorphisms $\theta : A/B \to C/D$ yielding the 4 different subproducts $\langle(1)\rangle \times \langle(1)\rangle, V_1 = \langle(1)\rangle \times S_2, V_2 = S_2 \times \langle(1)\rangle$ and $S_2 \times S_2$. If $|A/B| = 2$, one has $\langle(12)\rangle/\langle(1)\rangle$; therefore the isomorphism $\langle(12)\rangle/\langle(1)\rangle \to \langle(12)\rangle/\langle(1)\rangle$; gives the subgroup $V_3 = \big\{\big((1),(1)\big), \big((12),(12)\big)\big\}$. Let $\rho_0 = (\{1,1\}, S_2, S_2), \rho_1 = (V_1, S_2, S_2), \rho_2 = (V_2, S_2, S_2)$, $\rho_3 = (V_3, S_2, S_2), \rho = (S_2 \times S_2, S_2, S_2)$.



Hasse diagram of ρ

Definition 2.3. Given homomorphic relation $\rho = (R, A_1, A_2)$, we say that the corresponding $Q(\rho) = (A_1\kappa_1, \kappa_1, A_2\kappa_2, \kappa_2, \mu)$ of Theorem 2.1 is the Goursat quintuple for ρ .

Let V be a group. We call $\rho = (\theta : A_1\kappa_1/\kappa_1 \to A_2\kappa_2/\kappa_2)$ a V-relation of ρ if V is its Goursat type, i.e., if $A_i\kappa_i/\kappa_i \cong V, i = 1, 2$ and we denote by $S_\rho(V)$ the set of all V-relation of ρ and M_V the set of all isomorphis $\theta_i : A_i\kappa_i/\kappa_i \to V$. Given morphisms $\theta_i : A_i\kappa_i/\kappa_i \to V$ in $M_V, i = 1, 2$, composition yields an isomorphism $\theta = \theta_1\theta_2^{-1} : A_1\kappa_1/\kappa_1 \to A_2\kappa_2/\kappa_2$. Hence there is a map $\Pi : M_V \times M_V \to S_\rho(V)$ defined by

$$\Pi(\theta_1, \theta_2) = \theta_1\theta_2^{-1}.$$

Let V, V' be groups. We now describe and analyze the partial order of relation of $\rho = (L, A_1, A_2)$ in terms of pairs of morphisms.

Proposition 2.4. *Let $(\theta_i : A_i\kappa_i/\kappa_i \to V) \in M_V$ and $(\theta'_i : A_i\kappa'_i/\kappa'_i \to V') \in M_{V'}$, $i = 1, 2$, be morphisms, let $\theta = \Pi(\theta_1, \theta_2), \theta' = \Pi(\theta'_1, \theta'_2)$ with corresponding relation $\rho = (L, A_1, A_2), \rho' = (L', A_1, A_2)$. Then $\rho' \leqslant \rho$ if and only if*

(i) $(A_i\kappa'_i, \kappa'_i) \leqslant (A_i\kappa_i, \kappa_i)$ *and*

(ii) $\lambda_1 = \lambda_2$ *where* $\lambda_i = \theta_i\varphi_i(\theta'_i)^{-1}$, *and* $\varphi_i : A_i\kappa'_i/\kappa'_i \to A_i\kappa_i/\kappa_i$ *is the homomorphism defined by* $(a\kappa'_i)^{\varphi_i} = a\kappa_i$, *for* $a \in A_i\kappa'_i, i = 1, 2$.

$$\begin{array}{ccccccc}
A_1\kappa_1/\kappa_1 & \xrightarrow{\theta_1} & V & & V & \xleftarrow{\theta_2} & A_2\kappa_2/\kappa_2 \\
\varphi_1 \uparrow & & \uparrow \lambda_1 & & \lambda_2 \uparrow & & \uparrow \varphi_2 \\
A_1\kappa_1'/\kappa_1' & \xrightarrow{\theta_1'} & V' & & V' & \xleftarrow{\theta_2'} & A_2\kappa_2'/\kappa_2'
\end{array}$$

