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Some Results on the v-Analogue of Gamma Function

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Abstract

In this paper, some properties for the v-analogue of Gamma and digamma functions are investigated. Also, a celebrated Bohr-Mollerup type theorem related to the v-analogue of Gamma function is given. Furthermore, an expression for the v-digamma function is presented by using the v-analogue of beta function. Also, some limits for the v-analogue of Gamma and beta functions are given.

1 Introduction and Preliminaries

Gamma function Γ, which was introduced by Euler with the aim to generalize the factorial function to non integer values, is one of the most important special function [1, 2, 3]. It has played a notable role in many branches such as mathematics and physics. It has been the subject of many studies for over three hundred years. Many definitions have been given for Gamma function. Although they all describe the same function, it is not always easy to show that they are equivalent. For this reason, the features that characterize this function have been studied for a long time. The purpose of this is that instead of showing that the definitions given for Gamma function are equivalent, it is an easier way to find the properties that make this function unique and see if the given definitions provide these properties. In 1922, two mathematicians, Harald Bohr and Johannes Mollerup, were able to give a precise characterization of Gamma function. Today, this theorem is known as the Bohr-Mollerup theorem.

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Generalizations of Gamma function has attracted much attention from many mathematicians [4, 5, 9, 12, 13, 15, 17]. There are some remarkable achievements. Recently, in [6], the authors introduced a new one-parameter deformation of the classical Gamma function and called v-analogue (v-deformation or v-generalization) of Gamma function. It is defined for $x, v > 0$ as

$$\Gamma_v(x) = \int_0^\infty \left(\frac{t}{v}\right)^{\frac{x}{v}-1} e^{-t} dt. \tag{1}$$

Note that when $v = 1$, $\Gamma_v(x) = \Gamma(x)$. They introduced and proved some identities of the said function such as

$$\begin{aligned} \Gamma_v(x+v) &= xv^{-2}\Gamma_v(x), && (2)\\ \Gamma_v(x) &= v^{1-\frac{x}{v}}\Gamma\left(\frac{x}{v}\right), && (3)\\ \Gamma_v(v) &= 1. && (4) \end{aligned}$$

They also gived the relation

$$\Gamma_v(x) = \lim_{n\to\infty} \frac{n!\left(\frac{n}{v}\right)^{\frac{x}{v}} v^{n+2}}{x(x+v)(x+2v)\ldots(x+nv)}. \tag{5}$$

The v-analogue of Gamma function is expressed as

$$\frac{1}{\Gamma_v(x)} = v^{\frac{x}{v}-2} x e^{\frac{\gamma x}{v}} \prod_{k=1}^{\infty} \left(1 + \frac{x}{kv}\right) e^{-\frac{x}{kv}}, \tag{6}$$

where γ is the Euler-Mascheroni constant, [6]. The logarithmic derivative of $\Gamma_v(x)$ is called v-digamma or v-psi function and denoted by $\psi_v(x)$. The series representation of $\psi_v(x)$ is given in [6] by the relation

$$\Psi_v(x) = -\frac{\ln v + \gamma}{v} - \frac{1}{x} + \sum_{n=1}^{\infty} \left[\frac{1}{nv} - \frac{1}{x+nv}\right]. \tag{7}$$

In this work, we establish some properties involving the v-analogue of Gamma and digamma functions. Also, we give a Bohr-Mollerup type theorem related to the v-analogue of Gamma function. After defining the v-analogue of beta function, we give an expression for the v-digamma function. Finally, we give some limits for the v-analogue of Gamma and beta functions. We present our results in the following sections.

2 Main Results

This section contains three subsections. We prove the Bohr-Mollerup type theorem, giving some properties and an asymptotic expansion related to the v-analogue of the Gamma function. Also, we present some integral representations of the v-analogue of the digamma function.

