Earthline Journal of Mathematical Sciences
E-ISSN: 2581-8147
Volume 16, Number 6, November 2026, Pages 1201-1214
https://doi.org/10.34198/ejms.16626.76.12011214

Some Properties for Convolution Structures of Holomorphic Functions

Najah Ali Jiben Al-Ziadi[1,*], Zainab Aodeh A. Mohmmed[2] and Abbas Kareem Wanas[3]

[1] Department of Mathematics, College of Education, University of Al-Qadisiyah, Diwaniya, Iraq
e-mail: najah.ali@qu.edu.iq

[2] Department of Mathematics, College of Education, University of Al-Qadisiyah, Diwaniya, Iraq
e-mail: semh.Alisawi@qu.edu.iq

[3] Department of Mathematics, College of Education for Women, University of Al-Qadisiyah, Diwaniya, Iraq
e-mail: abbas.kareem.w@qu.edu.iq

Abstract

In the paper submitted, by employing the linear operator $I^{j}_{\alpha,\beta}$ defined using the convolution of univalent functions k and h belonging to the class $\mathbb{A}$, which consists of holomorphic functions in the disk $\mathcal{U}=\{s : s \in \mathbb{C} : |s|<1\}$ that satisfy the condition $k(0)=k^{'}(0)-1=0$, we proved that the convolution under this operator belongs to the new class $\mathfrak{S}_{(\upsilon,\beta,\alpha)(h)}(\Phi)$, according to the concept of the convex hull method of differential subordination, for any convex univalent function mapping the unit disk into the right half-plane, Φ, in $\mathcal{U}$ with $\Phi(0)=1$.

1 Introduction

For s belonging to the unit disk $\mathcal{U}=\{s : s \in \mathbb{C} : |s|<1\}$, the class of holomorphic functions $\mathbb{A}$ is defined by $\mathbb{A} = \left\{k : k \in \mathcal{H}(\mathcal{U}) : k(s)=s+\varepsilon_2 s^2+\varepsilon_3 s^3+\ldots \text{ with } k(0)=0 \text{ and } k'(0)=1\right\}$, where $\mathcal{H}(\mathcal{U})=\{k : k \text{ holomorphic in } \mathcal{U}\}$.

For two holomorphic functions $k(s)=s+\varepsilon_2 s^2+\varepsilon_3 s^3+\ldots$ and $h(s)=s+e_2 s^2+e_3 s^3+\ldots$, the special product of these functions, called the convolution, is indicated by $h*k$ and is symbolised by a power series of the form

$$(h*k)(s)=s+\sum_{i=2}^{\infty}\varepsilon_i e_i s^i=(k*h)(s). \tag{1.1}$$

Let $\mathfrak{B}(s)$ and $\mathfrak{S}(s)$ be two holomorphic functions in $\mathcal{U}$. The subordination relation [1] between $\mathfrak{B}(s)$ and $\mathfrak{S}(s)$ is expressed as $\mathfrak{B}(s) \prec \mathfrak{S}(s)$ if there is a holomorphic function $\mathfrak{T}(s)$ in $\mathcal{U}$ with the properties $\mathfrak{T}(0)=0$ and $|\mathfrak{T}(s)|<1$ $(s \in \mathcal{U})$, such that $\mathfrak{B}(s)=\mathfrak{S}(\mathfrak{T}(s))$. If $\mathfrak{S}(s)$ is univalent, then $\mathfrak{B}(s) \prec \mathfrak{S}(s)$ if and only if $\mathfrak{B}(0)=\mathfrak{S}(0)$ and $\mathfrak{B}(\mathcal{U}) \subset \mathfrak{S}(\mathcal{U})$.

Received: September 22, 2026; Revised & Accepted: October 22, 2026; Published online: November 2, 2026
2020 Mathematics Subject Classification: Primary 30C45; Secondary 30C50, 30C80.
Keywords and phrases: convex function, subordination, convolution, univalent functions, differential operator.
*Corresponding author
Copyright 2026 the Authors

A function $k \in \mathbb{A}$, as a member of the class $\mathbb{K}_\lambda(\Upsilon)$, is convex of complex order Υ ($\Upsilon \in \mathbb{C}\backslash\{0\}$) and type λ if and only if

$$\mathcal{R}e\left(1+\frac{1}{\Upsilon}\left(\frac{sk''(s)}{k'(s)}\right)\right) > \lambda, \qquad (s \in \mathcal{U},\ 0 \leq \lambda < 1).$$

A function $k \in \mathbb{A}$, as a member of the class $\mathbb{S}^*_\lambda(\Upsilon)$, is starlike of complex order Υ ($\Upsilon \in \mathbb{C}\backslash\{0\}$) and type λ if and only if

$$\mathcal{R}e\left(1+\frac{1}{\Upsilon}\left(\frac{sk'(s)}{k(s)}\right)\right) > \lambda, \qquad (s \in \mathcal{U},\ 0 \leq \lambda < 1).$$

Many studies have examined classes of functions derived from $\mathbb{K}_\lambda(\Upsilon)$ and $\mathbb{S}^*_\lambda(\Upsilon)$ according to the value of λ and the use of the concept of subordination. Among the studies that have focused on these topics, we mention the following: $\mathbb{S}^*$, introduced by Ajab et al. [2], $\mathbb{S}^*_\Upsilon(\Phi)$, investigated and studied by Ravichandran et al. [3]; and the collection $M_{0,\Upsilon}(\Phi)$ defined by Suchitra et al. [4]. Note that $\mathbb{S}^*_0(\Upsilon) = \mathbb{S}^*(\Upsilon)$ and $\mathbb{K}_0(\Upsilon) = \mathbb{K}(\Upsilon)$ (see [5,6]). Previously, results involving convolution were obtained by applying Alexander's theorem, which showed that if the condition $sk'(s) = k(s) * h(s)$ holds for a function h and if k is convex, this leads to the following important result: k is convex if and only if $k * h$ is starlike, where $h(s) = \left(\frac{s}{(1-s)^2}\right)$. These ideas have motivated the investigation of the class of functions k that satisfy the subordination condition

$$\frac{s(k*h)'(s)}{(k*h)(s)} \prec \Phi(s),$$

where $h(s) \in \mathbb{A}$ is a fixed function and Φ is a convex function mapping into the right half-plane.

The following subclasses of univalent holomorphic functions will be investigated. The function $h(s) \in \mathbb{A}$ is assumed to be fixed in the sequel, unless otherwise noted. The function $\Phi(s)$ is presumed to be a fixed, convex, univalent function mapping into the right half-plane, such that $\Phi(0) = 1$, and $\Phi'(0) > 0$.

For $k \in \mathbb{A}$, the differential operator $I^j_{\alpha,\beta}$ [7] is defined by

$$I^0_{\alpha,\beta}k(s) = k(s),$$

$$I^1_{\alpha,\beta}k(s) = \frac{\alpha k(s) + \beta s k'(s)}{\alpha+\beta} = s + \sum_{l=2}^{\infty}\left(\frac{\alpha+\beta l}{\alpha+\beta}\right)^1 \varepsilon_l s^l,$$

$$\vdots$$

$$I^j_{\alpha,\beta}k(s) = I_{\alpha,\beta}\left(I^{j-1}_{\alpha,\beta}k(s)\right) = s + \sum_{l=2}^{\infty}\left(\frac{\alpha+\beta l}{\alpha+\beta}\right)^j \varepsilon_l s^l, \tag{1.2}$$

where $\alpha \in \mathbb{R}$ and $\beta \geq 0$ with $\alpha + \beta > 0$, $j \in \mathbb{N}_0 = \mathbb{N}\cup[0]$.