Proof. We define ρ' and ρ as follows

$$a_1'\rho' a_2' \Leftrightarrow \theta_1'(a_1'\kappa_1') = \theta_2'(a_2'\kappa_2'),$$

and

$$a_1\rho a_2 \Leftrightarrow \theta_1(a_1\kappa_1) = \theta_2(a_2\kappa_2).$$

Then $\rho' \leq \rho$ if and only if $(A_i\kappa_i', \kappa_i') \leqslant (A_i\kappa_i, \kappa_i)$, $i = 1, 2$, and, for $a_1\rho' a_2$ we have $\theta_1(a_1\kappa_1) = \theta_2(a_2\kappa_2)$. But if $a_1\rho' a_2$, then

$$\theta_i(a_i\kappa_i) = \theta_i(\varphi_i(a_i\kappa_i')) = \lambda_i(\theta_i'(a_i\kappa_i')).$$

So $\theta_1(a_1\kappa_1) = \theta_2(a_2\kappa_2)$ if and only if $\lambda_1 = \lambda_2$. □

Corollary 2.5. *With the notation of Proposition 2.4, $\rho \leqslant \rho'$ if and only if*

(i) $(A_i\kappa_i', \kappa_i') \leqslant (A_i\kappa_i, \kappa_i)$,

(ii) $\varphi_1\theta = \theta'\varphi_2$.

3 Generalization to Other Algebraic Systems

By an n-ary operation f_n on a set A is understood a mapping which assigns to each n-tuple of elements of A a single element of A, n being some finite non-negative integer. In particular, a 0-ary operation is a constant. Let F be a set of operation symbols with prescribed subscripts. An algebra, in the sense of Birkhoff ([7]), is a representation of such a set of symbols as n-ary operations on a set A, and may be denoted by A. If A' is a subset of A closed under all the operations in F, the induced representation A' is called a subalgebra of A. The Cartesian product $A \times B$ of two similar algebras is turned into another algebra of the same kind, called the direct product of A and B. For all algebra variety, the following statements are equivalent: [7]:

$(M1)$ there exists ternary operation $m(x, y, z)$ satisfying the equation : $m(x, y, y) = x$ and $m(y, y, z) = z$.

$(M2)$ If R and S are congruence relations on any algebra, then $RS = RS$.

$(M3)$ If ρ is any homomorphic relation between two algebras : $\rho\rho^{-}\rho = \rho$.

An algebraic category satisfying any of these equivalent conditions is called a Mal'cev variety.

Example 3.1. $i)$ Groups are Mal'cev variety with $m(x, y, z) = xy^{-1}z$.

$ii)$ Rings, Modules and Boolean algebras are Mal'cev varieties.

$iii)$ Heyting algebras are Mal'cev variety where m can be given by:

$$m(x, y, z) = \big((z \to y) \to x\big) \wedge \big((x \to y) \to z\big)$$

The isomorphism theorem due to J. Lambek may be stated as follows:

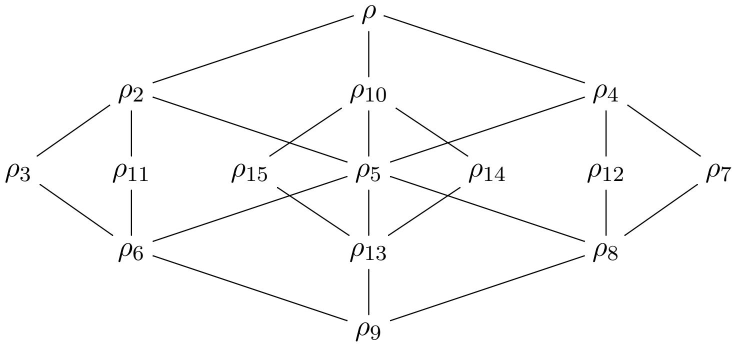
Theorem 3.2. *[11](Goursat's lemma)*
Every homomorphic relation $\rho = (R, A_1, A_2)$ between two algebra in a Mal'cev variety gives rise to an isomorphism between factors of subalgebras of A_1 and A_2 as follows:

$$\frac{A_1\rho}{\rho^{-}\rho} \cong \frac{A_2\rho^{-}}{\rho\rho^{-}}$$

as every isomorphism $\mu : A_1'/\theta \cong A_2'/\theta'$ where θ and θ' are congruence relations on subalgebra A_1' of A_1 and A_2' of A_2 respectively, gives rise to homomorphic relation ρ from A_1 to A_2 where we put $a\rho b$ if and only if $\theta(a) = \mu\theta'(b)$ and $\theta(a), \theta(b)$ are equivalence classes.