2.1 Some properties of the v-analogue of Gamma function

Firstly, we define the v-analogue of Gamma function for some negative values. From the relation (2), we have

$$\Gamma_v(x) = v^2 \frac{\Gamma_v(x+v)}{x}. \tag{8}$$

Thus, if $-v < x < 0$, the right-hand side of the equation (8) is well-defined. This means that we now have $\Gamma_v(x)$ defined for $-v < x < 0$. Note that, since $\Gamma_v(x)$ is positive for $x > 0$ we get $\Gamma_v(x)$ is negative for $-v < x < 0$. Now writing

$$\Gamma_v(x+2v) = \Gamma_v((x+v)+v) = (x+v)v^{-2}\Gamma_v(x+v) = (x+v)(v^{-2})^2 x\Gamma_v(x)$$

for $-2v < x < -v$ we get

$$\Gamma_v(x) = (v^2)^2 \frac{\Gamma_v(x+2v)}{x(x+v)}.$$

Again note that, since $\Gamma_v(x+2v)$ is positive for $-2v < x < -v$ we get $\Gamma_v(x)$ is positive for values $-2v < x < -v$. This process can be repeated to define $\Gamma_v(x)$ for values $-nv < x < -nv + v$. Hence we get the following theorem.

Theorem 1. *For $x, v > 0$ and $-nv < x < -nv + v$, $n = 1, 2, \ldots$ we have*

$$\Gamma_v(x) = v^{2n} \frac{\Gamma_v(x+nv)}{x(x+v)\ldots(x+(n-1)v)}.$$

In the following, we use regularization method given by Gel'fand and Shilov in [10] to obtain an expression for the v-analogue of Gamma function. We subtract enough terms of the Taylor's series of the function e^{-t} in (1) to make the integral converges. Then for $n = 1, 2, \ldots$, $x, v > 0$ and $-nv < x < -nv + v$, we have

$$\Gamma_v(x) = \int_0^\infty \left(\frac{t}{v}\right)^{\frac{x}{v}-1} \left[e^{-t} - \sum_{i=0}^{n} \frac{(-t)^i}{i!}\right] dt.$$

It is known that Bohr-Mollerup theorem, see for example [16], characterizes the Euler's Gamma function as the uiquely defined log-convex solution $f : (0, \infty) \to (0, \infty)$ and satisfies the functional equation $f(x + 1) = xf(x)$ and $f(1) = 1$. Now, we give a Bohr-Mollerup type theorem related to the v-analogue of Gamma function:

Theorem 2. *Let f be any positive valued function defined on $(0, \infty)$ satisfying the following three properties for $v > 0$:*
(a) $f(v) = 1$, (b) $f(x + v) = xv^{-2}f(x)$ and (c) $\ln f(x)$ is convex.
Then $f(x) = \Gamma_v(x)$ for all $x \in (0, \infty)$.

Proof. Since $\ln \Gamma_v$ is convex by using the Theorem 3.10 in [6], Γ_v satisfies (a), (b) and (c). Then, it is enough to prove that f is uniquely determined by (a), (b) and (c). Also by (b), it is enough to do this for $x \in (0, v]$. By using convexity of $\ln f$, we obtain

$$\begin{aligned} \frac{\ln f(nv + v) - \ln f(nv)}{v} &\leq \frac{\ln f(nv + v + x) - \ln f(nv + v)}{x} \\ &\leq \frac{\ln f(nv + 2v) - \ln f(nv + v)}{v}. \end{aligned}$$

That is,

$$\frac{1}{v} \ln\left(\frac{f(nv + v)}{f(nv)}\right) \leq \frac{1}{x} \ln\left(\frac{f(nv + v + x)}{f(nv + v)}\right) \leq \frac{1}{v} \ln\left(\frac{f(nv + 2v)}{f(nv + v)}\right). \quad (9)$$

By (b), we have

$$f(nv + v) = n!v^{-n} \quad (10)$$

and

$$f(nv + v + x) = (v^{-2})^{n+1}(nv + x)(nv + x - v)(nv + x - 2v) \ldots xf(x). \quad (11)$$

