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Results of Semigroup of Linear Operator Generating a Continuous Time Markov Semigroup

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Abstract

In this paper, we present results of ω-order preserving partial contraction mapping creating a continuous time Markov semigroup. We use Markov and irreducible operators and their integer powers to describe the evolution of a random system whose state changes at integer times, or whose state is only inspected at integer times. We concluded that a linear operator $P : \ell^1(X_+) \to \ell^1(X_+)$ is a Markov operator if its matrix satisfies $P_{x,y} \geqslant 0$ and $\sum_{x \in X_+} P_{x,y=1}$ for all $y \in X$.

1 Introduction

In the context of Markov semigroups the relevant norm is the ℓ^1 norm. All of the probabilistic properties of Markov operators are naturally formulated in terms of this norm, and they need not even be bounded with respect to the ℓ^2 norm. Because Markov semigroup is the study of dynamical systems with stochastic

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perturbations, we can make use of it to obtain a linear transformation of the space of integrable functions which preserves the set of densities. Suppose X is a Banach space, $X_n \subseteq X$ is a finite set, $T(t)$ the C_0-semigroup, $\omega - OCP_n$ the ω-order preserving partial contraction mapping, M_m be a matrix, $L(X)$ be a bounded linear operator on X, P_n a partial transformation semigroup, $\rho(A)$ a resolvent set, $\sigma(A)$ a spectrum of A and $A \in \omega - OCP_n$ is a generator of C_0-semigroup. This paper consists of results of ω-order preserving partial contraction mapping generating a continuous time Markov semigroup.

Akinyele *et al.* [1], obtained perturbation results of infinitesimal generator in semigroup of linear operator. Batty [2], introduced some spectral conditions for stability of one-parameter semigroup and also in [3] Batty *et al.*, showed some asymptotic behavior of semigroup of operator. Balakrishnan [4], obtained an operator calculus for infinitesimal generators of semigroup. Banach [5], established and introduced the concept of Banach spaces. Chill and Tomilov [6], deduced some resolvent approach to stability operator semigroup. Davies [7], established linear operators and their spectra. Engel and Nagel [8], introduced one-parameter semigroup for linear evolution equations. Rӑbiger and Wolf [9] introduced some spectral and asymptotic properties of dominated operator. Rauf and Akinyele [10], established ω-order preserving partial contraction mapping and established its properties, also in [11], Rauf *et al.* presented some results of stability and spectra properties on semigroup of linear operator. Vrabie [12], proved some results of C_0-semigroup and its applications. Yosida [13], obtained some results on differentiability and representation of one-parameter semigroup of linear operators.

2 Preliminaries

Definition 2.1 (C_0-semigroup) [8]
A C_0*-semigroup* is a strongly continuous one parameter semigroup of bounded linear operator on Banach space.

Definition 2.2 (ω-OCP_n) [11]
A transformation $\alpha \in P_n$ is called *ω-order preserving partial contraction mapping* if $\forall x, y \in$ Dom$\alpha : x \leq y \implies \alpha x \leq \alpha y$ and at least one of its transformation must satisfy $\alpha y = y$ such that $T(t+s) = T(t)T(s)$ whenever $t, s > 0$ and otherwise for $T(0) = I$.

Definition 2.3 (Perturbation) [1]
Let $A : D(A) \subseteq X \to X$ be the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ and consider a second operator $B : D(B) \subseteq X \to X$ such that the sum $A + B$ generates a strongly continuous semigroup $(S(t))_{t\geq 0}$. We say that A *is perturbed by operator* B or that B *is a perturbation of* A.

Definition 2.4 (Analytic Semigroup) [12]
We say that a C_0-semigroup $\{T(t); t \geq 0\}$ is *analytic* if there exists $0 < \theta \leq \pi$, and a mapping $S : \bar{\mathbb{C}}_\theta \to L(X)$ such that:
(i) $T(t) = S(t)$ for each $t \geq 0$;
(ii) $S(z_1 + z_2) = S(z_1)S(z_2)$ for $z_1, z_2 \in \bar{\mathbb{C}}_\theta$;
(iii) $lim_{z_1 \in \bar{\mathbb{C}}_\theta, z_1 \to 0} S(z_1)x = x$ for $x \in X$; and
(iv) the mapping $z_1 \to S(z_1)$ is analytic from $\bar{\mathbb{C}}_\theta$ to $L(X)$.
In addition, for each $0 < \delta < \theta$, the mapping $z_1 \to S(z_1)$ is bounded from $\mathbb{C}_\delta$ to $L(X)$, then the C_0-semigroup $\{T(t); t \geq 0\}$ is called *analytic and uniformly bounded.*

Definition 2.5 (Markov generator) [7]
Let $A : D(A) \subset \ell(X) \to \ell(X)$ be a linear operator, $(X, \|\|)$ be a locally compact normed space. We say that A is a *Markov generator* if:
(i) $D(A)$ is dense in $\ell(X)$;
(ii) A fulfills the positive maximum principle; and
(iii) $R(\lambda I - A) = \ell(X)$ for some $\lambda > 0$.