Now, for $\Upsilon \in (0,1]$ and for all $s \in \mathcal{U}$, we define the following new subclasses:

$$\mathfrak{S}_{(\Upsilon,\beta,\alpha)(h)}(\Phi) = \Big\{k \in \mathbb{A} : k \text{ satisfying } 1 + \frac{1}{\Upsilon}\left(\frac{s(I^j_{\alpha,\beta}(h*k))'(s)}{\left(I^j_{\alpha,\beta}(h*k)\right)(s)} - 1\right) \prec \Phi(s), \text{ where } \frac{\left(I^j_{\alpha,\beta}(h*k)\right)(s)}{s} \neq 0\Big\}, \tag{1.3}$$

$$\mathfrak{K}_{(\Upsilon,\beta,\alpha)(h)}(\Phi) = \Big\{k \in \mathbb{A} : k \text{ satisfying } 1 + \frac{1}{\Upsilon}\left(\frac{s(I^j_{\alpha,\beta}(h*k))''(s)}{(I^j_{\alpha,\beta}(h*k))'(s)}\right) \prec \Phi(s), \text{ where } \frac{\left(I^j_{\alpha,\beta}(h*k)\right)(s)}{s} \neq 0\ ,\ \left(I^j_{\alpha,\beta}(h*k)\right)'(s) \neq 0\Big\}, \tag{1.4}$$

$$\mathfrak{C}_{(\Upsilon,\beta,\alpha)(h)}(\Phi) = \Big\{k \in \mathbb{A} : k \text{ satisfying } 1 + \frac{1}{\Upsilon}\left(\frac{s(I^j_{\alpha,\beta}(h*k))'(s)}{\left(I^j_{\alpha,\beta}(h*\mathfrak{T})\right)(s)} - 1\right) \prec \Phi(s), \text{ for some } \mathfrak{T}(s) \in \mathfrak{S}_{(\gamma,\beta,\alpha)(h)}(\Phi)\Big\}, \tag{1.5}$$

$$\mathcal{G}^{\xi}_{(\Upsilon,\beta,\alpha)(h)}(\Phi) = \Big\{k \in \mathbb{A} : k \text{ satisfying } \xi\left(1 + \frac{1}{\Upsilon}\left(\frac{s(I^j_{\alpha,\beta}(h*k))''(s)}{(I^j_{\alpha,\beta}(h*k))'(s)}\right)\right) + (1-\xi)\left(1 + \frac{1}{\Upsilon}\left(\frac{s(I^j_{\alpha,\beta}(h*k))'(s)}{\left(I^j_{\alpha,\beta}(h*k)\right)(s)} - 1\right)\right) \prec \Phi, \text{ where } \left(I^j_{\alpha,\beta}(h*k)\right)'(s) \neq 0 \text{ and } \xi \text{ any real number}\Big\}, \tag{1.6}$$

$$\mathcal{Q}_{(\Upsilon,\beta,\alpha)(h)}(\Phi) = \Big\{k \in \mathbb{A} : k \text{ satisfying } 1 + \frac{1}{\Upsilon}\left(\frac{\left(s(I^j_{\alpha,\beta}(h*k))'\right)'(s)}{(I^j_{\alpha,\beta}(h*\mathfrak{T}))'(s)} - 1\right) \prec \Phi(s), \text{ for some } \mathfrak{T}(s) \in \mathfrak{K}_{(\Upsilon,\beta,\alpha)(h)}\Big\}, \tag{1.7}$$

$$\widetilde{\mathfrak{S}}_{(\Upsilon,\beta,\alpha)}(\Phi) = \Big\{k \in \mathbb{A} : k \text{ satisfying } \frac{1}{\Upsilon}\left(\frac{s(I^j_{\alpha,\beta}(k))'(s)}{\left(I^j_{\alpha,\beta}(k)\right)(s)}\right) \prec \Phi(s), \text{ where } \frac{I^j_{\alpha,\beta}(k)(s)}{s} \neq 0\Big\}. \tag{1.8}$$

Recall that Ruscheweyh and Sheil-Small [8], Shanmugam [9], Padmanabhan and Parvatham [10], and Goodman [11] studied a collection of functions that are convex, starlike, and close-to-convex, and they

proved that these classes are closed under convolution with convex functions. To understand the breadth of studies that have addressed the concept of subordination for functions with geometric properties using operators, we refer to the work of researchers in references [7, 12–20].

In the present submission, we extend the classes of starlike, convex, quasi-convex and close-to-convex functions to give various subclasses of $\mathbb{A}$ and consider several inclusion properties of functions associated with the operator $I^{j}_{\alpha,\beta}$. Some properties of the new classes are studied. We would like to point out that, during the research, we use the abbreviation "co. un." to denote a convex univalent function to avoid repetition of the term.

It is worth noting that the collection of pre-starlike functions is a set that is represented by the symbol $\mathcal{A}_{\delta}$ [21] which is defined as $\mathcal{A}_{\delta} = \left\{k \in \mathbb{A} : k * \frac{s}{(1-s)^{2-2\delta}} \in \mathbb{S}^{*}(\delta), \text{ where } \delta \leq 1\right\}$. Note that in this research, in order to denote the class using a single symbol, we will denote the class $\mathbb{S}^{*}(\delta)$ by $\mathbb{S}^{*}_{\delta}$.

To prove our primary conclusions, the following results are required for our investigation.

Theorem 1.1 ([21]). *Let $\delta \leq 1$. If $k \in \mathcal{A}_{\delta}$ and $h \in \mathbb{S}^{*}(\delta)$, then $\frac{k*\mathcal{F}_{h}}{k*h}(\mathcal{U}) \subset \overline{co}(\mathcal{F}(\mathcal{U}))$ for any holomorphic function $\mathcal{F} \in \mathcal{H}(\mathcal{U})$, where $\overline{co}(\mathcal{F}(\mathcal{U}))$ is the convex hull of $\mathcal{F}(\mathcal{U})$.*

Theorem 1.2 ([22]). *Let v_1, $v_2 \in \mathbb{C}$, $\Phi \in \mathcal{H}(\mathcal{U})$ be a co. un. function in $\mathcal{U}$, and $\mathcal{R}e(v_1\Phi(s) + v_2) > 0$. If $\mathsf{K}(s)$ is holomorphic with $\mathsf{K}(0) = \Phi(0)$ in $\mathcal{U}$, then*

$$\mathsf{K}(s) + \frac{s\mathsf{K}'(s)}{v_1\mathsf{K}(s) + v_2} \prec \Phi(s) \quad \textit{implies } \mathsf{K}(s) \prec \Phi(s).$$

Theorem 1.3 ([22]). *Let $\Phi \in \mathcal{H}(\mathcal{U})$ be a co. un. in $\mathcal{U}$, with $\Phi(0) = 1$, and let v_1, $v_2 \in \mathbb{C}$. Assume that the differential equation*

$$\Theta(s) + \frac{s\Theta'(s)}{v_1\Theta(s) + v_2} = \Phi(s)$$

has a univalent solution Θ that satisfies $\Theta(s) \prec \Phi(s)$. If

$$\mathsf{K}(s) = e + e_1 s + e_2 s^2 + \dots$$

satisfies

$$\mathsf{K}(s) + \frac{s\mathsf{K}'(s)}{v_1\mathsf{K}(s) + v_2} \prec \Phi(s),$$

then $\mathsf{K}(s) \prec \Theta(s)$, where $\Theta(s)$ is the best dominant.