Substituting the expressions (10) and (11) in the inequalities (9), we get

$$\begin{aligned} \frac{x}{v}\ln(nv^{-1}) &\leq \ln\left(\frac{(v^{-2})^{n+1}(nv+x)(nv+x-v)(nv+x-2v)\ldots xf(x)}{n!v^{-n}}\right) \\ &\leq \frac{x}{v}\ln\left((nv+v)v^{-2}\right). \end{aligned}$$

Then,

$$0 \leq \ln\left(\frac{(nv+x)(nv+x-v)(nv+x-2v)\ldots xf(x)}{n!v^{n+2}(nv^{-1})^{\frac{x}{v}}}\right) \leq \ln\left(\frac{nv+v}{nv}\right)^{\frac{x}{v}}.$$

Since the last expression tends to 0 as $n \to \infty$ we have

$$\lim_{n\to\infty} \ln\left(\frac{(nv+x)(nv+x-v)(nv+x-2v)\ldots xf(x)}{n!v^{n+2}(nv^{-1})^{\frac{x}{v}}}\right) = 0.$$

Now, interchanging ln with the limit on the left and applying the exponential to both sides yields $f(x) = \Gamma_v(x)$, and the proof is completed. □

Using the relation (3), we give the following formulas:

Proposition 3. *Let $v > 0$. Then the v-analogue of Gamma function satisfies the relations:*

(i) $\Gamma_v(x)\Gamma_v(v-x) = \frac{v\pi}{\sin\left(\frac{\pi x}{v}\right)}$, $x \neq 0, -v, -2v, \ldots, v, 2v, \ldots$ *(v-reflection),*

(ii) $\Gamma_v(x)\Gamma_v(x+\frac{v}{2}) = 2^{1-\frac{2x}{v}}\sqrt{v\pi}\,\Gamma_v(2x)$, $x \neq 0, -\frac{v}{2}, -v, -\frac{3v}{2}, -2v, \ldots$
(v-Legendre Duplication),

(iii) $\Gamma_v(x) = \sqrt{2\pi}v^{-\frac{2x}{v}+\frac{3}{2}}x^{\frac{x}{v}-\frac{1}{2}}e^{-\frac{x}{v}} + O\left(\frac{1}{x}\right)$, $x > 0$.

Proof. The relation (3) gives

$$\Gamma_v(x)\Gamma_v(v-x) = v^{1-\frac{x}{v}}\Gamma\left(\frac{x}{v}\right)v^{1-\frac{v-x}{v}}\Gamma\left(\frac{v-x}{v}\right) = v\Gamma\left(\frac{x}{v}\right)\Gamma\left(1-\frac{x}{v}\right).$$

Now using the Euler's reflection formula, see [7]:

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}, \quad x \notin \mathbb{Z},$$

we get the result (i). Again using equation (3) and Legendre duplication formula, see [14],

$$\Gamma(x)\Gamma\left(x+\frac{1}{2}\right) = 2^{1-2x}\sqrt{\pi}\Gamma(2x), \quad x \neq 0, -\frac{1}{2}, -1, -\frac{3}{2}\ldots,$$

gives that

$$\begin{aligned}\Gamma_v(x)\Gamma_v\left(x+\frac{v}{2}\right) &= v^{\frac{3}{2}-\frac{2x}{v}}\Gamma\left(\frac{x}{v}\right)\Gamma\left(\frac{x}{v}+\frac{1}{2}\right) = v^{\frac{3}{2}-\frac{2x}{v}}2^{1-\frac{2x}{v}}\sqrt{\pi}\Gamma(\frac{2x}{v}) \\ &= v^{\frac{3}{2}-\frac{2x}{v}}2^{1-\frac{2x}{v}}\sqrt{\pi}v^{\frac{2x}{v}-1}\Gamma_v(2x) = 2^{1-\frac{2x}{v}}\sqrt{v\pi}\Gamma_v(2x).\end{aligned}$$