Example 1
2×2 matrix $[M_m(\mathbb{N} \cup \{0\})]$

Suppose

$$A = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$$

and let $T(t) = e^{tA}$, then

$$e^{tA} = \begin{pmatrix} e^{2t} & e^{I} \\ e^{t} & e^{2t} \end{pmatrix}.$$

Example 2

3×3 matrix $[M_m(\mathbb{N} \cup \{0\})]$
Suppose

$$A = \begin{pmatrix} 2 & 2 & 3 \\ 2 & 2 & 2 \\ 1 & 2 & 2 \end{pmatrix}$$

and let $T(t) = e^{tA}$, then

$$e^{tA} = \begin{pmatrix} e^{2t} & e^{2t} & e^{3t} \\ e^{2t} & e^{2t} & e^{2t} \\ e^{t} & e^{2t} & e^{2t} \end{pmatrix}.$$

Example 3

3×3 matrix $[M_m(\mathbb{C})]$, we have for each $\lambda > 0$ such that $\lambda \in \rho(A)$ where $\rho(A)$ is a resolvent set on X.
Suppose we have

$$A = \begin{pmatrix} 2 & 2 & 3 \\ 2 & 2 & 2 \\ 1 & 2 & 2 \end{pmatrix}$$

and let $T(t) = e^{tA_\lambda}$, then

$$e^{tA_\lambda} = \begin{pmatrix} e^{2t\lambda} & e^{2t\lambda} & e^{3t\lambda} \\ e^{2t\lambda} & e^{2t\lambda} & e^{2t\lambda} \\ e^{t\lambda} & e^{2t\lambda} & e^{2t\lambda} \end{pmatrix}.$$

3 Main Results

This section presents results of continuous time Markov semigroup generated by ω-OCP_n using Markov and irreducible operators:

Theorem 3.1

Assume $A \in \omega - OCP_n$ is the infinitesimal generator of a continuous time Markov semigroup. Let $A_R : \ell^p_R(X) \to \ell^p_R(X)$ be real and let A_C be its complexification, where $1 \leqslant p < \infty$. Then A_C has the same norm as A_R.

Proof:

For $p = 1$, we have that

$$\|A\| = \sup_{y \in X} \left\{ \sum_{x \in X} |A_{x,y}| \right\} \tag{3.1}$$

so that an infinite matrix $A_{x,y}$ determines a bounded operator A on $\ell(X)$ if and only if the RHS of (3.1) is finite.

If $p > 1$, then the inequality

$$\|A_R\| \leqslant \|A_C\|$$

follows directly from the definition of the norm of an operator. To prove the converse we use the fact that

$$|a + ib|^p = c^{-1} \int_{-\pi}^{\pi} |a \cos\theta + b \sin\theta|^p d\theta \tag{3.2}$$

for all $a, b \in \mathbb{R}$, where

$$c := \int_{-\pi}^{\pi} |\cos\theta|^p d\theta. \tag{3.3}$$

If $f, g \in \ell^p_{\mathbb{R}}(X)$, then

$$\begin{aligned}
\|A_C(f+ig)\|^p &= \sum_{x\in X} |(A_R f)(x) + i(A_R g)(x)|^p \\
&= C^{-1} \sum_{x\in X} \int_{\pi}^{\pi} |(A_R f)(x)\cos(\theta) + (A_p g)(x)\sin(\theta)|^p d\theta \\
&= C^{-1} \int_{-\pi}^{\pi} \sum_{x\in X} |(A_R f)(x)\cos(\theta) + (A_R g)(x)\sin(\theta)|^p d\theta \\
&= C^{-1} \int_{-\pi}^{\pi} \|A_R(f\cos(\theta) + g\sin(\theta))\|^p d\theta \\
&\leqslant \|A_R\|^p C^{-1} \int_{-\pi}^{\pi} \|f(x)\cos(\theta) + g(x)\sin\|^p d\theta \\
&= \|A_R\|^p C^{-1} \int_{-\pi}^{\pi} \sum_{x\in X} |f(x)\cos(\theta) + g(x)\sin(\theta)|^p d\theta \\
&= \|A_R\|^p C^{-1} \sum_{x\in X} |f(x)\cos(\theta) + g(x)\sin(\theta)|^p d\theta \\
&= \|A_R\|^p \sum_{x\in X} |f(x) + ig(x)|^p \\
&= \|A_R\|^p \|f+ig\|^p. \qquad (3.4)
\end{aligned}$$

Hence, the proof is completed.