Theorem 1.4 ([23]). *Let Φ be a co. un. in $\mathcal{U}$, and let $\mathbb{p}$ be holomorphic in $\mathcal{U}$ with $\mathcal{R}e(\mathbb{p}(s)) > 0$. If $\mathsf{K}(s)$ is holomorphic in $\mathcal{U}$ and $\mathsf{K}(0) = \Phi(0)$, then $\mathsf{K}(s) + \mathbb{p}(s)\left(s\mathsf{K}'(s)\right) \prec \Phi(s)$ $(s \in \mathcal{U})$, it follows that $\mathsf{K}(s) \prec \Phi(s)$ $(s \in \mathcal{U})$.*

2 Main Results and Inclusion Properties Involving Differential Operator

Theorem 2.1. *Let Φ be a co. un. and right half plane function with $\Phi(0) = 1$ and $0 < \Upsilon \leq 1$, and let h be a fixed function in $\mathbb{A}$ such that, for every $k \in \mathbb{A}$ that satisfies $\frac{I^j_{\alpha,\beta}(h*k)(s)}{s} \neq 0$ and $\left(I^j_{\alpha,\beta}(h*k)(s)\right)' \neq 0$, the following statements hold:*

1) $\mathfrak{K}_{(\Upsilon,\beta,\alpha)(h)}(\Phi) \subset \mathfrak{S}_{(\Upsilon,\beta,\alpha)(h)}(\Phi)$.

2) $k \in \mathfrak{K}_{(\Upsilon,\beta,\alpha)(h)}(\Phi)$ if and only if $s(I^j_{\alpha,\beta}(k))'(s) \in \mathfrak{S}_{(\Upsilon,\beta,\alpha)(h)}(\Phi)$.

Proof of (1). Since $\frac{I^j_{\alpha,\beta}(h*k)(s)}{s} \neq 0$, let $m(s)$ be a holomorphic function in $\mathcal{U}$ defined by

$$m(s) \ = 1 + \frac{1}{\Upsilon}\left(\frac{s(I^j_{\alpha,\beta}(h*k))'(s)}{\left(I^j_{\alpha,\beta}(h*k)\right)(s)} - 1\right). \tag{2.1}$$

That is,

$$(\Upsilon\ m(s) \ + (\ 1 - \Upsilon) = \frac{s(I^j_{\alpha,\beta}(h*k))'(s)}{\left(I^j_{\alpha,\beta}(h*k)\right)(s)}. \tag{2.2}$$

Using the fact that $\left(I^j_{\alpha,\beta}(h*k)(s)\right)' \neq 0$ and the condition $Re\,(\Upsilon m(s) + (\ 1 - \Upsilon)) > 0$, we obtain

$$m(s) \ + \frac{sm'(s)}{\Upsilon m(s) + (\ 1 - \Upsilon)} = 1 + \frac{s(I^j_{\alpha,\beta}(h*k))''(s)}{\Upsilon(I^j_{\alpha,\beta}(h*k))'(s)}.$$

If $k \in \mathfrak{K}_{(\gamma,\beta,\alpha)(h)}(\Phi)$, where Φ is a function satisfying the hypotheses, it follows that k satisfies the subordination condition in (1.4). As a consequence, we get

$$m(s) \ + \frac{sm'(s)}{\Upsilon m(s) + (\ 1 - \Upsilon)} \prec \Phi(s).$$

Suppose $v_1 = \Upsilon$, $v_1 = 1 - \Upsilon$, and that $m(s)$ is holomorphic in $\mathcal{U}$ and satisfies the property $m(0) = \Phi(0)$. By applying Theorem 1.2, we get $m(s) \prec \Phi(s)$. Thus, $\mathfrak{K}_{(\gamma,\beta,\alpha)(h)}(\Phi) \subset \mathfrak{S}_{(\gamma,\beta,\alpha)(h)}(\Phi)$. □

Proof of (2). Suppose that there are two functions $h, k \in \mathbb{A}$ such that $\frac{\left(I^j_{\alpha,\beta}(h*k)\right)(s)}{s} \neq 0$, $\left(I^j_{\alpha,\beta}(h*k)\right)(s) \neq 0$.

Since

$$\begin{aligned} 1 + \frac{1}{\Upsilon}\left(\frac{s\left[h * s(I^j_{\alpha,\beta}(k))'\right]'(s)}{\left(h * s(I^j_{\alpha,\beta}(k))'\right)(s)} - 1\right) &= 1 + \frac{1}{\Upsilon}\left(\frac{\left[s(I^j_{\alpha,\beta}(h*k))'\right]'(s)}{\left(I^j_{\alpha,\beta}(h*k)\right)'(s)} - 1\right) \\ &= 1 + \frac{1}{\Upsilon}\left(\frac{s(I^j_{\alpha,\beta}(h*k))''(s)}{(I^j_{\alpha,\beta}(h*k))'(s)}\right). \end{aligned} \tag{2.3}$$

If $k(s) \in \mathfrak{K}_{(\Upsilon,\beta,\alpha)(h)}(\Phi)$, then, by (1.4) and (2.3), we have

$$1+\frac{1}{\Upsilon}\left(\frac{s\left[h*s(I^{j}_{\alpha,\beta}(k))^{'}\right]^{'}(s)}{\left(h*s(I^{j}_{\alpha,\beta}(k))^{'}\right)(s)}-1\right)\prec\Phi(s).$$

This implies that $s\left(I^{j}_{\alpha,\beta}(k)\right)^{'}(s)\in\mathfrak{S}_{(\Upsilon,\beta,\alpha)(h)}(\Phi)$.

Conversely, if $s\left(I^{j}_{\alpha,\beta}(k)\right)^{'}(s)\in\mathfrak{S}_{(\Upsilon,\beta,\alpha)(h)}(\Phi)$ then, from (1.3), we get

$$1+\frac{1}{\Upsilon}\left(\frac{s\left[h*s(I^{j}_{\alpha,\beta}(k))^{'}\right]^{'}(s)}{\left(h*s(I^{j}_{\alpha,\beta}(h*k))^{'}\right)(s)}-1\right)\prec\Phi(s).$$

Hence, by (2.3), we conclude that

$$1+\frac{1}{\Upsilon}\left(\frac{s(I^{j}_{\alpha,\beta}(h*k))^{''}(s)}{(I^{j}_{\alpha,\beta}(h*k))^{'}(s)}\right)\prec\Phi(s).$$

Therefore, $k\in\mathfrak{K}_{(\Upsilon,\beta,\alpha)(h)}(\Phi)$. □

Theorem 2.2. *Suppose that Φ be co. un. function fulfilling the condition*

$$\mathcal{R}e\left(\frac{sE^{'}(s)}{E(s)}\right)>\delta\qquad(0\leq\delta<1),\tag{2.4}$$

and $\mathcal{T}\in\mathbb{A}$, with $\mathcal{T}\in\mathcal{A}_{\delta}$. If $k\in\mathfrak{S}_{(\Upsilon,\beta,\alpha)(h)}(\Phi)$, then $(k*\mathcal{T})(s)\in\mathfrak{S}_{(\Upsilon,\beta,\alpha)(h)}(\Phi)$.