By Stirling's asymptotic formula for Gamma function, see [18]:

$$\Gamma(x) = \sqrt{2\pi}x^{x-\frac{1}{2}}e^{-x} + O\left(\frac{1}{x}\right)$$

and the relation (3) we get

$$\begin{aligned}\Gamma_v(x+v) &= v^{1-\frac{x+v}{v}}\Gamma\left(\frac{x}{v}+1\right) = v^{-\frac{x}{v}}\frac{x}{v}\Gamma\left(\frac{x}{v}\right) \\ &= v^{-\frac{x}{v}}\frac{x}{v}\sqrt{2\pi}x^{x-\frac{1}{2}}e^{-x} + O\left(\frac{1}{x}\right) \\ &= \sqrt{2\pi}x^{\frac{x}{v}+\frac{1}{2}}v^{-\frac{2x}{v}-\frac{1}{2}}e^{\frac{-x}{v}} + O\left(\frac{1}{x}\right).\end{aligned}$$

Now recalling the recurrence formula (2) we get the relation (*iii*). □

2.2 Asymptotic expansion of the v-analogue of Gamma function

Now, we use the following result which can be found in [8] to give an analogue of the Stirling's formula for the function Γ_v.

Theorem 4. *Assume that $f : (a, b) \to \mathbb{R}$, with $a, b \in [0, +\infty)$ attains a global minimum at a unique point $c \in (a, b)$, such that $f''(c) > 0$. Then one has*

$$\int_a^b g(x)e^{-\frac{f(x)}{h}}\,dx = h^{\frac{1}{2}}e^{-\frac{f(c)}{h}}\sqrt{2\pi}\frac{g(c)}{\sqrt{f''(c)}} + O(h). \tag{12}$$

Theorem 5. *For $x, v > 0$, the following identity holds:*

$$\Gamma_v(x+1) = \sqrt{2\pi}v^{-\frac{2x}{v}-\frac{1}{2}}x^{\frac{x}{v}+\frac{1}{2}}e^{-\frac{x}{v}} + O\left(\frac{1}{x}\right).$$

Proof. We have

$$\Gamma_v(x+1) = \int_0^\infty \left(\frac{t}{v}\right)^{\frac{x}{v}} e^{-t}dt.$$

By making the change of variable $t = ux$ we get

$$\Gamma_v(x+1) = v^{-\frac{x}{v}}x^{\frac{x}{v}+1}\int_0^\infty u^{\frac{x}{v}}e^{-ux}\,du = v^{-\frac{x}{v}}x^{\frac{x}{v}+1}\int_0^\infty e^{-x\left(u-\frac{1}{v}\ln u\right)}\,du.$$

Let $f(u) = u - \ln u^{\frac{1}{v}}$. Then $f'(u) = 0$ if and only if $u = \frac{1}{v}$. Also, $f''(\frac{1}{v}) = v > 0$. Then, from (12), we have

$$\begin{aligned}\int_0^\infty e^{-x\left(u-\frac{1}{v}\ln u\right)}\,du &= \left(\frac{1}{x}\right)^{\frac{1}{2}}e^{-\frac{x}{v}(1+\ln v)}\sqrt{2\pi}\frac{1}{\sqrt{v}} + O\left(\frac{1}{x}\right) \\ &= \sqrt{\frac{2\pi}{x}}\,v^{-\left(\frac{x}{v}+\frac{1}{2}\right)}e^{-\frac{x}{v}} + O\left(\frac{1}{x}\right).\end{aligned}$$

Therefore

$$\Gamma_v(x+1) = \sqrt{2\pi}v^{-\frac{2x}{v}-\frac{1}{2}}x^{\frac{x}{v}+\frac{1}{2}}e^{-\frac{x}{v}} + O\left(\frac{1}{x}\right),$$

and the proof is completed. □

2.3 v-Analogue of the digamma and beta functions

The next two theorems gives the integral representations of the function ψ_v. On the other hand,

Theorem 6. *We have*

$$\psi_v(x) = \frac{1}{v}\int_0^\infty \frac{1}{y}\left[e^{-y} - \frac{1}{\left(1+\frac{y}{v}\right)^{\frac{x}{v}}}\right] dy \tag{13}$$

for $x, v > 0$.