Example 3.3. A ring is an algebra $R = \langle R, +, ., -, 0\rangle$ where $+$ and . are binary, $-$ is unary and 0 is nullary operations. Consider ring $R = \mathbb{Z}_4 \times \mathbb{Z}_4$ we want to determine all subrings of R. It suffices to determine all subgroups of $\mathbb{Z}_4 \times \mathbb{Z}_4$. First, the subgroups of $\mathbb{Z}_4 \times \mathbb{Z}_4$ are $\langle 0\rangle$,$\langle 2\rangle$ and $\mathbb{Z}_4$. Consider the subnormal quotient groups A/B where $B \trianglelefteq A \subseteq \mathbb{Z}_4$. If $|A/B| = 1$, one has $\langle 0\rangle/\langle 0\rangle$, $\langle 2\rangle/\langle 2\rangle$, $\mathbb{Z}_4/\mathbb{Z}_4$. It has only the identity maps between the 3 different quotients;so there are 9 different isomorphisms $\theta : A/B \to$

C/D yielding the 9 different subproducts such that $H_1 = \mathbb{Z}_4 \times \mathbb{Z}_4$, with $\theta_1 : \mathbb{Z}_4/\mathbb{Z}_4 \to \mathbb{Z}_4/\mathbb{Z}_4, [0,1,2,3] \mapsto [0,1,2,3]$ similarly we have
$H_2 = \{(0,0),(1,0),(2,0),(0,2),(1,2),(2,2),(3,0),(3,2)\}$,
$H_3 = \{(0,0),(1,0),(2,0),(3,0)\}$,
$H_4 = \{(0,0),(0,1),(2,0),(2,1),(0,2),(2,2),(0,3),(2,3)\}$,
$H_5 = \{(0,0),(2,0),(0,2),(2,2)\}$,
$H_6 = \{(0,0),(2,0)\}$,
$H_7 = \{(0,0),(0,1),(0,2),(0,3)\}$,
$H_8 = \{(0,0),(0,2)\}$,
$H_9 = \{(0,0)\}$.
If $|A/B| = 2$, one has $\langle 2\rangle/\langle 0\rangle$, $\mathbb{Z}_4/\langle 2\rangle$, there are 4 different subproducts such that
$H_{10} = \{(0,0),(2,0),(0,2),(2,2),(1,1),(1,3),(3,1),(3,3)\}$,
$H_{11} = \{(0,0),(0,2),(2,1),(2,3)\}$,
$H_{12} = \{(0,0),(0,2),(2,1),(2,3)\}$,
$H_{13} = \{(0,0),(2,2)\}$.
If $|A/B| = 4$, one has $\mathbb{Z}_4/\langle 0\rangle$, there are 2 different subproducts such that
$H_{14} = \{(0,0),(1,1),(2,2),(3,3)\}$,
$H_{15} = \{(0,0),(1,3),(2,2),(3,1)\}$.
The number of subring of R is $N^{(s)}(2^2, 2^2) = 12$ (see [13] for instant) and $(0,3) = (2,1)(2,3) \notin H_{11}, (3,3) = (1,3)(3,1) \notin H_{15}$,$(0,3) = (2,1)(2,3) \notin H_{12}$. This allows us to determine all subring of $\mathbb{Z}_4 \times \mathbb{Z}_4$ it is $(H_i)_{1\le i\le 14}$ with $i \ne 11, 12, 15$. One has $(H_i)_{1\le i\le 9}, H_{13}$ ideals of R and H_1, H_{10}, H_{14} are unitary subrings. Let $\rho_i = (H_i, \mathbb{Z}_4, \mathbb{Z}_4)$.