Proof. For a function $k\in\mathfrak{S}_{(\Upsilon,\beta,\alpha)(h)}(\Phi)$, we define the function $m(s)$ as

$$m(s)=1+\frac{1}{\Upsilon}\left(\frac{s(I^{j}_{\alpha,\beta}(h*k))^{'}(s)}{\left(I^{j}_{\alpha,\beta}(h*k)\right)(s)}-1\right)\qquad(0<\Upsilon\leq1).$$

It is clear that $m(s)$ is holomorphic in $\mathcal{U}$ and $m(s)\prec\Phi(s)$. Suppose that $\mathcal{T}(s)$ is a function in $\mathcal{A}_{\delta}$. To prove that $E(s)=\left(I^{j}_{\alpha,\beta}(h*k)\right)(s)\in\mathfrak{S}_{(\Upsilon,\beta,\alpha)}(\delta)$, observe that, since $k\in\mathfrak{S}_{(\Upsilon,\beta,\alpha)(h)}(\Phi)$, and Φ is co. un. and satisfies condition (2.4), we have $\mathcal{R}e\left(\frac{sE^{'}(s)}{E(s)}\right)=\mathcal{R}e(1-\Upsilon+\Upsilon m(s))>\delta$. It follows that

$$\mathcal{R}e\left(1+\frac{1}{\Upsilon}\left(\frac{s\left(I^{j}_{\alpha.\beta}(h*k)\right)^{'}(s)}{\left(I^{j}_{\alpha,\beta}(h*k)\right)(s)}-1\right)\right)>\delta.$$

Thus, $E\in\mathbb{S}^{*}_{\delta}(\Upsilon)$. Since $\mathcal{T}\in\mathcal{A}_{\delta}$, $E\in\mathbb{S}^{*}_{\delta}(\Upsilon)\subset\mathbb{S}^{*}_{\delta}$ and since Φ is convex, Theorem 1.1 gives

$$\frac{(\mathcal{T}*Em)(s)}{(\mathcal{T}*E)(s)}\prec\Phi(s),\tag{2.5}$$

since $s\left(I^{j}_{\alpha,\beta}(h*k)\right)'(s) = (h*s(I^{j}_{\alpha,\beta}(k)')(s)$.

This implies

$$1+\frac{1}{\Upsilon}\left(\frac{s(I^{j}_{\alpha,\beta}(h*\mathcal{T}*k))'(s)}{\left(I^{j}_{\alpha,\beta}(h*\mathcal{T}*k)\right)(s)}-1\right)$$

$$=\frac{(\Upsilon-1)(\mathcal{T}*(I^{j}_{\alpha,\beta}(h*k))(s)+(1-\Upsilon)(\mathcal{T}*\left(I^{j}_{\alpha,\beta}(h*k)\right)(s)+\Upsilon(\mathcal{T}*m(I^{j}_{\alpha,\beta}(h*k))(s)}{\Upsilon\left(\mathcal{T}*I^{j}_{\alpha,\beta}(h*k)\right)(s)}$$

$$=\frac{(\mathcal{T}*Em)(s)}{(\mathcal{T}*E)(s)}.$$

Thus, the subordination relation

$$1+\frac{1}{\Upsilon}\left(\frac{s(I^{j}_{\alpha,\beta}(h*\mathcal{T}*k))'(s)}{\left(I^{j}_{\alpha,\beta}(h*\mathcal{T}*k)\right)(s)}-1\right)\prec\Phi(s)$$

holds. This implies $(k*\mathcal{T})(s)\in\mathfrak{S}_{(\Upsilon,\beta,\alpha)(h)}(\Phi)$. □

Corollary 2.1. *The inclusion relation* $\mathfrak{S}_{(\Upsilon,\beta,\alpha)(h)}(\Phi)\subseteq\mathfrak{S}_{(\Upsilon,\beta,\alpha)(\mathcal{T}*h)}(\Phi)$ *holds if both functions* Φ *and* $\mathcal{T}$ *fulfill the requirements of Theorem 2.2.*

Proof. Suppose that $k\in\mathfrak{S}_{(\Upsilon,\beta,\alpha)(h)}(\Phi)$. By Theorem 2.2, $k*\mathcal{T}\in\mathfrak{S}_{(\Upsilon,\beta,\alpha)(h)}(\Phi)$. By the associative property, $I^{j}_{\alpha,\beta}((\mathcal{T}*h)*k)=I^{j}_{\alpha,\beta}(h*(k*\mathcal{T}))$ which implies that $k\in\mathfrak{S}_{(\Upsilon,\beta,\alpha)(\mathcal{T}*h)}(\Phi)$. □

Corollary 2.2. *For* $k\in\mathfrak{K}_{(\Upsilon,\beta,\alpha)(h)}(\Phi)$, *the inclusion relation* $\mathfrak{K}_{(\Upsilon,\beta,\alpha)(h)}(\Phi)\subseteq\mathfrak{K}_{(\Upsilon,\beta,\alpha)(\mathcal{T}*h)}(\Phi)$ *holds, and* $(k*\mathcal{T})(s)\in\mathfrak{K}_{(\Upsilon,\beta,\alpha)(h)}(\Phi)$ *holds if both functions* Φ *and* $\mathcal{T}$ *fulfill the requirements of Theorem 2.2.*

Proof. If $k\in\mathfrak{K}_{(\Upsilon,\beta,\alpha)(h)}(\Phi)$ and satisfies the condition $\frac{(I^{j}_{\alpha,\beta}(h*k))(s)}{s}\neq 0$, $\left(I^{j}_{\alpha,\beta}(h*k)\right)'(s)\neq 0$, then, according to part (2) of Theorem 2.1 and Theorem 2.2, we obtain $\left(\mathcal{T}*s\left(I^{j}_{\alpha,\beta}(k)\right)'\right)(s)\in\mathfrak{S}_{(\Upsilon,\beta,\alpha)(h)}(\Phi)$, since $I^{j}_{\alpha,\beta}((\mathcal{T}*h)*k)=I^{j}_{\alpha,\beta}(h*(k*\mathcal{T}))$. Hence $(k*\mathcal{T})(s)\in\mathfrak{K}_{(\Upsilon,\beta,\alpha)(h)}(\Phi)$, and the first inclusion follows from Corollary 2.1. □

Remark 2.1. *If* $\Upsilon=1$, $j=0$, $h=\frac{s}{1-s}$, *and* $\delta=0$, *then, for* $\Phi(s)=\frac{1+s}{1-s}$. *Theorem 2.1 implies that* $\mathfrak{K}_{(\Upsilon,\beta,\alpha)(h)}(\Phi)=\mathbb{K}\subseteq\mathfrak{S}_{(\Upsilon,\beta,\alpha)(h)}(\Phi)=\mathbb{S}^{*}$ *and* $k\in\mathbb{K}$ *if and only if* $sk'\in\mathbb{S}^{*}$. *Also, Theorem 2.2 implies that, for* $k\in\mathbb{S}^{*}$, *and* $\mathcal{T}\in\mathbb{K}$, *we get* $(k*\mathcal{T})(s)\in\mathbb{S}^{*}$.