Theorem 3.2

Let $A : X \to X$ be a positive operator. Then

$$\|A\| = \sup\left\{\frac{\|Af\|}{\|f\|} : 0 \neq f \in X_+\right\}. \qquad (3.5)$$

If A is positive and has norm 1, then

$$\mathcal{L} := \{f \in X,\ A \in \omega - OCP_n : Af = f\}$$

is a linear sublattice of X.

Proof:

Suppose $f \in X$, then $-|f| \leqslant f \leqslant |f|$ and $A \geqslant 0$ together implies $-A(|f|) \leqslant$

$A(f) \leqslant A(|f|)$, so that

$$|A(f)| \leqslant A(|f|).$$

If c denotes the RHS of (3.5), then one can see immediately that $c \leqslant \|A\|$.

Conversely,

$$\|Af\| = \||A(f)|\| \leqslant \|A(|f|)\| \leqslant c\||f|\| = \|f\| \tag{3.6}$$

for all $f \in X$ and $A \in \omega - OCP_n$. Therefore, $\|A\| \leqslant c$.

If $\|A\| = 1$, and $Af = f$, then

$$\||f|\| = \|f\| = \|A(f)\| = \||A(f)|\| \leqslant \|A(|f|)\| \leqslant \||f|\|.$$

Since the two extreme quantities are equal, we have that if $0 \leqslant f \leqslant g \in X_+$. Then

$$\|f\| \leqslant \|g\|,$$

and

$$\|f\| = \|g\| \tag{3.7}$$

implies

$$f = g.$$

We define a linear sublattice $\mathcal{L}$ of X to be linear subspace such that $f \in \mathcal{L}$ implies $|f| \in \mathcal{L}$ and implies

$$A(|f|) = |A(f)| = |f|, \tag{3.8}$$

so $\mathcal{L}$ is a linear sublattice.

Given $f \in X$, we define:

$$\sup(f) := \{x \in X : f(x) \neq 0\}.$$

Assume A is a positive operator on X, we say that $E \subseteq X$ is an invariant set if for every $f \in X_+$ such that $\sup(f) \subseteq E$, and we have $\sup(Af) \subseteq E$.

This is equivalent to the condition that $x \in E$ and $(x, y) \in E$ implies $y \in E$. We say that A is irreducible if the only invariant sets are X and ϕ. This is equivalent to the operator-theoretic condition that for all $x, y \in X$ there exists $n > 0$ such that $(A^n)_{x,y} > 0$. From a graph-theoretic perspective, irreducibility demands for all $x, y \in X$ there exists a path

$$w := (y = x_0, x_1, \ldots, x_n = x) \tag{3.9}$$

such that $(x_{r-1}, x_r) \in E$ for all relevant r, and this achieved the proof.

Theorem 3.3

Suppose A is the infinitesimal generator of a continuous time Markov semigroup. If $A : X \to X$ is a positive, irreducible operator and $\|A\| = 1$, then the subspace $\mathcal{L} := \{f : Af = f\}$ is of dimension at most 1. If $\mathcal{L}$ is one-dimensional, then the associated eigenfunction satisfies $f(x) > 0$ for all $x \in X$ and $A \in \omega - OCP_n$. (Possible after replacing f by $-f$).

Proof:

Let $f \in \mathcal{L}_+ = \mathcal{L} \cap X_+$ and $f(y) > 0$ and $A_{x,y} > 0$, then

$$f(x) = (Af)(x) = \sum_{u \in X} A_{x,u} f(u) \geqslant A_{x,y} f(y) > 0. \tag{3.10}$$

This implies that the set $E := \sup(f)$ is invariant with respect to A. Using the irreducibility assumption, we deduce that $f \in \mathcal{L}_+$ implies $\sup(f) = X$, unless f vanishes identically. If $f \in \mathcal{L}$, then $f_+ \in \mathcal{L}$ because $\mathcal{L}$ is a sublattice. It follows that either $f_+ = 0$ or $f_- = 0$. This establishes that every non-zero $f \in \mathcal{L}$ is strictly positive, possibly after multiplying it by -1. If $f, g \in \mathcal{L}_+$ and $\lambda = f(a)/g(a)$ for some choice of $a \in X$, then $h = f - \lambda g$ lies in $\mathcal{L}$ and vanishes at a. Hence h is identically zero, and f, g are linearly dependent. We conclude that $\dim(\mathcal{L}) = 1$ and this achieved the proof.

Theorem 3.4

Let $A : \ell^1(X) \to \ell^1(X)$ be a bounded linear operator. Then e^{At} is a positive operator for all $t \geqslant 0$ if and ony if $A(x, y) \geqslant 0$ for all $x \neq y$. It is a Markov operator for all $t \geqslant 0$ if and only if in addition to the above condition, we have

$$A(y, y) = -\sum_{\{x : x \neq y\}} A(x, y)$$

for all $y \in X$ *and* $A \in \omega - OCP_n$.