Proof. Taking the first derivative of (1) with respect to x and using the identitiy

$$\ln\left(\frac{t}{v}\right) = \int_0^\infty \frac{e^{-x} - e^{-\frac{t}{v}x}}{x}\, dx$$

we get

$$\begin{aligned}
\Gamma_v'(x) &= \frac{1}{v}\int_0^\infty \left(\frac{t}{v}\right)^{\frac{x}{v}-1} \ln\left(\frac{t}{v}\right) e^{-t}\, dt \\
&= \frac{1}{v}\int_0^\infty \int_0^\infty \left(\frac{t}{v}\right)^{\frac{x}{v}-1} \frac{e^{-t-y} - e^{-t\left(1+\frac{y}{v}\right)}}{y}\, dy\, dt \\
&= \frac{1}{v}\int_0^\infty \int_0^\infty \left(\frac{t}{v}\right)^{\frac{x}{v}-1} \frac{e^{-t-y} - e^{-t\left(1+\frac{y}{v}\right)}}{y}\, dt\, dy \\
&= \frac{1}{v}\int_0^\infty \frac{1}{y}\left[e^{-y}\int_0^\infty \left(\frac{t}{v}\right)^{\frac{x}{v}-1} e^{-t}\, dt - \int_0^\infty \left(\frac{t}{v}\right)^{\frac{x}{v}-1} e^{-t\left(1+\frac{y}{v}\right)}\, dt\right] dy.
\end{aligned}$$

Now, using the definition of $\Gamma_v(x)$ and change of variable $\left(1+\frac{y}{v}\right)t = s$, we have

$$\begin{aligned}
\Gamma_v'(x) &= \frac{1}{v}\int_0^\infty \frac{1}{y}\left[e^{-y}\,\Gamma_v(x) - \frac{1}{\left(1+\frac{y}{v}\right)^{\frac{x}{v}}}\Gamma_v(x)\right] dy \\
&= \frac{1}{v}\Gamma_v(x)\int_0^\infty \frac{1}{y}\left[e^{-y} - \frac{1}{\left(1+\frac{y}{v}\right)^{\frac{x}{v}}}\right] dy,
\end{aligned}$$

and since $\psi_v(x) = \frac{\Gamma_v'(x)}{\Gamma_v(x)}$ the result follows. □

Theorem 7. *We have*

$$\psi_v(x) = \frac{1}{v}\int_0^\infty \left[\frac{e^{-t}}{t} - \frac{e^{-\frac{tx}{v}}}{1-e^{-t}}\right] dt \tag{14}$$

for $x, v > 0$.

Proof. Using the change of variable $1 + \frac{y}{v} = e^t$ in the equation (13), we have

$$\begin{aligned}
\psi_v(x) &= \frac{1}{v}\lim_{t\to 0^+}\left[\int_t^\infty \frac{e^{-y}}{y}\,dy - \int_{\ln(\frac{t}{v}+1)}^\infty \frac{e^{-\frac{tx}{v}}}{1-e^{-t}}\,dt\right] \\
&= \frac{1}{v}\lim_{t\to 0^+}\left[\int_t^{\ln(\frac{t}{v}+1)} \frac{e^{-y}}{y}\,dy + \int_{\ln(\frac{t}{v}+1)}^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-\frac{tx}{v}}}{1-e^{-t}}\right) dt\right]
\end{aligned}$$

and since

$$\left|\int_t^{\ln(\frac{t}{v}+1)} \frac{e^{-y}}{y}\,dy\right| \le \int_t^{\ln(\frac{t}{v}+1)} \left|\frac{e^{-y}}{y}\right| dy \le \int_{\ln(\frac{t}{v}+1)}^t \frac{dy}{y} \to 0$$

as $t \to 0$, we get the result. □

Remark 8. By letting $v = 1$ in the Theorems 6 and 7, we obtain the integral representations of ψ due to Dirichlet and Gauss respectively given in [2].