Theorem 2.3. *For* $k\in\mathfrak{C}_{(\Upsilon,\beta,\alpha)(h)}(\Phi)$, $k_1\in\mathfrak{S}_{(\Upsilon,\beta,\alpha)(h)}(\Phi)$, *and both functions* Φ *and* $\mathcal{T}$ *fulfill the requirements of Theorem (2.2), we have* $(\mathcal{T}*k)(s)\in\mathfrak{C}_{(\Upsilon,\beta,\alpha)(h)}(\Phi)$ *and* $(\mathcal{T}*k_1)(s)\in\mathfrak{S}_{(\Upsilon,\beta,\alpha)(h)}(\Phi)$.

Proof. Suppose that $m(s)$ is a function defined by

$$m(s)=1+\frac{1}{\Upsilon}\left(\frac{s(I^{j}_{\alpha,\beta}(h*k))^{'}(s)}{\left(I^{j}_{\alpha,\beta}(h*k_1)\right)(s)}-1\right),$$

with a function $\Psi=\mathcal{T}$ and $E(s)=\left(I^{j}_{\alpha,\beta}(h*k_1)\right)(s)$. Then $\Psi\in\mathcal{A}_{\delta}$ and $E\in\mathbb{S}^{*}_{\delta}(\Upsilon)\subset\mathbb{S}^{*}_{\delta}$. An application of Theorem 1.1 implies that $\frac{(\mathcal{T}*Em)(s)}{(\mathcal{T}*E)(s)}\in\overline{co}(m(\mathcal{U}))$. By the convex property of $\Phi(s)$ and since $m(s)\prec\Phi(s)$, it follows that

$$\frac{(\mathcal{T}*Em)(s)}{(\mathcal{T}*E)(s)}\prec\Phi(s). \tag{2.6}$$

A simple calculation, as in Theorem 2.2, reveals the following:

$$1+\frac{1}{\Upsilon}\left(\frac{s\left(I^{j}_{\alpha,\beta}(h*\mathcal{T}*k)\right)^{'}(s)}{\left(I^{j}_{\alpha,\beta}(h*k_1)\right)(s)}-1\right)$$

$$=\frac{(\Upsilon-1)\left(\mathcal{T}*(I^{j}_{\alpha,\beta}(h*k_1)\right)(s)-\left[(\Upsilon-1)\left(\mathcal{T}*(I^{j}_{\alpha,\beta}(h*k_1)\right)(s)-\Upsilon\left(\mathcal{T}*m(I^{j}_{\alpha,\beta}(h*k_1))\right)(s)\right]}{\Upsilon\left(\mathcal{T}*(I^{j}_{\alpha,\beta}(h*k_1))\right)(s)}$$

$$=\frac{(\mathcal{T}*(Em))(s)}{\left(\mathcal{T}*(I^{j}_{\alpha,\beta}(h*k_1))\right)(s)}=\frac{(\mathcal{T}*Em)(s)}{(\mathcal{T}*E)(s)}.$$

Thus, by the subordination in (2.6), we get

$$1+\frac{1}{\Upsilon}\left(\frac{s(I^{j}_{\alpha,\beta}(h*\mathcal{T}*k)^{'}(s)}{\left(I^{j}_{\alpha,\beta}(h*\mathcal{T}*k_1)\right)(s)}-1\right)\prec\Phi(s).$$

This yields $\mathcal{T}*k\in\mathfrak{C}_{(\Upsilon,\beta,\alpha)(h)}(\Phi)$ with respect to $(\mathcal{T}*k_1)(s)\in\mathfrak{S}_{(\Upsilon,\beta,\alpha)(h)}(\Phi)$. □

Corollary 2.3. *The inclusion relation* $\mathfrak{C}_{(\Upsilon,\beta,\alpha)(h)}(\Phi)\subseteq\mathfrak{C}_{(\Upsilon,\beta,\alpha)(\mathcal{T}*h)}(\Phi)$ *hold, if both functions* Φ *and* $\mathcal{T}$ *fulfill requirements of Theorem 2.2.*

Proof. If $k\in\mathfrak{C}_{(\Upsilon,\beta,\alpha)(h)}(\Phi)$ with $k_1\in\mathfrak{S}_{(\Upsilon,\beta,\alpha)(h)}(\Phi)$, then, by Theorem 2.3, we obtain

$$1+\frac{1}{\Upsilon}\left(\frac{s(I^{j}_{\alpha,\beta}(h*\mathcal{T}*k)^{'}(s)}{\left(I^{j}_{\alpha,\beta}(h*\mathcal{T}*k_1)\right)(s)}-1\right)\prec\Phi(s).$$

Thus, $k\in\mathfrak{C}_{(\Upsilon,\beta,\alpha)(\mathcal{T}*h)}(\Phi)$. This concludes that $\mathfrak{C}_{(\Upsilon,\beta,\alpha)(h)}(\Phi)\subseteq\mathfrak{C}_{(\Upsilon,\beta,\alpha)(\mathcal{T}*h)}(\Phi)$. □

Theorem 2.4. *Suppose* Φ *be co. un. and right half plane function with* $\Phi(0)=1$ *and* $0<\Upsilon\leq 1$ *. Then*

1) $\mathcal{G}^{\xi}_{(\Upsilon,\beta,\alpha)(h)}(\Phi)\subseteq\mathfrak{S}_{(\Upsilon,\beta,\alpha)(h)}(\Phi)$ *where* $\xi>0$.

2) $\mathcal{G}^{\xi}_{(\Upsilon,\beta,\alpha)(h)}(\Phi) \subseteq \mathcal{G}^{\rho}_{(\Upsilon,\beta,\alpha)(h)}(\Phi)$ *where* $0 \leq \rho < \xi$.

Proof of (1). Let

$$\mathcal{X}(\xi, k(s)) = \xi\left(1 + \frac{s(I^{j}_{\alpha,\beta}(h*k))^{''}(s)}{\Upsilon\left(I^{j}_{\alpha,\beta}(h*k)\right)^{'}(s)}\right) + (1-\xi)\left(1 + \frac{1}{\Upsilon}\left(\frac{s(I^{j}_{\alpha,\beta}(h*k)^{'}(s)}{\left(I^{j}_{\alpha,\beta}(h*k)\right)(s)} - 1\right)\right),$$

and define the holomorphic function $m(s)$ in $\mathcal{U}$ by $m(s) = 1 + \frac{1}{\Upsilon}\left(\frac{s(I^{j}_{\alpha,\beta}(h*k)^{'}(s)}{(I^{j}_{\alpha,\beta}(h*k))(s)} - 1\right)$.