Proof:

Assume e^{At} is positive and $x \neq y$. Then

$$A(x,y) = \lim_{t \to 0_+} t^{-1} \langle e^{At} \delta y \delta x \rangle$$

and the RHS is non-negative. Conversely, suppose that $A(x,y) \geqslant 0$ for all $x \neq y$. We may write $A := B + cl$ where $B \geqslant 0$ and $c := \inf\{A(x,x) : x \in X\}$. Note that $|c| \leqslant \|A\|$. It follows that

$$e^{At} = e^{ct} \sum_{n=0}^{\infty} \frac{t^n B^n}{n!} \geqslant 0$$

for all $t \geqslant 0$. If e^{At} is positive for all $t \geqslant 0$ and $A \in \omega - OCP_n$, then it is a Markov semigroup if and only if

$$\langle e^{At} f, 1 \rangle = \langle f, 1 \rangle$$

for all $f \in \ell^1(X)$ and $t \geqslant 0$. Differentiating this at $t = 0$ implies that $\langle Af, 1 \rangle = 0$ for all $f \in \ell(x)$, or equivalently that

$$\sum_{x \in X} A(x,y) = 0$$

for all $x, y \in X$ and $A \in \omega - OCP_n$.

Conversely, if this holds, then

$$\langle e^{At} f, 1 \rangle = \sum_{n=0}^{\infty} \frac{t^n}{n!} \langle A^n f, 1 \rangle = \langle f, 1 \rangle \tag{3.11}$$

for all $t \in \mathbb{R}$.

We need to note that a continuous time Markov semigroup $P_t := e^{At}$ acting on $\ell^1(X)$ with a bounded generator A is actually defined for all $t \in \mathbb{C}$, not just for $t \geqslant 0$. The Markov property implies that $\|P_t\| = 1$ for all $t \geqslant 0$. However, for $t < 0$ the operators P_t are generally not positie. If X is finite, then all of the eigenvalue of A must satisfy $Re(\lambda) \leqslant 0$ because $|e^{\lambda t}| \leqslant \|e^{At}\| = 1$ for all $t \geqslant 0$.

By evaluating the trace of A one sees that atleast one eigenvalue λ must satisfy $Re(\lambda) < 0$, unless A is identically zero. Since $\|e^{At}\| \geqslant |e^{\lambda t}|$, it follows that the norm grows exponentially as $t \to -\infty$. Hence, the proof is complete.

Theorem 3.5

If A is the bounded infinitesimal generator of a continuouns time Markov semigroup P_t acting on $\ell^1(X)$ and Q is a Markov operator acting on $\ell^1(X)$ and c is a positive constant operator, then $A := c(Q - I)$ is the generator of a Markov semigroup P_t.

Proof:

First, we have to check that $A(x, y) \geqslant 0$ for all $x \neq y$ and that $\langle Af, 1\rangle = 0$ for all $f \in \ell^1(X)$ and $A \in \omega - OCP_n$. Suppose the semigroup operator is given by

$$P_t := e^{ctQ - ctI} = e^{-ct} \sum_{n=0}^{\infty} \frac{c^n t^n Q^n}{n!}.$$

Expressing further, we have

$$P_t = \sum_{n=0}^{\infty} a(t, n) Q^n$$

where $a(t, n) > 0$ for all $t, n \geqslant 0$ and $\sum_{n=o}^{\infty} a(t, n) = 1$. Thus equation can be interpreted as describing a particle which makes jumps at random real times according to the Poisson law $a(t, n)$ and when it jumps it does so from one point of X to another according to the law of Q. If $c := \sup\{-A(x, x) : x \in X\}$, then $0 \leqslant c \leqslant \|A\|$. The case $c = 0$ implies $A = 0$, for which we can make any choice of Q. Now suppose that $c > 0$, it is immediate that $B := A + cI$ is a positive operator and that

$$\langle Bf, 1\rangle = \langle Af, 1\rangle + c\langle f, 1\rangle = c\langle f, 1\rangle \tag{3.12}$$

for all $\ell^1(X)$ and $A, B \in \omega - OCP_n$. Therefore $Q := c^{-1}B$ is a Markov operator and $A = c(Q - I)$.

If $P_t(x, y) > 0$ for all $x, y \in X$, $A, B \in \omega - OCP_n$ and $t > 0$, we say that P_t is irreducible. Hence, the proof is achieved.

Conclusion

In this paper, it has been established that ω-order preserving partial contraction mapping generates a continuous time Markov semigroup using a Markov and irreducible in Banach space.

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