Now, we introduce the v-analogue of beta function in a similar way to the classical beta function B.

Definition 9. Let $v > 0$. Then the *v-analogue* (also called *v-deformation* or *v-generalization*) of beta function is defined as

$$B_v(x, y) = \frac{\Gamma_v(x)\Gamma_v(y)}{\Gamma_v(x+y)} \tag{15}$$

for $x, y > 0$.

Since by using the equation (3) we have

$$B_v(x, y) = vB\left(\frac{x}{v}, \frac{y}{v}\right),$$

then we get the integral representation of B_v as

$$B_v(x,y) = v\int_0^1 t^{\frac{x}{v}-1}(1-t)^{\frac{y}{v}-1}\,dt$$

for $x, y, v > 0$.

Now, we give an expression for the v-digamma function by using the v-analogue of beta function.

Theorem 10. *Let $x, v > 0$. Then the v-digamma function has the series representation*

$$\psi_v(x) = -\frac{\gamma}{v} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(nv)n!} \prod_{k=1}^{n} \left(\frac{x}{v} - k\right).$$

Proof. By the definition of derivative we have

$$\Gamma_v'(x) = \lim_{y\to 0} \frac{\Gamma_v(x+y) - \Gamma_v(x)}{y}.$$

Then, using the equation (15) and the symmetry of B_v i.e. $B_v(x,y) = B_v(y,x)$ we get

$$\begin{aligned} \psi_v(x) &= \frac{\Gamma_v'(x)}{\Gamma_v(x)} = \frac{1}{\Gamma_v(x)} \lim_{y\to 0} \frac{\Gamma_v(x+y) - \Gamma_v(x)}{y} \\ &= \lim_{y\to 0} \frac{\Gamma_v(y) - B_v(x,y)}{yB_v(x,y)} \\ &= \lim_{y\to 0} \frac{\Gamma_v(y) - B_v(y,x)}{yB_v(y,x)}. \end{aligned} \tag{16}$$

Now, using v-Weierstrass canonical product (6) and the serie representation of the exponential function, [11],

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

we have

$$\begin{aligned}
\lim_{y\to 0}\Gamma_v(y) &= \lim_{y\to 0}\left[\frac{1}{y}v^{2-\frac{y}{v}}e^{-\frac{\gamma y}{v}}\prod_{n=1}^{\infty}\left(1+\frac{y}{nv}\right)^{-1}e^{\frac{y}{nv}}\right] \\
&= \lim_{y\to 0}\frac{1}{y}\left(1-\frac{\gamma y}{v}+O(y^2)\right)\lim_{y\to 0}\left(v^{2-\frac{y}{v}}\prod_{n=1}^{\infty}\left(1+\frac{y}{nv}\right)^{-1}e^{\frac{y}{nv}}\right) \\
&= v^2\left(\lim_{y\to 0}\frac{1}{y}-\frac{\gamma}{v}\right) \\
&= v^2\lim_{y\to 0}\frac{1}{y}-\gamma v, \qquad (17)
\end{aligned}$$

and using the power series, [11],

$$(1+x)^q = 1+qx+\frac{q(q-1)}{2!}x^2+\ldots+\frac{q(q-1)\ldots(q-k+1)}{k!}x^k+\ldots$$

in the integral representation of B_v for $x=-t$ and $q=\frac{y}{v}-1$, we have

$$\begin{aligned}
B_v(x,y) &= v\int_0^1 t^{\frac{x}{v}-1}(1-t)^{\frac{y}{v}-1}\,dt \\
&= v\int_0^1 t^{\frac{x}{v}-1}\left[1-\left(\frac{y}{v}-1\right)t+\left(\frac{y}{v}-1\right)\left(\frac{y}{v}-2\right)\frac{t^2}{2!}-\ldots\right]dt \\
&= v\left[\frac{v}{x}-\left(\frac{y}{v}-1\right)\frac{v}{x+v}+\left(\frac{y}{v}-1\right)\left(\frac{y}{v}-2\right)\frac{v}{2!(x+2v)}-\ldots\right]. \qquad (18)
\end{aligned}$$