By differentiating the last equation with respect to (s) and performing simple calculations, we obtain

$$\frac{\xi s m^{'}(s)}{\Upsilon m(s) + (1-\Upsilon)} = \frac{\xi s\left(I^{j}_{\alpha,\beta}(h*k)\right)^{''}(s)}{\Upsilon\left(I^{j}_{\alpha,\beta}(h*k)\right)^{'}(s)} + \frac{\xi}{\Upsilon} - \frac{\xi s\left(I^{j}_{\alpha,\beta}(h*k)\right)^{'}(s)}{\Upsilon\left[\left(I^{j}_{\alpha,\beta}(h*k)\right)(s)\right]}.$$

That is,

$$\frac{\xi s m^{'}(s)}{\Upsilon m(s) + (1-\Upsilon)} - \frac{1}{\Upsilon} + \frac{s\left[\left(I^{j}_{\alpha,\beta}(h*k)\right)^{'}(s)\right]}{\Upsilon\left[\left(I^{j}_{\alpha,\beta}(h*k)\right)(s)\right]} + 1$$

$$= \xi\left(1 + \frac{s\left(I^{j}_{\alpha,\beta}(h*k)\right)^{''}(s)}{\Upsilon\left(I^{j}_{\alpha,\beta}(h*k)\right)^{'}(s)}\right) + (1-\xi)\left(1 + \frac{s\left[\left(I^{j}_{\alpha,\beta}(h*k)\right)^{'}(s)\right]}{\Upsilon\left[\left(I^{j}_{\alpha.\beta}(h*k)\right)(s)\right]} - \frac{1}{\Upsilon}\right).$$

That is,

$$\frac{\xi s m'(s)}{\Upsilon m(s) + (1-\Upsilon)} + \left[1 + \frac{s\left[\left(I^{j}_{\alpha,\beta}(h*k)\right)^{'}(s)\right]}{\Upsilon\left[\left(I^{j}_{\alpha,\beta}(h*k)\right)(s)\right]} - \frac{1}{\Upsilon}\right] = \mathcal{X}(\xi, k(s)).$$

The condition $\mathcal{R}e\left(\Upsilon m(s) + (1-\Upsilon)\right) > 0$, implies that

$$\frac{\xi s m'(s)}{\Upsilon m(s) + (1-\Upsilon)} + m(s) = \mathcal{X}(\xi, k(s)).$$

Thus, let $k \in \mathcal{G}^{\xi}_{(\Upsilon,\beta,\alpha)(h)}(\Phi)$, so that $\mathcal{X}(\xi, k(s)) \prec \Phi(s)$. Set $v_1 = \frac{\Upsilon}{\xi}$ and $v_1 = \frac{1-\Upsilon}{\xi}$, where $\xi > 0$. Since $m(s)$ is holomorphic in $\mathcal{U}$ and satisfies the property $m(0) = \Phi(0)$, Theorem 1.2, together with the right part of the last equation, ensures that $\mathcal{G}(s) \prec \Phi(s)$. Therefore, $k \in \mathfrak{S}_{(\Upsilon,\beta,\alpha)(h)}(\Phi)$. □

Proof of (2). When $\rho = 0$, case (1) applies. Now, suppose that $\rho > 0$. We have

$$\mathcal{X}(\rho, k(s)) = \rho\left(1 + \frac{s\left(I^{j}_{\alpha,\beta}(h*k)\right)^{''}(s)}{\Upsilon\left(I^{j}_{\alpha,\beta}(h*k)\right)^{'}(s)}\right) + (1-\rho)\left(1 + \frac{s\left(I^{j}_{\alpha,\beta}(h*k)\right)^{'}(s)}{\Upsilon\left(I^{j}_{\alpha,\beta}(h*k)\right)(s)} - \frac{1}{\Upsilon}\right)$$

$$= \left(1-\frac{\rho}{\xi}\right)\left(1+\frac{s\left(I^{j}_{\alpha,\beta}(h*k)\right)'(s)}{\Upsilon\left(I^{j}_{\alpha,\beta}(h*k)\right)(s)}-\frac{1}{\Upsilon}\right) + \frac{\rho}{\xi}\left[\xi\left(1+\frac{s\left(I^{j}_{\alpha,\beta}(h*k)\right)''(s)}{\Upsilon\left(I^{j}_{\alpha,\beta}(h*k)\right)'(s)}\right)\right] + (1-\xi)\left(1+\frac{1}{\Upsilon}\left(\frac{s\left(I^{j}_{\alpha,\beta}(h*k)\right)'(s)}{\left(I^{j}_{\alpha,\beta}(h*k)\right)(s)}-1\right)\right)$$

$$= \left(1-\frac{\rho}{\xi}\right)\left(1+\frac{1}{\Upsilon}\left(\frac{s\left(I^{j}_{\alpha,\beta}(h*k)\right)'(s)}{\left(I^{j}_{\alpha,\beta}(h*k)\right)(s)}-1\right)\right)+\frac{\rho}{\xi}\left[\mathcal{X}\left(\xi,k(s)\right)\right].$$

That is, $\mathcal{X}\left(\rho,k(s)\right)=\left(1-\frac{\rho}{\xi}\right)m(s)+\frac{\rho}{\xi}\left[\mathcal{X}\left(\xi,k(s)\right)\right]$ is a convex combination of two functions subordinate to the convex function $\Phi(s)$.

Since, by part (1),

$$1+\frac{1}{\Upsilon}\left(\frac{s\left[I^{j}_{\alpha,\beta}(h*k)'(s)\right]}{\left[I^{j}_{\alpha,\beta}(h*k)(s)\right]}-1\right)\prec\Phi(s)\ \textit{and}\ \ \mathcal{X}\left(\xi,k(s)\right)\prec\Phi(s),$$

this implies that $\mathcal{X}\left(\rho,k(s)\right)\prec\Phi(s)$. Thus, $k(s)\in\mathcal{G}^{\rho}_{(\Upsilon,\beta,\alpha)(h)}(\Phi)$. □

Theorem 2.5. *Let Φ be a co. un. function mapping into the right half-plane, with $\Phi(0)=1$ and $0<\Upsilon\leq 1$. Then*

1) $\mathfrak{K}_{(\Upsilon,\beta,\alpha)(h)}(\Phi)\subseteq\mathcal{Q}_{(\Upsilon,\beta,\alpha)(h)}(\Phi)\subseteq\mathfrak{C}_{(\Upsilon,\beta,\alpha)(h)}(\Phi)$.

2) $k\in\mathcal{Q}_{(\Upsilon,\beta,\alpha)(h)}(\Phi)$ *if and only if* $s(I^{j}_{\alpha,\beta}k)'(s)\in\mathfrak{C}_{(\Upsilon,\beta,\alpha)(h)}(\Phi)$.

Proof of (1). If we put $k=\mathfrak{T}$, it follows from Definition 1.7 that $\mathfrak{K}_{(\Upsilon,\beta,\alpha)(h)}(\Phi)\subseteq\mathcal{Q}_{(\Upsilon,\beta,\alpha)(h)}(\Phi)$. To prove the other inclusion, assume that

$$\mathcal{F}(s)=1+\frac{1}{\Upsilon}\left(\frac{s\left(I^{j}_{\alpha,\beta}(h*k)\right)'(s)}{\left(I^{j}_{\alpha,\beta}\left(h*\mathfrak{T}\right)\right)(s)}-1\right).$$

After differentiating the function $\mathcal{F}(s)$ and performing some calculations, we get

$$\frac{\Upsilon\mathcal{F}'(s)}{\left(\Upsilon\mathcal{F}(s)+(1-\Upsilon)\right.}=\frac{\left(s\left(I^{j}_{\alpha,\beta}(h*k)\right)'\right)'(s)}{s\left(I^{j}_{\alpha,\beta}(h*k)\right)'(s)}-\frac{\left(\left(I^{j}_{\alpha,\beta}\left(h*\mathfrak{T}\right)\right)\right)'(s)}{\left(I^{j}_{\alpha,\beta}\left(h*\mathfrak{T}\right)\right)(s)}$$

$$\frac{\mathcal{F}^{'}(s)}{(\Upsilon\mathcal{F}(s)+(1-\Upsilon)}=\frac{\left(\ s\left(I^{j}_{\alpha,\beta}(h*k)\right)^{'}\right)^{'}(s)}{\Upsilon\left(\Upsilon\mathcal{F}(s)+(1-\Upsilon)\right)\left(I^{j}_{\alpha,\beta}\left(h*\mathfrak{T}\right)\right)(s)}-\frac{1}{\Upsilon}\left(\frac{\left(I^{j}_{\alpha,\beta}\left(h*\mathfrak{T}\right)\right)^{'}(s)}{\left(I^{j}_{\alpha,\beta}\left(h*\mathfrak{T}\right)\right)(s)}\right)$$