Hasse diagram of ρ

Now we shall recall a generalization of Lambek Lemma for module theory due to B. Davvaz [8].

Lemma 3.4. *(A Generalization of Lambek Lemma). Let*

$$\begin{array}{ccccc} A' & \xrightarrow{\alpha_1} & A & \xrightarrow{\alpha_2} & A'' \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\theta} \\ B' & \xrightarrow[\beta_1]{} & B & \xrightarrow[\beta_2]{} & B'' \end{array}$$

be a commutative diagram such that the first row is U-exact $(Im\alpha_1 = \alpha_2^{-1}(U))$ and the second row is U'-exact $(Im\beta_1 = \beta_2^{-1}(U'))$. Then φ induces an isomorphism

$$\frac{Im\varphi \cap Im\beta_1}{Im\varphi\alpha_1} \cong \frac{(\theta\alpha_2)^{-1}(U')}{\alpha_2^{-1}(U) + \varphi^{-1}(0)}.$$

Definition 3.5. A sequence of algebras and homomorphisms

$$A \xrightarrow{\lambda} B \xrightarrow{\mu} C$$

is said to be θ-exact (where $\theta \in Con(C)$) at B if $Im\lambda = \ker_\theta \mu = \mu^- \theta\mu$.

Let us consider the following diagram

$$\begin{array}{ccccc} A & \overset{\lambda_1}{\underset{\lambda_2}{\rightrightarrows}} & B & \xrightarrow{\mu} & C \\ {\scriptstyle\alpha_2}\downarrow\downarrow{\scriptstyle\alpha_1} & & \downarrow{\scriptstyle\beta} & & {\scriptstyle\gamma_2}\downarrow\downarrow{\scriptstyle\gamma_1} \\ D & \xrightarrow{\lambda} & E & \overset{\mu_1}{\underset{\mu_2}{\rightrightarrows}} & F \end{array}$$

or if we prefer:

$$\begin{array}{ccccc} A+A & \xrightarrow{[\lambda_1,\lambda_2]} & B & \xrightarrow{\mu} & C \\ {\scriptstyle [\alpha_1,\alpha_2]}\downarrow & & {\scriptstyle \beta}\downarrow & & \downarrow{\scriptstyle <\gamma_1,\gamma_2>} \\ D & \xrightarrow[\lambda]{} & E & \xrightarrow[<\mu_1,\mu_2>]{} & F\times F \end{array}$$

and assume that $\beta[\lambda_1, \lambda_2] = \lambda[\alpha_1, \alpha_2]$ (that is, $\beta\lambda_i = \lambda\alpha_i, i = 1, 2$) and $<\gamma_1, \gamma_2 > \mu =< \mu_1, \mu_2 > \beta$, (that is, $\gamma_i\mu = \mu_i\beta, i = 1, 2$). Then

$$Im(\lambda_1 - \lambda_2) = \ker_\theta \mu, \;\; Im\lambda = \ker_{\theta'}(\mu_1 - \mu_2).$$

Here

$$Im\lambda = \{\lambda d | d \in D\}$$

is the usual image of λ and $\ker_\theta \mu = \mu^- \theta\mu$; $\mu\mu^- = 1$, with the graph

$$\{(b_1, b_2) \in B \times B | (\mu b_1, \mu b_2) \in \theta\}$$

and $\ker_{\theta'}(\mu_1 - \mu_2) = \{e \in E | (\mu_1 e, \mu_2 e) \in \theta'\}$ we also write $Im(\lambda_1 - \lambda_2) = \lambda_1\lambda_2^-$ for the relation B with graph

$$\{(\lambda_1 a, \lambda_2 a) | a \in A\}.$$

Proposition 3.6. *In the diagram as above, if the first row is θ-exact and the second row is θ'-exact, in malcev variety. Let $\mu\mu^- = 1$, then*

$$\frac{Im(B \to E) \cap Im(D \to E)}{Im(A \rightrightarrows E)} \cong \frac{\ker_{\theta'}(B \to F)}{Ker_\theta(B \to C) \vee Ker(B \to E)}$$

where the congruence relation in the denominator of right hand side is assumed to be restricted to algebra in the numerator.
Here $\vee$ denotes the join in lattice of congruence relation on B.