Now, using the equations (16), (17) and (18), we have

$$\begin{aligned}
\psi_v(x) &= \lim_{y\to 0}\frac{\Gamma_v(y)-B_v(y,x)}{yB_v(y,x)} \\
&= \frac{v^2\lim\limits_{y\to 0}\frac{1}{y}-\gamma v-v^2\lim\limits_{y\to 0}\frac{1}{y}+v^2\left(\frac{x}{v}-1\right)\frac{1}{y+v}-\frac{v^2}{2!}\left(\frac{x}{v}-1\right)\left(\frac{x}{v}-2\right)\frac{1}{y+2v}+\ldots}{\lim\limits_{y\to 0}y\left[\frac{v^2}{y}-\frac{v^2}{y+v}\left(\frac{x}{v}-1\right)+\frac{v^2}{2!(y+2v)}\left(\frac{x}{v}-1\right)\left(\frac{x}{v}-2\right)-\ldots\right]} \\
&= \frac{-\gamma v+v^2\left(\frac{x}{v}-1\right)\frac{1}{v}-\frac{v^2}{2!}\left(\frac{x}{v}-1\right)\left(\frac{x}{v}-2\right)\frac{1}{2v}+\ldots}{v^2} \\
&= -\frac{\gamma}{v}+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(nv)n!}\prod_{k=1}^{n}\left(\frac{x}{v}-k\right),
\end{aligned}$$

and the result follows. □

3 A Limit for the v-Analogue of Gamma and Beta Function

In this section, we give a limit for the v-analogue of Gamma and beta function by using the Proposition 3.

Theorem 11. *For $x, v > 0$ and $a,b > 0$ we have*

$$\lim_{x\to\infty}\left(\frac{x}{v}\right)^{\frac{b-a}{v}}\left[\frac{\Gamma_v(ax+v)}{\Gamma_v(bx+v)}\right]^{\frac{1}{x}} = \left(\frac{a^a}{b^b}\right)^{\frac{1}{v}}(ve)^{\frac{b-a}{v}}.$$

Proof. By using the equations (2) and (iii) of the Proposition 3 we get

$$\Gamma_v(ax+v) = \sqrt{2\pi}v^{-\frac{2ax}{v}-\frac{1}{2}}(ax)^{\frac{ax}{v}+\frac{1}{2}}e^{-\frac{ax}{v}}\left(1+O\left(\frac{1}{ax}\right)\right) \tag{19}$$

and

$$\Gamma_v(bx+v) = \sqrt{2\pi}v^{-\frac{2bx}{v}-\frac{1}{2}}(bx)^{\frac{bx}{v}+\frac{1}{2}}e^{-\frac{bx}{v}}\left(1+O\left(\frac{1}{bx}\right)\right). \tag{20}$$

Then from (19) and (20) we obtain

$$\left[\frac{\Gamma_v(ax+v)}{\Gamma_v(bx+v)}\right]^{\frac{1}{x}} = \frac{(ax)^{\frac{a}{v}+\frac{1}{2x}}v^{-\frac{2a}{v}-\frac{1}{2x}}e^{-\frac{a}{v}}}{(bx)^{\frac{b}{v}+\frac{1}{2x}}v^{-\frac{2b}{v}-\frac{1}{2x}}e^{-\frac{b}{v}}} \times \left[\frac{\left(1+O\left(\frac{1}{ax}\right)\right)}{\left(1+O\left(\frac{1}{bx}\right)\right)}\right]^{\frac{1}{x}}. \tag{21}$$