$$\mathcal{F}^{'}(s)=\left(\frac{\left(I^{j}_{\alpha,\beta}(h*\mathfrak{T})\right)^{'}(s)}{\left(I^{j}_{\alpha,\beta}(h*\mathfrak{T})\right)(s)}\right)\left[\frac{1}{\Upsilon}\left(\frac{\left(\ s\left(I^{j}_{\alpha,\beta}(h*k)\right)^{'}\right)^{'}(s)}{\left(\left(I^{j}_{\alpha,\beta}\left(h*\mathfrak{T}\right)\right)^{'}(s)\right.}\right)-\frac{1}{\Upsilon}\left(\Upsilon\mathcal{F}(s)+(1-\Upsilon)\right)\right],$$

that is,

$$\mathcal{F}(s)+\frac{s\mathcal{F}^{'}(s)}{\left(\frac{s\left(I^{j}_{\alpha,\beta}(h*\mathfrak{T})\right)^{'}(s)}{\left(I^{j}_{\alpha,\beta}(h*\mathfrak{T})\right)(s)}\right)}=\left(1+\frac{1}{\Upsilon}\left(\frac{\left(\ s\left(I^{j}_{\alpha,\beta}(h*k)\right)^{'}\right)^{'}(s)}{\left(\left(I^{j}_{\alpha,\beta}\left(h*\mathfrak{T}\right)\right)^{'}(s)\right.}-1\right)\right).$$

Thus, there exists a function $\mathcal{T}\in\mathfrak{K}_{(\Upsilon,\beta,\alpha)(h)}(\Phi)$ such that

$$\left(1+\frac{1}{\Upsilon}\left(\frac{s\left(I^{j}_{\alpha,\beta}(h*k)\right)^{'}(s)}{\left(I^{j}_{\alpha,\beta}\left(h*\mathfrak{T}\right)\right)(s)}-1\right)\right)\prec\Phi(s)\text{ when }k\in\mathcal{Q}_{(\Upsilon,\beta,\alpha)(h)}(\Phi).$$

Also, by applying Theorem 2.1, part (1), we have $\mathfrak{T}\in\mathfrak{S}_{(\Upsilon,\beta,\alpha)(h)}(\Phi)$, this implies

$$\mathcal{R}e\left(\frac{s(I^{j}_{\alpha,\beta}(h*\mathfrak{T})^{'}(s)}{I^{j}_{\alpha,\beta}\left(h*\mathfrak{T}\right)(s)}\right)>0.$$

We conclude, by applying Theorem 1.4, that $\mathcal{F}(s)\prec\Phi(s)$.

Thus, $k\in\mathfrak{C}_{(\Upsilon,\beta,\alpha)(h)}(\Phi)$. □

Proof of (2). Since

$$\left(1+\frac{1}{\Upsilon}\left(\frac{\left(\ s\left(I^{j}_{\alpha,\beta}(h*k)\right)^{'}\right)^{'}(s)}{\left(I^{j}_{\alpha,\beta}\left(h*\mathfrak{T}\right)\right)^{'}(s)}-1\right)\right)=1+\frac{1}{\Upsilon}\left(\frac{s\left(h*\left(\ s\left(I^{j}_{\alpha,\beta}(k)\right)^{'}\right)\right)^{'}(s)}{h*\left(I^{j}_{\alpha,\beta}\left(\mathfrak{T}\right)\right)^{'}(s)}-1\right).\tag{2.7}$$

Thus, if $k\in\mathcal{Q}_{(\Upsilon,\beta,\alpha)(h)}(\Phi)$ with respect to a function $\mathfrak{T}\in\mathfrak{K}_{(\Upsilon,\beta,\alpha)(h)}(\Phi)$, then

$$1+\frac{1}{\Upsilon}\left(\frac{\left(\ s\left(I^{j}_{\alpha,\beta}(h*k)\right)^{'}\right)^{'}(s)}{\left(I^{j}_{\alpha,\beta}\left(h*\mathfrak{T}\right)\right)^{'}(s)}-1\right)\prec\Phi(s).$$

By Theorem 2.1, part (2), and by the definition of (1.5), we get $s\left(I^{j}_{\alpha,\beta}(k)\right)^{'}(s)\in\ \mathfrak{C}_{(\Upsilon,\beta,\alpha)(h)}(\Phi)$.

Conversely, if $s\left(I^{j}_{\alpha,\beta}(k)\right)^{'}(s) \in \mathfrak{C}_{(\Upsilon,\beta,\alpha)(h)}(\Phi)$, then there exists a function $\mathfrak{T}_1 \in \mathfrak{S}_{(\Upsilon,\beta,\alpha)(h)}(\Phi)$ such that $\frac{\left(I^{j}_{\alpha,\beta}(h*\mathfrak{T}_1)\right)^{'}(s)}{s} \neq 0$ and such that it satisfies $1+\frac{1}{\Upsilon}\left(\frac{\left(s\left(h*s\left(I^{j}_{\alpha,\beta}(k)\right)^{'}\right)\right)^{'}(s)}{I^{j}_{\alpha,\beta}(h*\mathfrak{T}_1)(s)}-1\right) \prec \Phi(s)$.

Suppose that there exists $\mathfrak{T} \in \mathfrak{K}_{(\Upsilon,\beta,\alpha)(h)}(\Phi)$ such that $s\mathfrak{T}^{'}=\mathfrak{T}_1$. Then the right-hand side of (2.7) is subordinate to $\Phi(s)$. This implies that the left-hand side of (2.7) is subordinate to $\Phi(s)$, which implies that $k \in \mathcal{Q}_{(\Upsilon,\beta,\alpha)(h)}(\Phi)$. □

Corollary 2.4. *If $k \in \mathcal{Q}_{(\Upsilon,\beta,\alpha)(h)}(\Phi)$ and the functions Φ and $\mathcal{T}$ fulfill the requirements of Theorem 2.2, then $\mathcal{T}*k \in \mathcal{Q}_{(\Upsilon,\beta,\alpha)(h)}(\Phi)$.*

Proof. If $k \in \mathcal{Q}_{(\Upsilon,\beta,\alpha)(h)}(\Phi)$, then, by part (2) of Theorem 2.5, $s\left(I^{j}_{\alpha,\beta}(k)\right)^{'}(s) \in \mathfrak{C}_{(\Upsilon,\beta,\alpha)(h)}(\Phi)$. Thus, Theorem 2.3 gives $s\left(I^{j}_{\alpha,\beta}(\mathcal{T}*k)\right)^{'}(s) \in \mathfrak{C}_{(\Upsilon,\beta,\alpha)(h)}(\Phi)$. Again, by part (2) of Theorem 2.5, we obtain $(\mathcal{T}*k)(s) \in \mathcal{Q}_{(\Upsilon,\beta,\alpha)(h)}(\Phi)$. □

Corollary 2.5. *The inclusion relation $\mathcal{Q}_{(\Upsilon,\beta,\alpha)(h)}(\Phi) \subseteq \mathcal{Q}_{(\Upsilon,\beta,\alpha)(\mathcal{T}*h)}(\Phi)$ holds if both functions Φ and $\mathcal{T}$ satisfy the requirements of Theorem 2.2.*