Proof. We consider the homomorphic relation ρ from B to E defined as follows:

$$e\rho b \Leftrightarrow \exists b' \in B(e = \beta b' \wedge (\mu b', \mu b) \in \theta \wedge (\mu_1 e, \mu_2 e) \in \theta').$$

Note that equation $(\mu_1 e, \mu_2 e) \in \theta'$ on the right can be replaced by $(\mu_1\beta b, \mu_2\beta b) \in \theta'$, since

$$\mu_i e = \mu_i\beta b' = \gamma_i\mu b' = \gamma_i\mu b = \mu_i\beta b, (i = 1, 2).$$

We now calculate:

$$\begin{aligned}
e \in \rho\rho^- E = \rho B &\Leftrightarrow \exists b, b' \in B(e = \beta b' \wedge (\mu b', \mu b) \in \theta \wedge (\mu_1 e, \mu_2 e) \in \theta') \\
&\Leftrightarrow \exists b' \in B(e = \beta b' \wedge e \in \ker_{\theta'}(\mu_1 - \mu_2)) \\
&\Leftrightarrow e \in Im\beta \wedge e \in Im\lambda; \\
e\rho\rho^- e' &\Leftrightarrow e\beta\mu^-\theta\mu\mu^-\theta\mu\beta^- e' \wedge (\mu_1 e, \mu_2 e) \in \theta' \wedge (\mu_1 e', \mu_2 e') \in \theta' \\
&\Leftrightarrow e\beta\mu^-\theta\mu\beta^- e' \wedge e, e' \in \ker_{\theta'}(\mu_1 - \mu_2) \quad \text{since } \mu\mu^- = 1 \\
&\Leftrightarrow e\beta Im(\lambda_1 - \lambda_2)\beta^- e' \wedge e, e' \in Im\lambda \\
&\Leftrightarrow eIm(\beta\lambda_1 - \beta\lambda_2)e'.
\end{aligned}$$

Note that

$$\begin{aligned}
\beta Im(\lambda_1 - \lambda_2)\beta^- &= \beta\lambda_2\lambda_2^-\beta^- \\
&= Im(\beta\lambda_1 - \beta\lambda_2) \\
&= Im(\lambda\alpha_1 - \lambda\alpha_2) \\
&= \lambda\alpha_1\alpha_2^-\lambda^-
\end{aligned}$$

so that the condition $e, e' \in Im\lambda$ is automatically fulfilled.

$$\begin{aligned}
b \in \rho^-\rho B = \rho^- E &\Leftrightarrow \exists e \in E, \exists b' \in B(e = \beta b' \wedge (\mu b', \mu b) \in \theta \wedge \\
&\quad\;\; (\mu_1\beta b, \mu_2\beta b) \in \theta') \\
&\Leftrightarrow (\mu_1\beta b, \mu_2\beta b) \in \theta' \\
&\Leftrightarrow b \in \ker_{\theta'}(\mu_1\beta - \mu_2\beta)
\end{aligned}$$

$$\begin{aligned}
b\rho^-\rho b' &\Leftrightarrow b\mu^-\theta\mu\beta^-\beta\mu^-\theta\mu b' \wedge (\mu_1\beta b, \mu_2\beta) \in \theta' \wedge (\mu_1\beta b', \mu_2\beta b') \in \theta' \\
&\Leftrightarrow b(\ker_\theta \mu \vee \ker \beta)b' \wedge b, b' \in \ker_{\theta'}(\mu_1\beta - \mu_2\beta).
\end{aligned}$$

The last step in the proof uses the fact that the congruence relation $\mu^-\theta\mu$ and $\beta^-\beta$ on B commute and that their relative product is their join in the lattice of congruence relations on B.

If $Im1$ and $\ker 2$ denote the two sides of the isomorphism in Proposition 3.6, we have $Im1 \cong \ker 2$, use Theorem 3.2. □

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