Since

$$\lim_{x\to\infty}\left(\frac{A}{B}\right)^{\frac{1}{x}} = 1, \;\; 0 < A, B < \infty$$

and

$$\lim_{x\to\infty}\left[1+O\left(\frac{1}{Ax}\right)\right]^{\frac{1}{x}} = 1, \;\; A > 0$$

we have

$$\lim_{x\to\infty} x^{\frac{b-a}{v}}\left[\frac{\Gamma_v(ax+v)}{\Gamma_v(bx+v)}\right]^{\frac{1}{x}} = \frac{a^{\frac{a}{v}}}{b^{\frac{b}{v}}}v^{-\frac{2(a-b)}{v}}e^{-\frac{a-b}{v}},$$

and the result follows. □

Theorem 12. *For $x, v > 0$ and $a,b,c,d > 0$, we have*

$$\lim_{x\to\infty} [B_v(ax+c, bx+d)]^{\frac{1}{x}} = \frac{a^{\frac{a}{v}} b^{\frac{b}{v}}}{(a+b)^{\frac{a+b}{v}}}.$$

Proof. By relations (2), (15) and (*iii*) of Proposition (3), we can write

$$B_v(ax+c, bx+d) = \frac{\Gamma_v(ax+c)\Gamma_v(bx+d)}{\Gamma_v((a+b)x+c+d)}$$

$$= \sqrt{2\pi} v^{-\frac{2(ax+c)}{v}+\frac{3}{2}} (ax+c)^{\frac{ax+c}{v}-\frac{1}{2}} e^{-\frac{ax+c}{v}} \left(1+O\left(\frac{1}{ax+c}\right)\right) \times$$

$$\times \sqrt{2\pi} v^{-\frac{2(bx+d)}{v}+\frac{3}{2}} (bx+d)^{\frac{bx+d}{v}-\frac{1}{2}} e^{-\frac{bx+d}{v}} \left(1+O\left(\frac{1}{bx+d}\right)\right) \times$$

$$\times \frac{1}{\sqrt{2\pi} v^{-\frac{2(a+b)x+c+d}{v}+\frac{3}{2}} ((a+b)x+c+d)^{\frac{(a+b)x+c+d}{v}-\frac{1}{2}} e^{-\frac{(a+b)x+c+d}{v}} \left(1+O\left(\frac{1}{(a+b)x+c+d}\right)\right)}.$$

Now observing that

$$\lim_{x\to\infty} (Bx+C)^{\frac{1}{x}} = 1, \ \ B > 0, C \geq 0,$$

$$\lim_{x\to\infty} \left(\frac{Ax+B}{Cx+D}\right) = \frac{A}{C}, \ \ A, C > 0, B, D \geq 0$$

and

$$\lim_{x\to\infty} \left(1+O\left(\frac{1}{Dx+E}\right)\right) = 1, \ \ D > 0, E \geq 0$$

we get

$$\lim_{x\to\infty} [B_v(ax+c, bx+d)]^{\frac{1}{x}} = \frac{v^{\frac{-2a}{v}} v^{\frac{-2b}{v}}}{v^{\frac{-2(a+b)}{v}}} \left(\frac{a}{a+b}\right)^{\frac{a}{v}} \left(\frac{b}{a+b}\right)^{\frac{b}{v}} = \frac{a^{\frac{a}{v}} b^{\frac{b}{v}}}{(a+b)^{\frac{a+b}{v}}},$$

and the result follows. □

4 Conclusions

Some properties, Bohr-Mollerup type theorem and an asymptotic expansion related to the v-analogue of the Gamma function have been given. Some integral representations of the v-analogue of the digamma function have been presented by defining the v-analogue of beta function. Also, some limits for the v-analogue of Gamma and beta functions have been given. Using the results of this paper, one can investigate and find new results on the said functions.

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