Proof. Let $\mathcal{T}$ and $\mathcal{M}$ be univalent functions in $\mathbb{A}$, and let $k \in \mathcal{Q}_{(\Upsilon,\beta,\alpha)(h)}(\Phi)$. By Corollary 2.4, we have $\mathcal{T}*k \in \mathcal{Q}_{(\Upsilon,\beta,\alpha)(h)}(\Phi)$, provided that $\mathcal{T}*\mathcal{M} \in \mathfrak{K}_{(\Upsilon,\beta,\alpha)(h)}(\Phi)$ according to Corollary 2.2. Thus, we get

$$1+\frac{1}{\Upsilon}\left(\frac{\left(s\left(I^{j}_{\alpha,\beta}(h*\mathcal{T}*k)\right)^{'}\right)^{'}(s)}{\left(I^{j}_{\alpha,\beta}(h*\mathcal{T}*\mathcal{M})\right)^{'}(s)}-1\right) \prec \Phi(s).$$

Hence $k \in \mathcal{Q}_{(\Upsilon,\beta,\alpha)(\mathcal{T}*h)}(\Phi)$. This implies that $\mathcal{Q}_{(\Upsilon,\beta,\alpha)(h)}(\Phi) \subseteq \mathcal{Q}_{(\Upsilon,\beta,\alpha)(\mathcal{T}*h)}(\Phi)$. □

Remark 2.2. *If $\Upsilon=1$ in Definitions 1.3-1.8, we have a new collection of functions $\mathfrak{S}_{(\beta,\alpha)(h)}$, $\mathfrak{K}_{(\beta,\alpha)(h)}$, $\mathfrak{C}_{(\beta,\alpha)(h)}$, $\mathcal{Q}_{(\beta,\alpha)(h)}$ and $\mathcal{G}^{\xi}_{(\beta,\alpha)(h)}$.*

3 Conclusion

Expanded subclasses of starlike, convex, quasi-convex, and close-to-convex functions that are equivalent to the corresponding classes were introduced by the researcher Shanmugam. These classes contributed to the development of known convolution properties of univalent holomorphic functions. The properties of univalent functions related to a linear operator were derived using the convex hull method. We demonstrated that these classes are closed under convolution with convex functions. Results involving inclusion relations were also proved by applying some theorems on the differential subordination theory

developed by Miller and Mocanu, as well as Alexander's theorem, based on the fact that "every convex univalent function is starlike," which is equivalent to "$\mathbb{K} \subset \mathbb{S}^*$".

References

[1] Goodman, A. W. (1983a). *Univalent functions* (Vols. 1–2). Polygonal Publishing House.

[2] Akbarally, A., Soh, S. C., & Ismail, M. (2013). Coefficient bounds for a class multivalent function defined by Salagean operator. *International Journal of Pure and Applied Mathematics*, *83*, 417–423. https://doi.org/10.12732/ijpam.v83i3.4

[3] Ravichandran, V., Polatoglu, Y., Bolcal, M., & Sen, A. (2005). Certain subclasses of starlike and convex functions of complex order. *Hacettepe Journal of Mathematics and Statistics*, *34*, 9–15.

[4] Suchithra, K., Stephen, B. A., & Sivasubramanian, S. (2006). A coefficient inequality for certain classes of analytic functions of complex order. *Journal of Inequalities in Pure and Applied Mathematics*, *7*, 1–6.

[5] Nasr, M., & Aouf, M. (1985). Starlike function of complex order. *Journal of Natural Sciences and Mathematics*, *25*, 1–12.

[6] Wiatrowski, P. (1970). On the coefficients of some family of holomorphic functions. *Scientific Papers of the University of Lodz, Mathematical and Natural Sciences*, *2*, 75–85.

[7] Swamy, S. (2012). Inclusion properties of certain subclasses of analytic functions. *International Mathematical Forum*, *7*, 1751–1760.

[8] Ruscheweyh, S., & Sheil-Small, T. (1973). Hadamard products of Schlicht functions and the Pólya–Schoenberg conjecture. *Commentarii Mathematici Helvetici*, *48*, 119–135.

[9] ShanShanmugam, T. (1989). Convolution and differential subordination. *International Journal of Mathematics and Mathematical Sciences*, *12*, 333–340. https://doi.org/10.1155/S0161171289000384

[10] Padmanabhan, K., & Parvatham, R. (1985). Some applications of differential subordination. *Bulletin of the Australian Mathematical Society*, *32*, 321–330. https://doi.org/10.1017/S0004972700002410

[11] Goodman, A. W. (1983b). *Univalent functions* (Vol. 2). Mariner Publishing Company.

[12] Oyekan, E., & Opoola, T. (2014). On a subordination result for analytic functions defined by convolution. *Annales Universitatis Mariae Curie-Skłodowska, Sectio A, Mathematica*, *68*, 59–66. https://doi.org/10.17951/a.2014.68.1.59

[13] Babalola, K. O. (2010). Quasi-convolution of analytic functions with applications. *arXiv*. https://doi.org/10.48550/arXiv.1004.2658

[14] Cho, N. E., & Kim, J. A. (2006). Inclusion properties of certain subclasses of analytic functions defined by a multiplier transformation. *Computers & Mathematics with Applications*, *52*, 323–330. https://doi.org/10.1016/j.camwa.2006.08.022

[15] Xu, N., & Raina, R. K. (2019). Some properties of a class of analytic functions. *Boletim da Sociedade Paranaense de Matemática*, *37*, 173–186. https://doi.org/10.5269/bspm.v37i4.34901

[16] El-Ashwah, R., Drbuk, M., & Aouf, M. (2016). A new class defined by subordination for γ-spirallike functions. *Journal of the Egyptian Mathematical Society*, *24*, 355–360. https://doi.org/10.1016/j.joems.2015.08.005

[17] Barnard, R. W., & Kellogg, C. (1980). Applications of convolution operators to problems in univalent function theory. *Michigan Mathematical Journal*, *27*, 81–94. https://doi.org/10.1307/mmj/1029002312

[18] Choi, J. H., Kim, Y. C., & Sugawa, T. (2000). Coefficient bounds and convolution properties for certain classes of close-to-convex functions. *Proceedings of the Japan Academy, Series A, Mathematical Sciences*, *76*, 95–98. https://doi.org/10.3792/pjaa.76.95

[19] Odeh, Z., & Jassim, K. (2019). Subordination for certain analytic multivalent functions defined by using differential operator. *Journal of Physics: Conference Series*, *1234*, Article 012110. https://doi.org/10.1088/1742-6596/1234/1/012110

[20] Hussain, A., Aouf, M., & Mostafa, A. (2023). Subordination results of analytic functions defined by convolution. *Sirte University Scientific Journal*, *13*, 55–59. https://doi.org/10.37375/susj.v13i1.1373

[21] Ruscheweyh, S. (1982). *Convolutions in geometric function theory*. University of Montreal Press.

[22] Miller, S. S., & Mocanu, P. T. (2000). *Differential subordinations: Theory and applications*. CRC Press.

[23] Miller, S. S., & Mocanu, P. T. (1981). Differential subordinations and univalent functions. *Michigan Mathematical Journal*, *28*, 157–171.

This is an open access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0/), which permits unrestricted, use, distribution and reproduction in any medium, or format for any purpose, even commercially provided the work is properly cited.